TEMPLE OF NEPTUNE, AT PAESTUM.

THE AMERICAN HOUSE CARPENTER.

A TREATISE

ON THE

ART OF BUILDING.

COMPRISING

STYLES OF ARCHITECTURE, STRENGTH OF MATERIALS,

AND

THE THEORY AND PRACTICE OF THE CONSTRUCTION OF FLOORS, FRAMED GIRDERS, ROOF TRUSSES, ROLLED-IRON BEAMS, TUBULAR-IRON GIRDERS, CAST-IRON GIRDERS, STAIRS, DOORS, WINDOWS, MOULDINGS, AND CORNICES;

TOGETHER WITH

A COMPEND OF MATHEMATICS.

A MANUAL FOR THE PRACTICAL USE OF

ARCHITECTS, CARPENTERS, STAIR-BUILDERS,

AND OTHERS.

EIGHTH EDITION,
REWRITTEN AND ENLARGED.

BY

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PREFACE.

SINCE the publication of the first edition of this work, six subsequent editions have been issued; but, although from time to time many additions to its pages and revisions of its subject-matter have been made, still its several issues have always been printed substantially from the original stereotype plates. In this edition, however, the book has been extensively remodelled and expanded, the greater portion of it rewritten, and the whole put in a new dress by being newly set up in type uniform in style with that of the late author's recent work, *Transverse Strains*. To this revision—a labor of love to him—he devoted all the time he could spare from his other pressing engagements for a year or more, and by close and arduous application brought the book to a successful termination, notwithstanding the engrossing nature of his customary business avocations. Although essentially an elementary work, and intended originally for a class of minds not generally favored with opportunities for securing a very extended form of education, either in the store of information acquired or in the discipline of mind which culture confers, still it has been his aim to embody in its pages so complete and exhaustive a treatment of the various subjects discussed, and so practical and useful a collection of data and the rules governing their application, as to make it also not unworthy the attention of those who have been more highly favored in that respect.

In all the various trades connected with building it is the intelligent workman that commands the greatest respect, and who receives in all cases the highest remuneration. As apprentice, journeyman, and master-builder, his course is upward and onward, and success crowns his efforts in all that he undertakes. There is a kind of freemasonry in the very air that surrounds the skilful, intelligent man, that gives him a pass at once into the appreciation and recognition of all those whose regard is valuable. We admire and respect the plodding toil of the honest, patient laborer, whose humble task may tax his muscles though not his mind, but we yield a deeper homage to the skilful hand and tutored eye that accomplish wonders in art and science through perseverance in aspiring studies. It was to excite in the minds of workmen like these an ambition to excel in their calling, and to point out to them the surest path to that consummation, that the preparation of this volume was undertaken; that all its tendencies are in that direction, and that it cannot well fail

of its purpose when judiciously used, must be the conviction of all who will take the trouble to examine its pages.

In the first part of the book matters more particularly relating to building are treated of. The first section is in the nature of an introduction, serving by its historical references to excite an interest in the general subject, while in the second are presented the methods of erecting edifices in accordance with the acknowledged principles of sound construction. In the remaining sections of Part I. the several well-defined branches of house-building, as stairs, doors and windows, etc., are illustrated and explained. In the second part the more useful rules and simple problems of mathematics are reduced to an easily acquired form, and adapted to the necessities of the ordinary workman. By studying the latter, the young mechanic may not only improve and strengthen his mind, but grow more self-reliant daily, demonstrating in his own experience that scientific knowledge gives power. By carefully studying this part of the book he will see how easy it is to acquire the knowledge of solving problems by signs and symbols, commonly called Algebra (although looked upon by the uninitiated as almost incomprehensible), and thus find it easy to understand all the illustrations of the various subjects wherein those condensed forms of expression are used. Useful problems in geometry, described in simple language, and hints upon the subject of drawing and shading, are also to be found in Part II. A glossary of architectural terms and many useful tables are provided in the Appendix, and finally, an Index is added to aid in referring to special subjects. The full-plate illustrations are inserted to make it attractive to the general reader, and at the same time to serve as explanatory of the historical portion of the volume.

It will not be denied that the class of information herein furnished is one of the most instructive and useful that can be presented to the practical mind of a workingman, or to any mind engaged in mechanical pursuits. The impress stamped upon it by the author's peculiar line of study is not to be effaced, but this has given it characteristics of originality and strength not to be found in a mere compilation.

THE EDITOR.

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PART I.

SECTION I.—ARCHITECTURE.

ART. **1.—Building Defined.**—Building and Architecture are technical terms by some thought to be synonymous; but there is a distinction. Architecture has been defined to be—"the art of building;" but more correctly it is—"the art of designing and constructing buildings, in accordance with such principles as constitute stability, utility, and beauty." The literal signification of the Greek word *architecton*, from which the word *architect* is derived, is chief-carpenter; and the architect who designs and builds well may truly be considered the chief *builder*. Of the three classes into which architecture has been divided—viz., Civil, Military, and Naval—the first is that which refers to the construction of edifices known as dwellings, churches, and other public buildings, bridges, etc., for the accommodation of civilized man—and is the subject of the remarks which follow.

2.—Antique Buildings; Tower of Babel.—Building is one of the most ancient of the arts: the Scriptures inform us of its existence at a very early period. Cain, the son of Adam, "builded a city, and called the name of the city after the name of his son, Enoch;" but of the peculiar style or manner of building we are not informed. It is presumed that it was not remarkable for beauty, but that utility and perhaps stability were its characteristics. Soon after the deluge—that memorable event, which removed from existence all traces of the works of man—the Tower of Babel

was commenced. This was a work of such magnitude that the gathering of the materials, according to some writers, occupied three years; the period from its commencement until the work was abandoned was twenty-two years; and the bricks were like blocks of stone, being twenty feet long, fifteen broad, and seven thick. Learned men have given it as their opinion that the tower in the temple of Belus at Babylon was the same as that which in the Scriptures is called the Tower of Babel. The tower of the temple of Belus was square at its base, each side measuring one furlong, and consequently half a mile in circumference. Its form was that of a pyramid, and its height was 660 feet. It had a winding passage on the outside from the base to the summit, which was wide enough for two carriages.

3.—Ancient Cities and Monuments.—Historical accounts of ancient cities, such as Babylon, Palmyra, and Nineveh of the Assyrians; Sidon, Tyre, Aradus, and Serepta of the Phœnicians; and Jerusalem, with its splendid temple, of the Israelites—show that architecture among them had made great advances. Ancient monuments of the art are found also among other nations; the subterraneous temples of the Hindoos upon the islands Elephanta and Salsetta; the ruins of Persepolis in Persia; pyramids, obelisks, temples, palaces, and sepulchres in Egypt—all prove that the architects of those early times were possessed of skill and judgment highly cultivated. The principal characteristics of their works are gigantic dimensions, immovable solidity, and, in some instances, harmonious splendor. The extraordinary size of some is illustrated in the pyramids of Egypt. The largest of these stands not far from the city of Cairo: its base, which is square, covers about 11¼ acres, and its height is nearly 500 feet. The stones of which it is built are immense—the smallest being full thirty feet long.

4.—Architecture in Greece.—Among the Greeks, architecture was cultivated as a fine art. Dignity and grace were added to stability and magnificence. In the Doric order, their first style of building, this is fully exemplified. Phidias, Ictinus, and Calicrates are spoken of as masters in

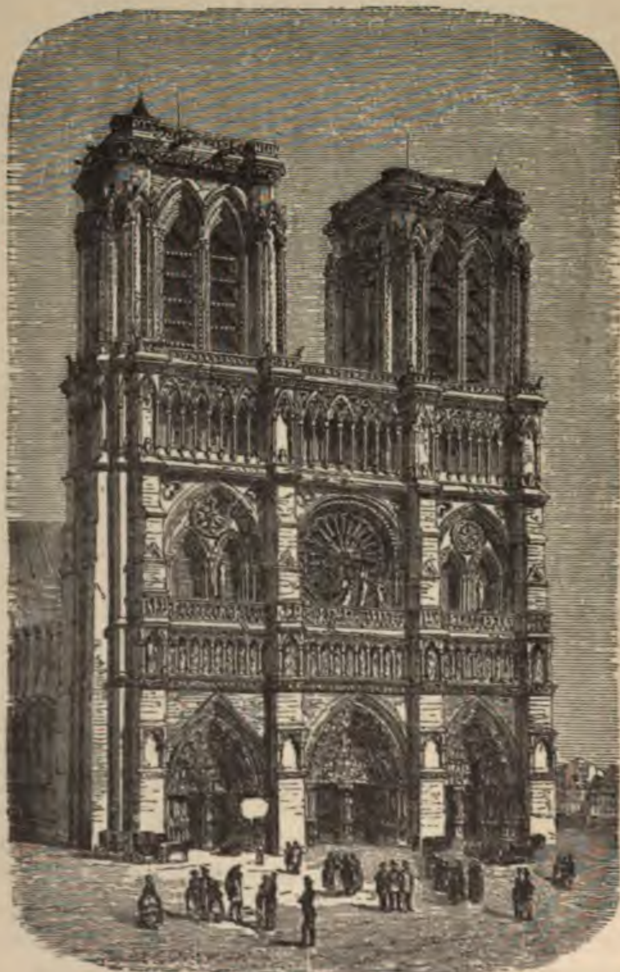
THE PANTHEON ROME.

the art at this period: the encouragement and support of Pericles stimulated them to a noble emulation. The beautiful temple of Minerva, called the Parthenon, erected upon the acropolis of Athens, the Propyleum, the Odeum, and others, were lasting monuments of their success. The Ionic and Corinthian orders were added to the Doric, and many magnificent edifices arose. These exemplified, in their chaste proportions, the elegant refinement of Grecian taste. Improvement in Grecian architecture continued to advance until perfection seems to have been attained. The specimens which have been partially preserved exhibit a combination of elegant proportion, dignified simplicity, and majestic grandeur. Architecture among the Greeks was at the height of its glory at the period immediately preceding the Peloponnesian war; after which the art declined. An excess of enrichment succeeded its former simple grandeur; yet a strict regularity was maintained amid the profusion of ornament. After the death of Alexander, 323 B.C., a love of gaudy splendor increased: the consequent decline of the art was visible, and the Greeks afterwards paid but little attention to the science.

5.—Architecture in Rome.—While the Greeks illustrated their knowledge of architecture in the erection of their temples and other public buildings, the Romans gave their attention to the science in the construction of the many aqueducts and sewers with which Rome abounded; building no such splendid edifices as adorned Athens, Corinth, and Ephesus, until about 200 years B.C., when their intercourse with the Greeks became more extended. Grecian architecture was introduced into Rome by Sylla; by whom, as also by Marius and Cæsar, many large edifices were erected in various cities of Italy. But under Cæsar Augustus, at about the beginning of the Christian era, the art arose to the greatest perfection it ever attained in Italy. Under his patronage Grecian artists were encouraged, and many emigrated to Rome. It was at about this time that Solomon's temple at Jerusalem was rebuilt by Herod—a Roman. This was 46 years in the erection, and was most probably of the Grecian style of building—perhaps of the

Corinthian order. Some of the stones of which it was built were 46 feet long, 21 feet high, and 14 thick; and others were of the astonishing length of 82 feet. The porch rose to a great height; the whole being built of white marble exquisitely polished. This is the building concerning which it was remarked: "Master, see what manner of stones, and what buildings are here." For the construction of private habitations also, finished artists were employed by the Romans: their dwellings being often built with the finest marble, and their villas splendidly adorned. After Augustus, his successors continued to beautify the city, until the reign of Constantine, who, having removed the imperial residence to Constantinople, neglected to add to the splendor of Rome; and the art, in consequence, soon fell from its high excellence.

6.—Rome and Greece.—Thus Rome was indebted to Greece for her knowledge of architecture—not only for the knowledge of its principles, but also for many of the best buildings themselves; these having been originally erected in Greece, and stolen by the unprincipled conquerors—taken down and removed to Rome. Greece was thus robbed of her best monuments of architecture. Touched by the Romans, Grecian architecture lost much of its elegance and dignity. The Romans, though justly celebrated for their scientific knowledge as displayed in the construction of their various édifices, were not capable of appreciating the simple grandeur, the refined elegance of the Grecian style; but sought to improve upon it by the addition of luxurious enrichment, and thus deprived it of true elegance. In the days of Nero, whose palace of gold is so celebrated, buildings were lavishly adorned. Adrian did much to encourage the art; but not satisfied with the simplicity of the Grecian style, the artists of his time aimed at inventing new ones, and added to the already redundant embellishments of the previous age. Hence the origin of the pedestal, the great variety of intricate ornaments, the convex frieze, the round and the open pediments, etc. The rage for luxury continued until Alexander Severus, who made some im-



CATHEDRAL OF NOTRE DAME, PARIS.

provement; but very soon after his reign the art began rapidly to decline, as particularly evidenced in the mean and trifling character of the ornaments.

7.—Architecture Debased.—The Goths and Vandals overran Italy, Greece, Asia, and Africa, destroying most of their works of ancient architecture. Cultivating no art but that of war, these savage hordes could not be expected to take any interest in the beautiful forms and proportions of their habitations. From this time architecture assumed an entirely different aspect. The celebrated styles of Greece were unappreciated and forgotten; and modern architecture made its first appearance on the stage of existence. The Goths, in their conquering invasions, gradually extended it over Italy, France, Spain, Portugal, and Germany, into England. From the reign of Galienus may be reckoned the total extinction of the arts among the Romans. From this time until the sixth or seventh century, architecture was almost entirely neglected. The buildings which were erected during this suspension of the arts were very rude. Being constructed of the fragments of the edifices which had been demolished by the Visigoths in their unrestrained fury, and the builders being destitute of a proper knowledge of architecture, many sad blunders and extensive patch-work might have been seen in their construction—entablatures inverted, columns standing on their wrong ends, and other ridiculous arrangements characterized their clumsy work. The vast number of columns which the ruins around them afforded they used as piers in the construction of arcades—which by some is thought, after having passed through various changes, to have been the origin of the plan of the Gothic cathedral. Buildings generally, which are not of the classical styles, and which were erected after the fall of the Roman empire, have by some been indiscriminately included under the term *Gothic*. But the changes which architecture underwent during the Mediæval age show that there were then several distinct modes of building.

8.—The Ostrogoths.—Theodoric, a friend of the arts, who reigned in Italy from A.D. 493 to 525, endeavored to

restore and preserve some of the ancient buildings; and erected others, the ruins of which are still seen at Verona and Ravenna. Simplicity and strength are the characteristics of the structures erected by him; they are, however, devoid of grandeur and elegance, or fine proportions. These are properly of the GOTHIC style; by some called the *old* Gothic, to distinguish it from the pointed Gothic.

9.—The Lombards, who ruled in Italy from A.D. 568, had no taste for architecture nor respect for antiquities. Accordingly, they pulled down the splendid monuments of classic architecture which they found standing, and erected in their stead huge buildings of stone which were greatly destitute of proportion, elegance, or utility—their characteristics being scarcely anything more than stability and immensity combined with ornaments of a puerile character. Their churches were decorated with rows of small columns along the cornice of the pediment, small doors and windows with circular heads, roofs supported by arches having arched buttresses to resist their thrust, and a lavish display of incongruous ornaments. This kind of architecture is called the LOMBARD style, and was employed in the seventh century in Pavia, the chief city of the Lombards; at which city, as also at many other places, a great many edifices were erected in accordance with its peculiar forms.

10.—The Byzantine Architects, of Byzantium, Constantinople, erected many spacious edifices; among which are included the cathedrals of Bamberg, Worms, and Mentz, and the most ancient part of the minster at Strassburg; in all of these they combined the classic styles with the crude Lombardian. This style is called the LOMBARD-BYZANTINE. To the last style there were afterwards added cupolas similar to those used in the East, together with numerous slender pillars with elaborate capitals, and the many minarets which are the characteristics of the proper *Byzantine*, or *Oriental* style.

11.—The Moors.—When the Arabs and Moors destroyed the kingdom of the Goths, the arts and sciences were mostly

MOSQUE AT CAIRO.

in possession of the Musselmen-conquerors; at which time there were three kinds of architecture practised; viz.: the Arabian, the Moorish, and the Lombardian. The ARABIAN style was formed from Greek models, having circular arches added, and towers which terminated with globes and minarets. The MOORISH is very similar to the Arabian, being distinguished from it by arches in the form of a horseshoe. It originated in Spain in the erection of buildings with the ruins of Roman architecture, and is seen in all its splendor in the ancient palace of the Mohammedan monarchs at Grenada, called the *Alhambra*, or *red-house*. The style which was originated by the Visigoths in Spain by a combination of the Arabian and Moorish styles, was introduced by Charlemagne into Germany. On account of the changes and improvements it there underwent, it was, at about the 13th or 14th century, termed the *German* or *romantic* style. It is exhibited in great perfection in the towers of the minster of Strassburg, the cathedral of Cologne and other edifices. The most remarkable features of this lofty and aspiring style are the lancet or pointed arch, clustered pillars, lofty towers, and flying buttresses. It was principally employed in ecclesiastical architecture, and in this capacity introduced into France, Italy, Spain, and England.

12.—The Architecture of England: is divided into the *Norman*, the *Early-English*, the *Decorated*, and the *Perpendicular* styles. The Norman is principally distinguished by the character of its ornaments—the chevron, or *zigzag*, being the most common. Buildings in this style were erected in the 12th century. The Early-English is celebrated for the beauty of its edifices, the chaste simplicity and purity of design which they display, and the peculiarly graceful character of its foliage. This style is of the 13th century. The Decorated style, as its name implies, is characterized by a great profusion of enrichment, which consists principally of the crocket, or feathered-ornament, and ball-flower. It was mostly in use in the 14th century. The Perpendicular style, which dates from the 15th century, is distinguished by its high towers, and parapets surmounted with spires similar in number and grouping to oriental minarets.

13.—Architecture Progressive.—The styles erroneously termed *Gothic* were distinguished by peculiar characteristics as well as by different names. The first symptoms of a desire to return to a pure style in architecture, after the ruin caused by the Goths, was manifested in the character of the art as displayed in the church of St. Sophia at Constantinople, which was erected by Justinian in the 6th century. The church of St. Mark at Venice, which arose in the 10th or 11th century, is a most remarkable building; a compound of many of the forms of ancient architecture. The cathedral at Pisa, a wonderful structure for the age, was erected by a Grecian architect in 1016. The marble with which the walls of this building were faced, and of which the four rows of columns that support the roof are composed, is said to be of an excellent character. The Campanile, or leaning-tower as it is usually called, was erected near the cathedral in the 12th century. Its inclination is generally supposed to have arisen from a poor foundation; although by some it is said to have been thus constructed originally, in order to inspire in the minds of the beholder sensations of sublimity and awe. In the 13th century, the science in Italy was slowly progressing; many fine churches were erected, the style of which displayed a decided advance in the progress towards pure classical architecture. In other parts of Europe, the Gothic, or pointed style was prevalent. The cathedral at Strassburg, designed by Irwin Steinbeck, was erected in the 13th and 14th centuries. In France and England during the 14th century, many very superior edifices were erected in this style.

14.—Architecture in Italy.—In the 14th and 15th centuries, architecture in Italy was greatly revived. The masters began to study the remains of ancient Roman edifices; and many splendid buildings were erected, which displayed a purer taste in the science. Among others, St. Peter's of Rome, which was built about this time, is a lasting monument of the architectural skill of the age. Giocondo, Michael Angelo, Palladio, Vignola, and other celebrated architects, each in their turn, did much to restore the art to its

INTERIOR OF ST. SOPHIA, CONSTANTINOPLE.

former excellence. In the edifices which were erected under their direction, however, it is plainly to be seen that they studied not from the pure models of Greece, but from the remains of the deteriorated architecture of Rome. The high pedestal, the coupled columns, the rounded pediment, the many curved-and-twisted enrichments, and the convex frieze, were unknown to pure Grecian architecture. Yet their efforts were serviceable in correcting, to a good degree, the very impure taste that had prevailed since the overthrow of the Roman empire.

15.—The Renaissance.—The Italian masters and numerous artists who had visited Italy for the purpose, spread the Roman style over various countries of Europe; which was gradually received into favor in place of the pointed Gothic. This fell into disuse; although it has of late years been again cultivated. It requires a building of great magnitude and complexity for a perfect display of its beauties. In America, the pure Grecian style was at first more or less studied; and perhaps the simplicity of its principles would be better adapted to a republican country than the more intricate mediæval styles; yet these, during the last quarter of a century, have been extensively studied, and now wholly supersede the Grecian styles.

16.—Styles of Architecture.—It is generally acknowledged that the various styles in architecture were the results of necessity, and originated in accordance with the different pursuits of the early inhabitants of the earth; and were brought by their descendants to their present state of perfection, through the propensity for imitation and desire of emulation which are found more or less among all nations. Those that followed agricultural pursuits, from being employed constantly upon the same piece of land, needed a permanent residence, and the wooden *hut* was the offspring of their wants; while the shepherd, who followed his flocks and was compelled to traverse large tracts of country for pasture, found the *tent* to be the most portable habitation; again, the man devoted to hunting and fishing—an idle and vagabond way of living—is naturally supposed to have been

content with the *cavern* as a place of shelter. The latter is said to have been the origin of the Egyptian style; while the curved roof of Chinese structures gives a strong indication of their having had the tent for their model; and the simplicity of the original style of the Greeks (the Doric) shows quite conclusively, as is generally conceded, that its original was of wood. The pointed, or ecclesiastical style, is said to have originated in an attempt to imitate the bower, or grove of trees, in which the ancients performed their idol-worship. But it is more probably the result of repeated scientific attempts to secure real strength with apparent lightness; thus giving a graceful, aspiring effect.

17.—Orders: or styles, in architecture are numerous; and a knowledge of the peculiarities of each is important to the student in the art. An ORDER, in architecture, is composed of three principal parts, viz.: the Stylobate, the Column, and the Entablature. This appertains chiefly to the classic styles.

18.—The Stylobate: is the substructure, or basement, upon which the columns of an order are arranged. In Roman architecture—especially in the interior of an edifice—it frequently occurs that each column has a separate substructure; this is called a *pedestal*. If possible, the pedestal should be avoided in all cases; because it gives to the column the appearance of having been originally designed for a small building, and afterwards pieced out to make it long enough for a larger one.

19.—The Column: is composed of the base, shaft, and capital.

20.—The Entablature: above and supported by the columns, is horizontal; and is composed of the architrave, frieze, and cornice. These principal parts are again divided into various members and mouldings.

21.—The Base: of a column is so called from *basis*, a foundation or footing.

INTERIOR OF ST. STEPHENS, PARIS.

22.—The Shaft: the upright part of a column standing upon the base and crowned with the capital, is from *shafto*, to dig—in the manner of a well, whose inside is not unlike the form of a column.

23.—The Capital: from *kephale* or *caput*, the head, is the uppermost and crowning part of the column.

24.—The Architrave: from *archi*, chief or principal, and *trabs*, a beam, is that part of the entablature which lies in immediate connection with the column.

25.—The Frieze: from *fibron*, a fringe or border, is that part of the entablature which is immediately above the architrave and beneath the cornice. It was called by some of the ancients *zophorus*, because it was usually enriched with sculptured animals.

26.—The Cornice: from *corona*, a crown, is the upper and projecting part of the entablature—being also the uppermost and crowning part of the whole order.

27.—The Pediment: above the entablature, is the triangular portion which is formed by the inclined edges of the roof at the end of the building. In Gothic architecture, the pediment is called a *gable*.

28.—The Tympanum: is the perpendicular triangular surface which is enclosed by the cornice of the pediment.

29.—The Attic: is a small order, consisting of pilasters and entablature, raised above a larger order, instead of a pediment. An attic story is the upper story, its windows being usually square.

30.—Proportions in an Order.—An order has its several members proportioned to one another by a scale of 60 equal parts, which are called minutes. If the height of buildings were always the same, the scale of equal parts would be a fixed quantity—an exact number of feet and inches. But as buildings are erected of different heights, the column and

its accompaniments are required to be of different dimensions. To ascertain the scale of equal parts, it is necessary to know the height to which the whole order is to be erected. This must be divided by the number of diameters which is directed for the order under consideration. Then the quotient obtained by such division is the length of the scale of equal parts—and is, also, the diameter of the column next above the base. For instance, in the Grecian Doric order the whole height, including column and entablature, is 8 diameters. Suppose now it were desirable to construct an example of this order, forty feet high. Then 40 feet divided by 8 gives 5 feet for the length of the scale; and this being divided by 60, the scale is completed. The upright columns of figures, marked *H* and *P*, by the side of the drawings illustrating the orders, designate the height and the projection of the members. The projection of each member is reckoned from a line passing through the axis of the column, and extending above it to the top of the entablature. The figures represent minutes, or 60ths, of the major diameter of the shaft of the column.

31.—Grecian Styles.—The original method of building among the Greeks was in what is called the *Doric* order: to this were afterwards added the *Ionic* and the *Corinthian*. These three were the only styles known among them. Each is distinguished from the other two by not only a peculiarity of some one or more of its principal parts, but also by a particular destination. The character of the Doric is robust, manly, and Herculean-like; that of the Ionic is more delicate, feminine, matronly; while that of the Corinthian is extremely delicate, youthful, and virgin-like. However they may differ in their general character, they are alike famous for grace and dignity, elegance and grandeur, to a high degree of perfection.

32.—The Doric Order: (*Fig.* 2,) is so ancient that its origin is unknown—although some have pretended to have discovered it. But the most general opinion is, that it is an improvement upon the original wooden buildings of the

FACADE OF ST. PETER'S, ROME.

Grecians. These no doubt were very rude, and perhaps not unlike the following figure.

FIG. 1.—SUPPOSED ORIGIN OF DORIC TEMPLE.

The trunks of trees, set perpendicularly to support the roof, may be taken for columns; the tree laid upon the tops of the perpendicular ones, the architrave; the ends of the cross-beams which rest upon the architrave, the triglyphs; the tree laid on the cross-beams as a support for the ends of the rafters, the bed-moulding of the cornice; the ends of the rafters which project beyond the bed-moulding, the mutules; and perhaps the projection of the roof in front, to screen the entrance from the weather, gave origin to the portico.

The peculiarities of the Doric order are the triglyphs—those parts of the frieze which have perpendicular channels cut in their surface; the absence of a base to the column—as also of fillets between the flutings of the column; and the plainness of the capital. The triglyphs should be so disposed that the width of the metopes—the space between the triglyphs—shall be equal to their height.

33.—The Intercolumniation: or space between the columns, is regulated by placing the centres of the columns under the centres of the triglyphs—except at the angle of the building; where, as may be seen in *Fig.* 2, one edge of

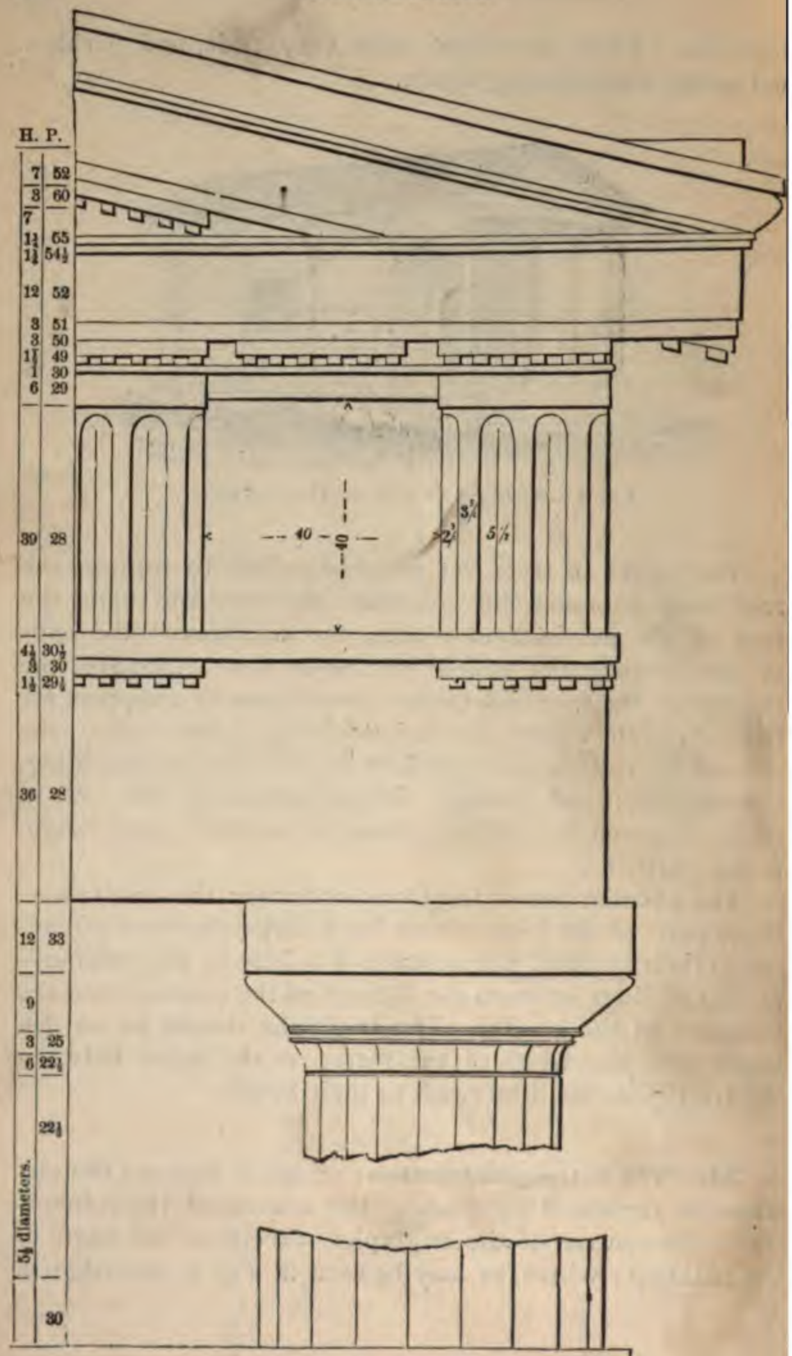


FIG. 2.—GRECIAN DORIC.

the triglyph must be over the centre of the column.* Where the columns are so disposed that one of them stands beneath every other triglyph, the arrangement is called *mono-triglyph* and is most common. When a column is placed beneath every third triglyph, the arrangement is called *diastyle;* and when beneath every fourth, *aræostyle.* This last style is the worst, and is seldom adopted.

34.—The Doric Order: is suitable for buildings that are destined for national purposes, for banking-houses, etc. Its appearance, though massive and grand, is nevertheless rich and graceful. The Patent Office at Washington, and the Treasury at New York, are good specimens of this order.

35.—The Ionic Order. (*Fig.* 3.)—The Doric was for some time the only order in use among the Greeks. They gave their attention to the cultivation of it, until perfection seems to have been attained. Their temples were the prin-

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* GRECIAN DORIC ORDER. *When the width to be occupied by the whole front is limited, to determine the diameter of the column.*

The relation between the parts may be expressed thus :

$$x = \frac{60\,a}{d(b + c) + (60 - c)}$$

Where a equals the width in feet occupied by the columns, and their intercolumniations taken collectively, measured at the base ; b equals the width of the metope, in minutes ; c equals the width of the triglyphs in minutes ; d equals the number of metopes, and x equals the diameter in feet.

Example.—A front of six columns—hexastyle—61 feet wide ; the frieze having one triglyph over each intercolumniation, or mono-triglyph. In this case, there being five intercolumniations and two metopes over each, therefore there are $5 \times 2 = 10$ metopes. Let the metope equal 42 minutes and the triglyph equal 28. Then $a = 61$; $b = 42$; $c = 28$; and $d = 10$; and the formula above becomes

$$x = \frac{60 \times 61}{10\,(42 + 28) + (60 - 28)} = \frac{60 \times 61}{10 \times 70 + 32} = \frac{3660}{732} = 5 \text{ feet} = \text{the diameter}$$
required.

Example.—An octastyle front, 8 columns, 184 feet wide, three metopes over each intercolumniation, 21 in all, and the metope and triglyph 42 and 28 as before. Then

$$x = \frac{60 \times 184}{21\,(42 + 28) + (60 - 28)} = \frac{11040}{1502} = 7.35\tfrac{30}{1502} \text{ feet} = \text{the diameter required.}$$

cipal objects upon which their skill in the art was displayed; and as the Doric order seems to have been well fitted, by its massive proportions, to represent the character of their male deities rather than the female, there seems to have been a necessity for another style which should be emblematical of feminine graces, and with which they might decorate such temples as were dedicated to the goddesses. Hence the origin of the Ionic order. This was invented, according to historians, by Hermogenes of Alabanda; and he being a native of Caria, then in the possession of the Ionians, the order was called the Ionic.

The distinguishing features of this order are the *volutes* or spirals of the capital; and the *dentils* among the bed-mouldings of the cornice: although in some instances dentils are wanting. The volutes are said to have been designed as a representation of curls of hair on the head of a matron, of whom the whole column is taken as a semblance.

The Ionic order is appropriate for churches, colleges, seminaries, libraries, all edifices dedicated to literature and the arts, and all places of peace and tranquillity. The front of the Custom-House, New York City, is a good specimen of this order.

36.—The Intercolumniation: of this and the other orders—both Roman and Grecian, with the exception of the Doric—are distinguished as follows. When the interval is one and a half diameters, it is called *pycnostyle*, or columns thick-set; when two diameters, *systyle;* when two and a quarter diameters, *eustyle;* when three diameters, *diastyle;* and when more than three diameters, *aræostyle*, or columns thin-set. In all the orders, when there are four columns in one row, the arrangement is called *tetrastyle;* when there are six in a row, *hexastyle;* and when eight, *octastyle*.

37.—To Describe the Ionic Volute.—Draw a perpendicular from a to s (*Fig.* 4), and make $a\ s$ equal to 20 min. or to $\frac{4}{7}$ of the whole height, $a\ c$; draw $s\ o$ at right angles to $s\ a$, and equal to $1\frac{1}{4}$ min.; upon o, with $2\frac{1}{2}$ min. for radius,

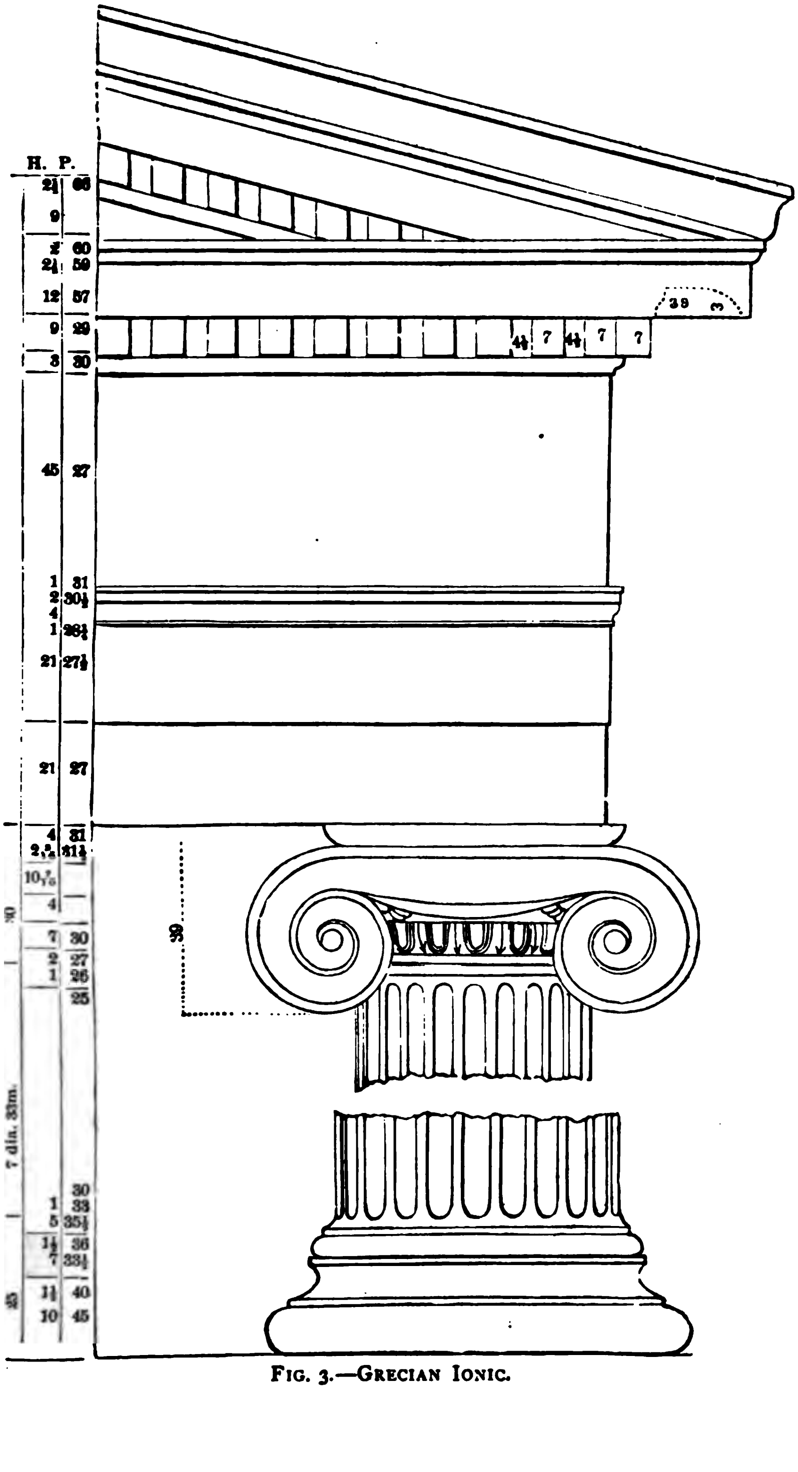


FIG. 3.—GRECIAN IONIC.

describe the eye of the volute; about *o*, the centre of the eye, draw the square, *r t* 1 2, with sides equal to half the diameter of the eye, viz. 2½ min., and divide it into 144 equal parts, as shown at *Fig.* 5. The several centres in rotation are at the angles formed by the heavy lines, as figured, 1, 2, 3, 4, 5, 6, etc. The position of these angles is determined by commencing at the point, 1, and making each heavy line one part less in length than the preceding one. No. 1 is the

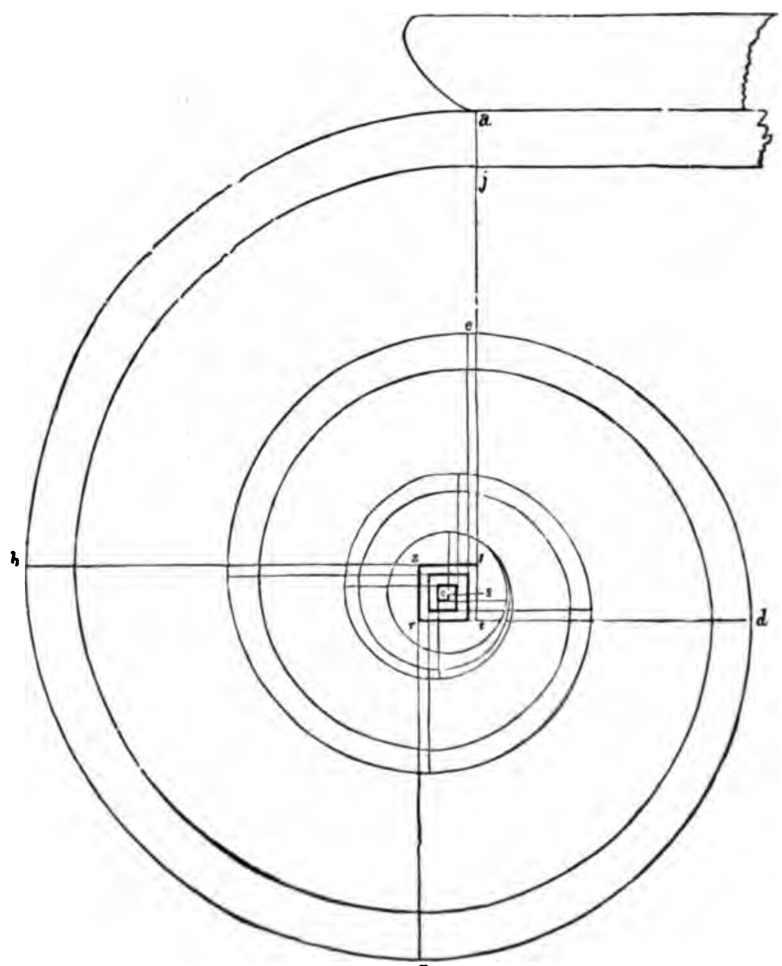


FIG. 4.—IONIC VOLUTE.

centre for the arc *a b* (*Fig.* 4;) 2 is the centre for the arc *b c;* and so on to the last. The inside spiral line is to be described from the centres, *x*, *x*, *x*, etc. (*Fig.* 5), being the centre of the first small square towards the middle of the eye from the centre for the outside arc. The breadth of the fillet at *a j* is to be made equal to $2\frac{3}{10}$ min. This is for a spiral of *three* revolutions; but one of any number of revolutions, as 4 or 6, may be drawn, by dividing *o f* (*Fig.* 5) into a corresponding number of equal parts. Then divide the part nearest the centre, *o*, into two parts, as at *h;* join *o* and 1, also *o* and 2; draw *h* 3 parallel to *o* 1, and *h* 4 parallel to *o*

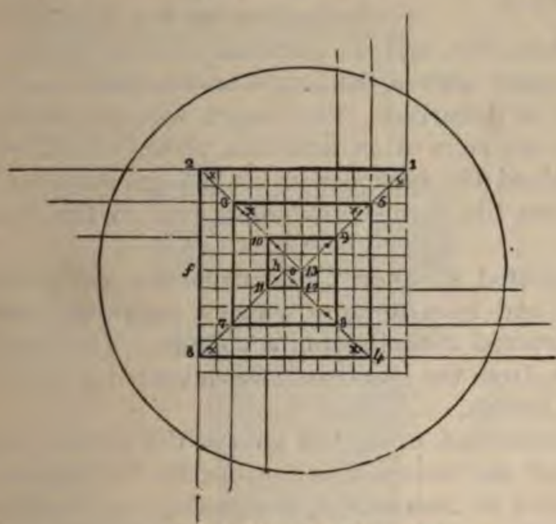


FIG. 5.—EYE OF VOLUTE.

2; then the lines *o* 1, *o* 2, *h* 3, *h* 4 will determine the length of the heavy lines, and the place of the centres. (See *Art.* 288.)

38.—The Corinthian Order: (*Fig.* 7,) is in general like the Ionic, though the proportions are lighter. The Corinthian displays a more airy elegance, a richer appearance; but its distinguishing feature is its beautiful capital. This is generally supposed to have had its origin in the capitals

of the columns of Egyptian temples, which, though not approaching it in elegance, have yet a similarity of form with the Corinthian. The oft-repeated story of its origin which is told by Vitruvius—an architect who flourished in Rome in the days of Augustus Cæsar—though pretty generally considered to be fabulous, is nevertheless worthy of being again recited. It is this: A young lady of Corinth was sick, and finally died. Her nurse gathered into a deep basket such trinkets and keepsakes as the lady had been fond of when alive, and placed them upon her grave, covering the basket with a flat stone or tile, that its contents might not be disturbed. The basket was placed accidentally upon the stem of an acanthus plant, which, shooting forth, enclosed the basket with its foliage, some of which, reaching the tile, turned gracefully over in the form of a volute.

Fig. 6.

A celebrated sculptor, Calimachus, saw the basket thus decorated, and from the hint which it suggested conceived and constructed a capital for a column. This was called Corinthian, from the fact that it was invented and first made use of at Corinth.

The Corinthian being the gayest, the richest, the most lovely of all the orders, it is appropriate for edifices which are dedicated to amusement, banqueting, and festivity—for all places where delicacy, gayety, and splendor are desirable.

39.—Persians and Caryatides.—In addition to the three regular orders of architecture, it was customary among the Greeks and other nations to employ representations of the human form, instead of columns, to support entablatures; these were called *Persians* and *Caryatides*.

40.—Persians: are statues of men, and are so called in commemoration of a victory gained over the Persians by Pausanias. The Persian prisoners were brought to Athens

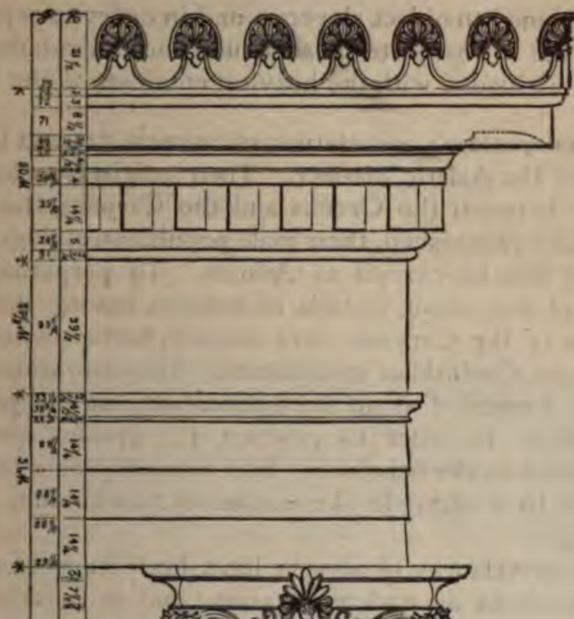


FIG. 7.—GRECIAN CORINTHIAN.

and condemned to abject slavery; and in order to represent them in the lowest state of servitude and degradation, the statues were loaded with the heaviest entablature, the Doric.

41.—Caryatides: are statues of women dressed in long robes after the Asiatic manner. Their origin is as follows: In a war between the Greeks and the Caryans, the latter were totally vanquished, their male population extinguished, and their females carried to Athens. To perpetuate the memory of this event, statues of females, having the form and dress of the Caryans, were erected, and crowned with the Ionic or Corinthian entablature. The caryatides were generally formed of about the human size, but the persians much larger, in order to produce the greater awe and astonishment in the beholder. The entablatures were proportioned to a statue in like manner as to a column of the same height.

These semblances of slavery have been in frequent use among moderns as well as ancients; and, as a relief from the stateliness and formality of the regular orders, are capable of forming a thousand varieties; yet in a land of liberty such marks of human degradation ought not to be perpetuated.

42.—Roman Styles.—Strictly speaking, Rome had no architecture of her own; all she possessed was borrowed from other nations. Before the Romans exchanged intercourse with the Greeks, they possessed some edifices of considerable extent and merit, which were erected by architects from Etruria; but Rome was principally indebted to Greece for what she acquired of the art. Although there is no such thing as an architecture of Roman invention, yet no nation, perhaps, ever was so devoted to the cultivation of the art as the Roman. Whether we consider the number and extent of their structures, or the lavish richness and splendor with which they were adorned, we are compelled to yield to them our admiration and praise. At one time, under the consuls and emperors, Rome employed 400 architects. The public works — such as theatres, circuses, baths, aqueducts, etc. — were, in extent and grandeur, be-

PORTICO OF THE ERECTHEUM, ATHENS.

yond anything attempted in modern times. Aqueducts were built to convey water from a distance of 60 miles or more. In the prosecution of this work rocks and mountains were tunnelled, and valleys bridged. Some of the latter descended 200 feet below the level of the water; and in passing them the canals were supported by an arçade, or succession of arches. Public baths are spoken of as large as cities, being fitted up with numerous conveniences for exercise and amusement. Their decorations were most splendid; indeed, the exuberance of the ornaments alone was offensive to good taste. So overloaded with enrichments were the baths of Diocletian that on one occasion of public festivity great quantities of sculpture fell from the ceilings and entablatures, killing many of the people.

43.—Grecian Orders modified by the Romans.—The orders of Greece were introduced into Rome in all their perfection. But the luxurious Romans, not satisfied with the simple elegance of their refined proportions, sought to improve upon them by lavish displays of ornament. They transformed in many instances the true elegance of the Grecian art into a gaudy splendor, better suited to their less refined taste. The Romans remodelled each of the orders: the Doric (*Fig.* 8) was modified by increasing the height of the column to 8 diameters; by changing the echinus of the capital for an ovolo, or quarter-round, and adding an astragal and neck below it; by placing the *centre*, instead of one edge, of the first triglyph over the centre of the column; and introducing horizontal instead of inclined mutules in the cornice, and in some instances dispensing with them altogether. The Ionic was modified by diminishing the size of the volutes, and, in some specimens, introducing a new capital in which the volutes were diagonally arranged (*Fig.* 9). This new capital has been termed *modern* Ionic. The favorite order at Rome and her colonies was the Corinthian (*Fig.* 10). But this order the Roman artists, in their search for novelty, subjected to many alterations—especially in the foliage of its capital. Into the upper part of this they introduced the modified Ionic capital; thus

combining the two in one. This change was dignified wi the importance of an *order*, and received the appellati

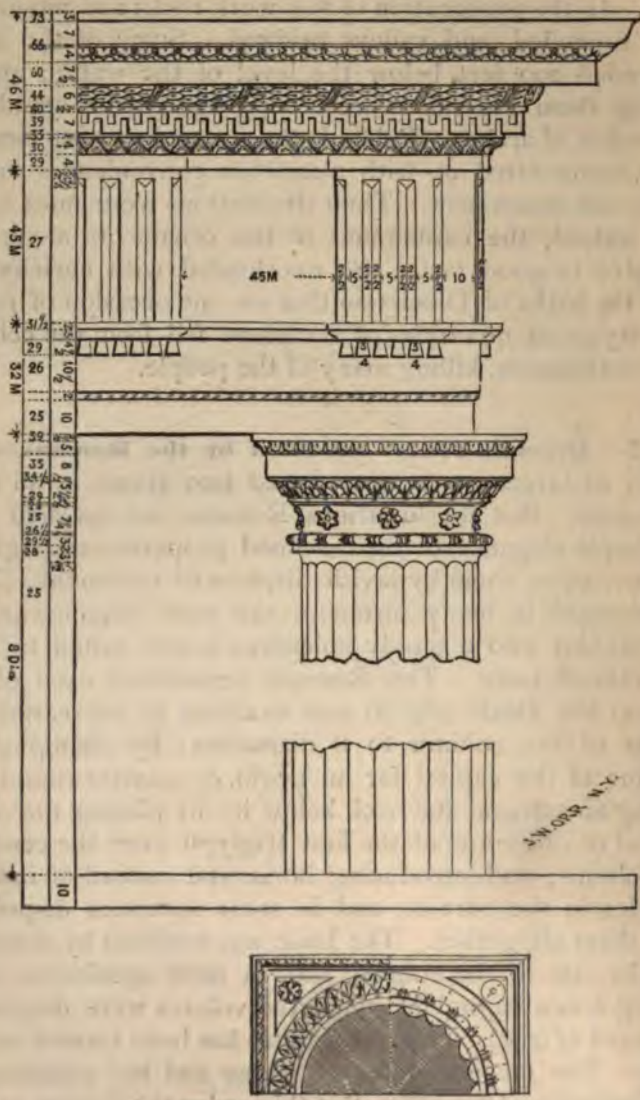


FIG. 8.—ROMAN DORIC.

of COMPOSITE, or *Roman:* the best specimen of which found in the Arch of Titus (*Fig.* 11). This style was n

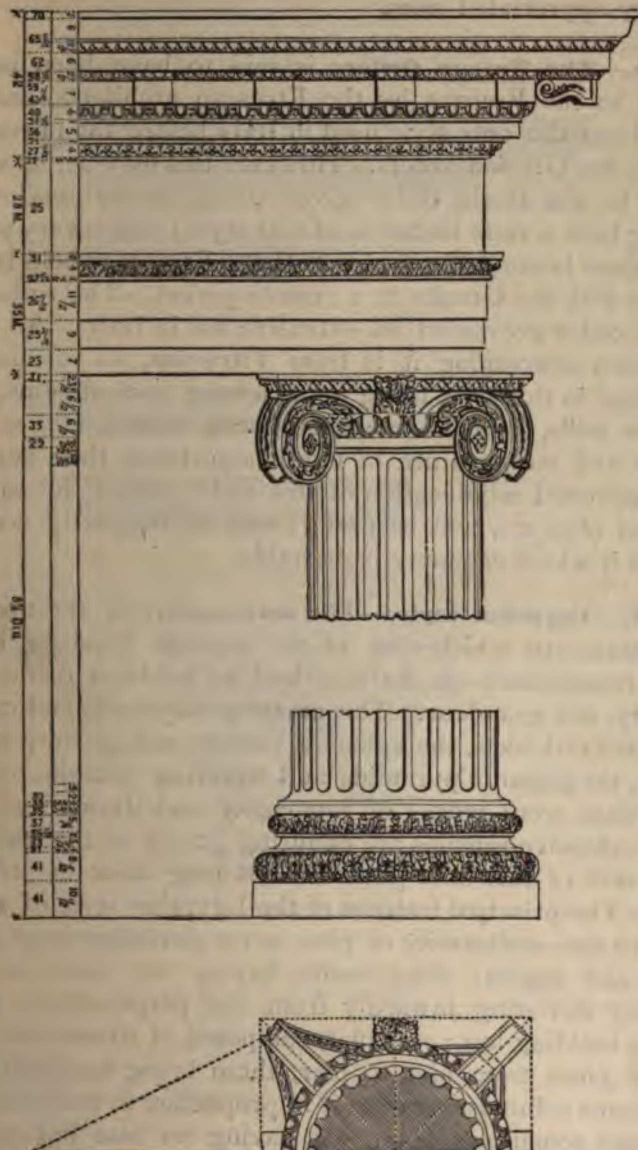


FIG. 9.—ROMAN IONIC.

much used among the Romans themselves, and is but slightly appreciated now.

44.—The Tuscan Order: is said to have been introduced to the Romans by the Etruscan architects, and to have been the only style used in Italy before the introduction of the Grecian orders. However this may be, its similarity to the Doric order gives strong indications of its having been a rude imitation of that style: this is very probable, since history informs us that the Etruscans held intercourse with the Greeks at a remote period. The rudeness of this order prevented its extensive use in Italy. All that is known concerning it is from Vitruvius, no remains of buildings in this style being found among ancient ruins.

For mills, factories, markets, barns, stables, etc., where utility and strength are of more importance than beauty, the improved modification of this order, called the *modern* Tuscan (*Fig.* 12), will be useful; and its simplicity recommends it where economy is desirable.

45.—Egyptian Style.—The architecture of the ancient Egyptians—to which that of the ancient Hindoos bears some resemblance—is characterized by boldness of outline, solidity, and grandeur. The amazing labyrinths and extensive artificial lakes, the splendid palaces and gloomy cemeteries, the gigantic pyramids and towering obelisks, of the Egyptians were works of immensity and durability; and their extensive remains are enduring proofs of the enlightened skill of this once-powerful but long since extinct nation. The principal features of the Egyptian style of architecture are—uniformity of plan, never deviating from right lines and angles; thick walls, having the outer surface slightly deviating inwardly from the perpendicular; the whole building low; roof flat, composed of stones reaching in one piece from pier to pier, these being supported by enormous columns, very stout in proportion to their height; the shaft sometimes polygonal, having no base but with a great variety of handsome capitals, the foliage of these being of the palm, lotus, and other leaves; entablatures having simply an architrave, crowned with a huge cavetto orna-

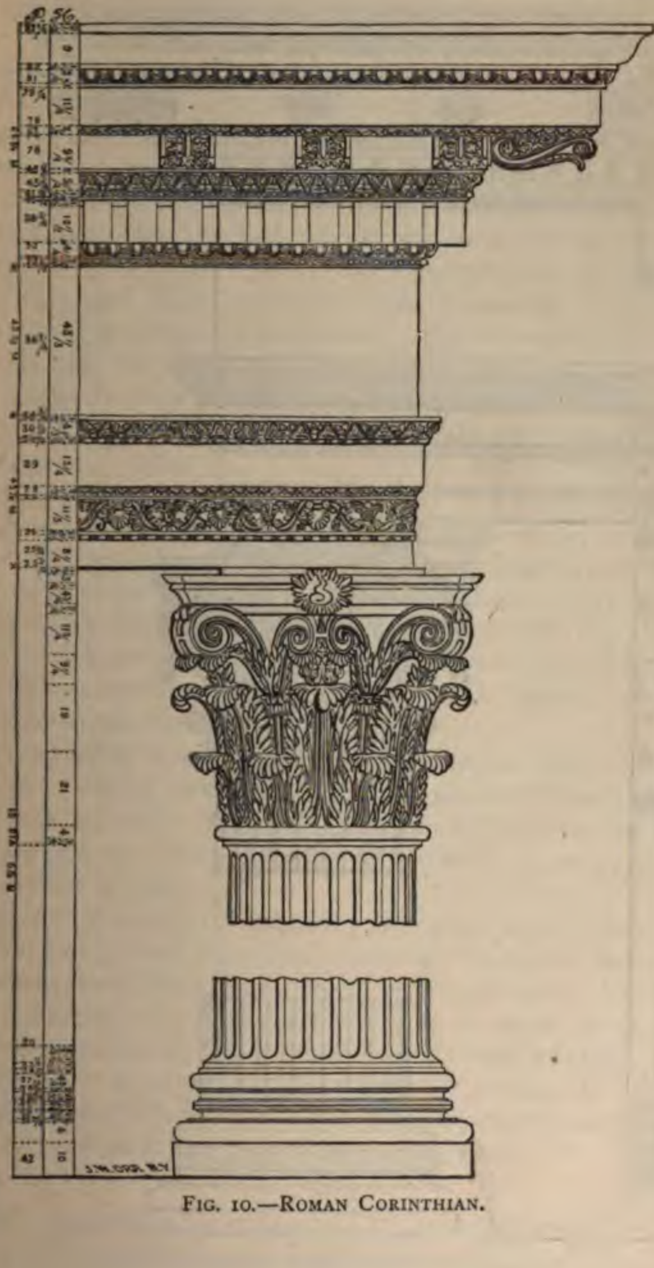


FIG. 10.—ROMAN CORINTHIAN.

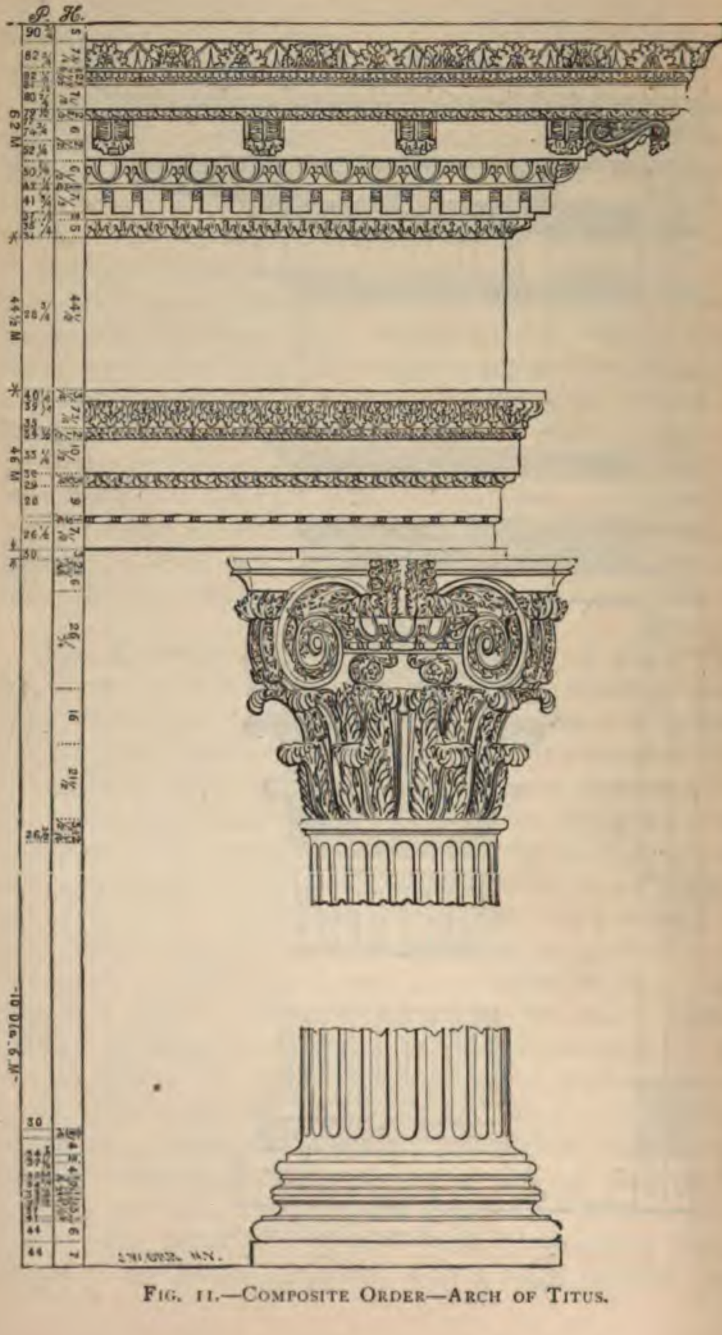


FIG. 11.—COMPOSITE ORDER—ARCH OF TITUS.

mented with sculpture; and the intercolumniation very narrow, usually 1½ diameters and seldom exceeding 2½. In the remains of a temple the walls were found to be 24 feet thick; and at the gates of Thebes, the walls at the foundation were 50 feet thick and perfectly solid. The immense stones of which these, as well as Egyptian walls generally, were built, had both their inside and outside surfaces faced, and the oints throughout the body of the wall as perfectly close as upon the outer surface. For this reason, as well as that the buildings generally partake of the pyramidal form, arise their great solidity and durability. The dimensions and extent of the buildings may be judged from the temple of Jupiter at Thebes, which was 1400 feet long and 300 feet wide—exclusive of the porticos, of which there was a great number.

It is estimated by Mr. Gliddon, U. S. Consul in Egypt, that not less than 25,000,000 tons of hewn stone were employed in the erection of the Pyramids of Memphis alone—or enough to construct 3000 Bunker Hill monuments. Some of the blocks are 40 feet long, and polished with emery to a surprising degree. It is conjectured that the stone for these pyramids was brought, by rafts and canals, from a distance of six or seven hundred miles.

The general appearance of the Egyptian style of architecture is that of solemn grandeur—amounting sometimes to sepulchral gloom. For this reason it is appropriate for cemeteries, prisons, etc.; and being adopted for these purposes, it is gradually gaining favor.

A great dissimilarity exists in the proportion, form, and general features of Egyptian columns. In some instances, there is no uniformity even in those of the same building, each differing from the others either in its shaft or capital. For practical use in this country, *Fig.* 13 may be taken as a standard of this style. The Halls of Justice in Centre Street, New York City, is a building in general accordance with the principles of Egyptian architecture.

46.—Buildings in General.—In selecting a style for an edifice, its peculiar requirements must be allowed to govern.

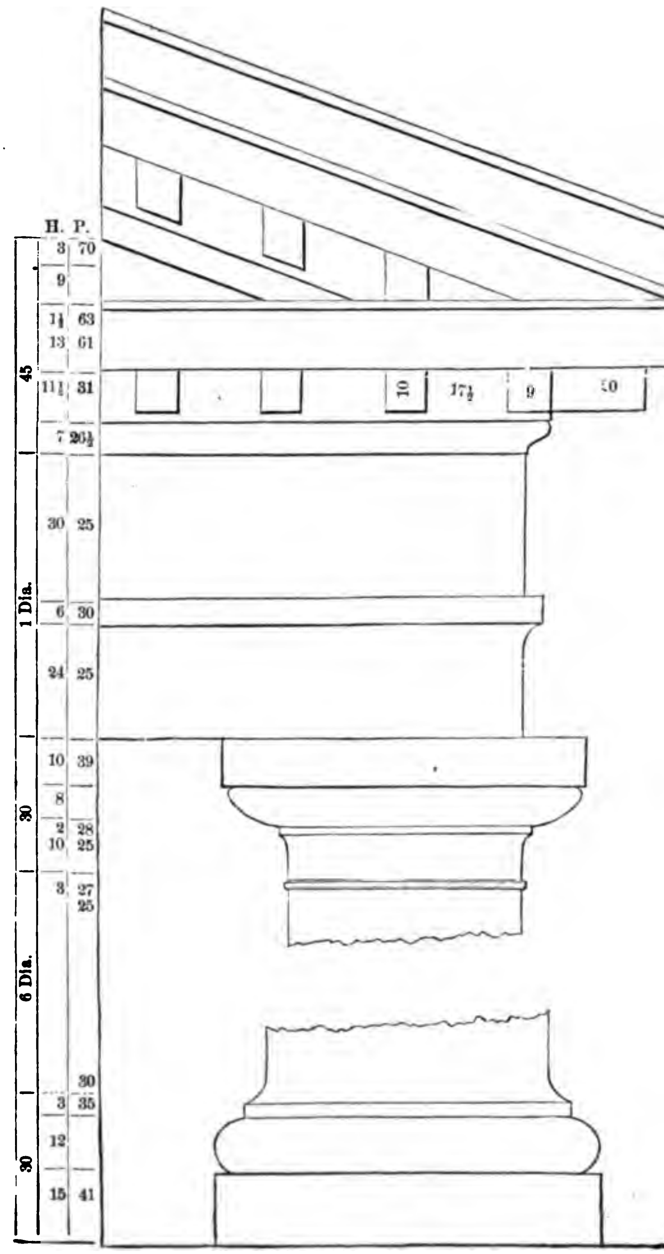


FIG. 12.—MODIFIED TUSCAN ORDER.

That style of architecture is to be preferred in which utility, stability, and regularity are gracefully blended with grandeur and elegance. But as an arrangement designed for a warm country would be inappropriate for a colder climate, it would seem that the style of building ought to be modified to suit the wants of the people for whom it is designed. High roofs to resist the pressure of heavy snows, and arrangements for artificial heat, are indispensable in northern climes; while they would be regarded as entirely out of place in buildings at the equator.

Among the Greeks, architecture was employed chiefly upon their temples and other large buildings; and the proportions of the orders, as determined by them, when executed to such large dimensions, have the happiest effect. But when used for small buildings, porticos, porches, etc., especially in country places, they are rather heavy and clumsy; in such cases, more slender proportions will be found to produce a better effect. The English cottage-style is rather more appropriate, and is becoming extensively practised for small buildings in the country.

47.—Expression.—Every building should manifest its destination. If it be intended for national purposes, it should be magnificent—grand; for a private residence, neat and modest; for a banqueting-house, gay and splendid; for a monument or cemetery, gloomy—melancholy; or, if for a church, majestic and graceful—by some it has been said, "somewhat dark and gloomy, as being favorable to a devotional state of feeling;" but such impressions can only result from a misapprehension of the nature of true devotion. "Her ways are ways of *pleasantness*, and all her paths are peace." The church should rather be a type of that brighter world to which it leads. Simply for purposes of contemplation, however, the glare of the noonday light should be excluded, that the worshipper may, with Milton—

"Love the high, embowèd roof,
With antique pillars massy proof,
And storied windows richly dight,
Casting a dim, religious light."

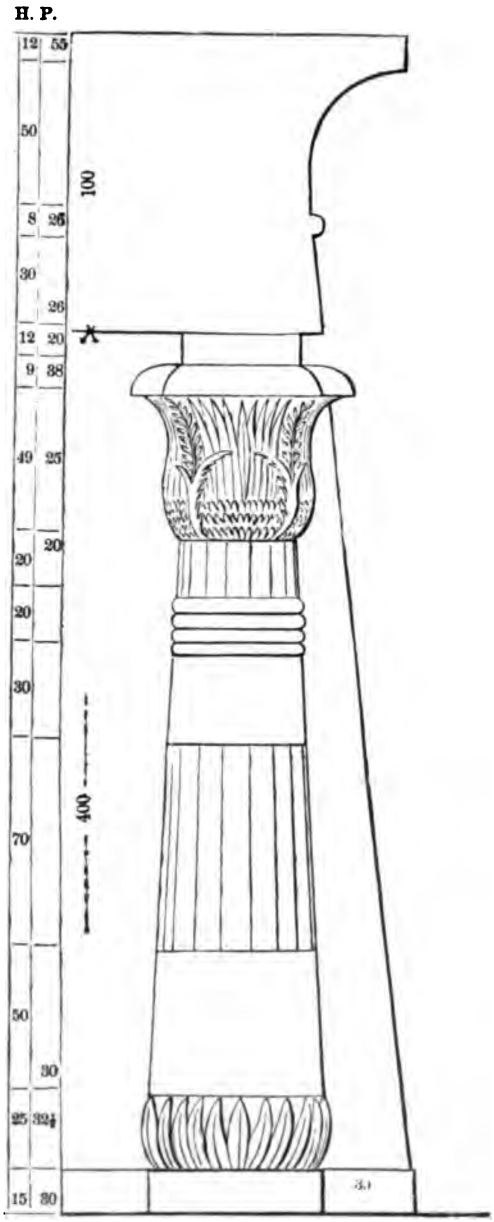


Fig. 13.—Egyptian Architecture.

However happily the several parts of an edifice may be disposed, and however pleasing it may appear as a whole, yet much depends upon its *site*, as also upon the character and style of the structures in its immediate vicinity, and the degree of cultivation of the adjacent country. A splendid country-seat should have the out-houses and fences in the same style with itself, the trees and shrubbery neatly trimmed, and the grounds well cultivated.

48.—Durability.—Europeans express surprise that we build so much with wood. And yet, in a new country, where wood is plenty, that this should be so is no cause for wonder. Still the practice should not be encouraged. Buildings erected with brick or stone are far preferable to those of wood: they are more durable; not so liable to injury by fire, nor to need repairs; and will be found in the end quite as economical. A wooden house is suitable for a temporary residence only; and those who would bequeath a dwelling to their children will endeavor to build with a more durable material. Wooden cornices and gutters, attached to brick houses, are objectionable—not only on account of their frail nature, but also because they render the building liable to destruction by fire.

49.—Dwelling-Houses: are built of various dimensions and styles, according to their destination; and to give designs and directions for their erection, it is necessary to know their situation and object. A dwelling intended for a gardener would require very different dimensions and arrangements from one intended for a retired gentleman—with his servants, horses, etc.; nor would a house designed for the city be appropriate for the country. For city houses, arrangements that would be convenient for one family might be very inconvenient for two or more. *Figs.* 14, 15, 16, 17, 18, and 19 represent the *ichnographical projection*, or ground-plan, of the floors of an ordinary city house, designed to be occupied by one family only. *Fig.* 21 is an *elevation*, or front view, of the same house. All these plans are drawn at the same scale—which is that at the bottom of *Fig.* 21.

Fig. 14 is a Plan of the Under-Cellar.

a, is the coal-vault, 6 by 10 feet.
b, is the furnace for heating the house.
c, *d*, are front and rear areas.

Fig. 15 is a Plan of the Basement.

a, is the library, or ordinary dining-room, 15 by 20 feet.
b, is the kitchen, 15 by 22 feet.
c, is the store-room, 6 by 9 feet.
d, is the pantry, 4 by 7 feet.
e, is the china closet, 4 by 7 feet.
f, is the servants' water-closet.
g, is a closet.
h, is a closet with a dumb-waiter to the first story above.
i, is an ash closet under the front stoop.
j, is the kitchen-range.
k, is the sink for washing and drawing water.
l, are wash-trays.

Fig. 16 is a Plan of the First Story.

a, is the parlor, 15 by 34 feet.
b, is the dining-room, 16 by 23 feet.
c, is the vestibule.
e, is the closet containing the dumb-waiter from the basement.
f, is the closet containing butler's sink.
g, *g*, are closets.
h, is a closet for hats and cloaks.
i, *j*, are front and rear balconies.

Fig. 17 is the Second Story.

a, *a*, are chambers, 15 by 13 feet.
b, is a bed-room, 7½ by 13 feet.
c, is the bath-room, 7½ by 13 feet.
d, *d*, are dressing-rooms, 6 by 7½ feet.
e, *e*, are closets.
f, *f*, are wardrobes.
g, *g*, are cupboards.

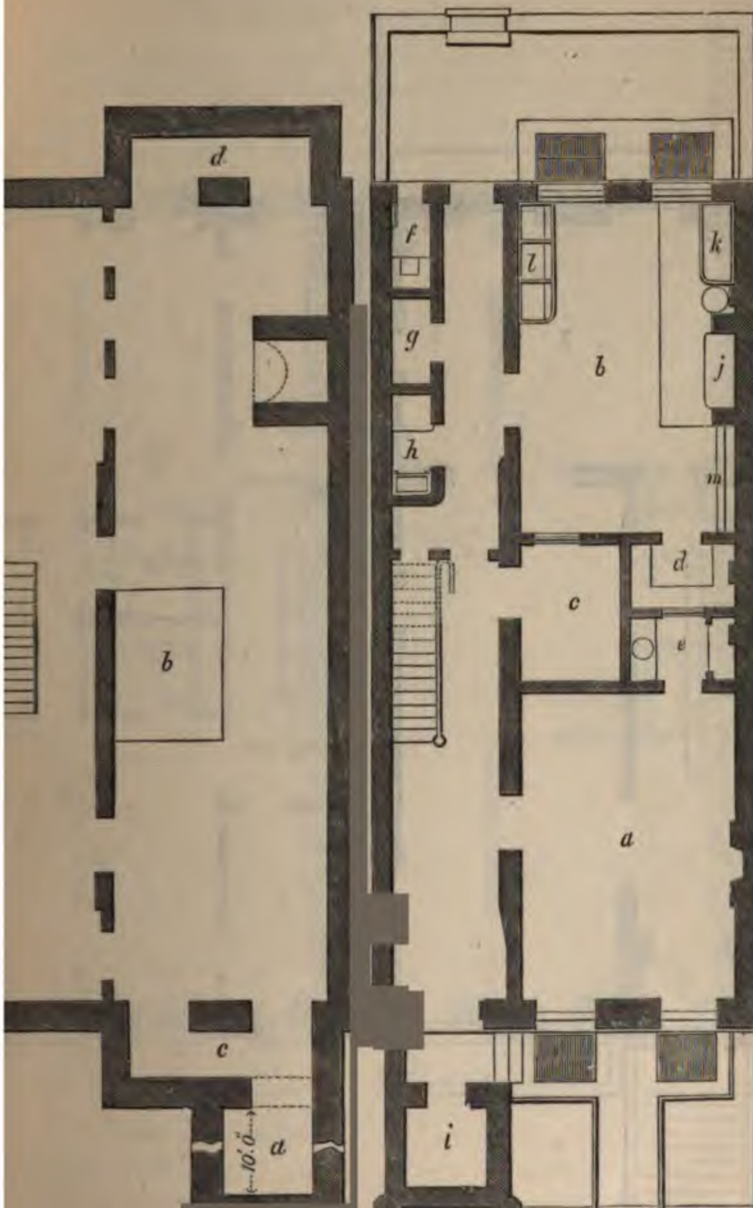


FIG. 14.
UNDER-CELLAR.

FIG. 15.
BASEMENT.

CITY DWELLING.

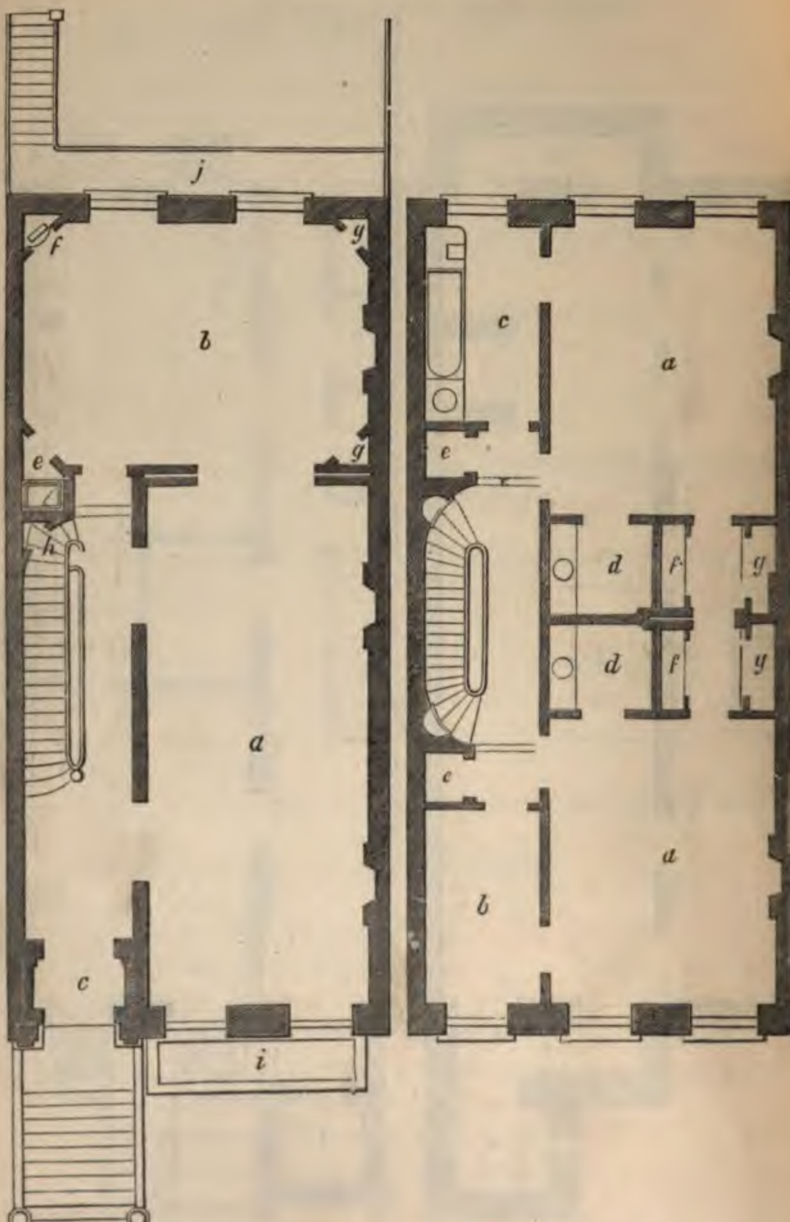


FIG. 16.
FIRST STORY.

FIG. 17.
SECOND STORY.

CITY DWELLING.

Fig. 18 is the Third Story.

a, *a*, are chambers, 15 by 19 feet.
b, *b*, are bed-rooms, $7\frac{1}{2}$ by 13 feet.
c, *c*, are closets.
d, is a linen-closet, 5 by 7 feet.

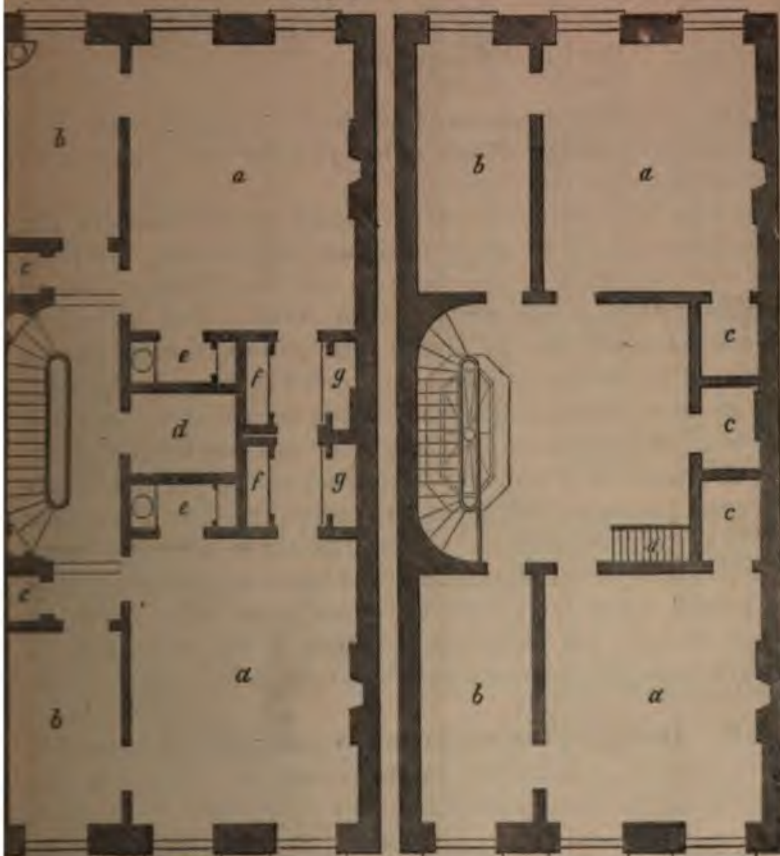


FIG. 18.
THIRD STORY.

FIG. 19.
FOURTH STORY.

CITY DWELLING.

e, *e*, are dressing-closets.
f, *f*, are wardrobes.
g, *g*, are cupboards.

Fig. 19 is the Fourth Story.

a, *a*, are chambers, 14 by 17 feet.

b, *b*, are bed-rooms, $8\frac{1}{2}$ by 17 feet.
c, *c*, *c*, are closets.
d, is the step-ladder to the roof.

Fig. 20 is the Section of the House showing the heights of the several stories.

Fig. 21 is the Front Elevation.

The size of the house is 25 feet front by 55 feet deep; this is about the average depth, although some are extended to 60 and 65 feet in depth.

These are introduced to give some general ideas of the principles to be followed in designing city houses. In placing the chimneys in the parlors, set the chimney-breasts equidistant from the ends of the room. The basement chimney-breasts may be placed nearly in the middle of the side of the room, as there is but one flue to pass through the chimney-breast above; but in the second story, as there are two flues; one from the basement and one from the parlor, the breast will have to be placed nearly perpendicular over the parlor breast, so as to receive the flues within the jambs of the fire-place. As it is desirable to have the chimney-breast as near the middle of the room as possible, it may be placed a few inches towards that point from over the breast below. So in arranging those of the stories above, always make provision for the flues from below.

50.—Arranging the Stairs and Windows.—There should be at least as much room in the passage at the side of the stairs as upon them; and in regard to the length of the passage in the second story, there must be room for the doors which open from each of the principal rooms into the hall, and more if the stairs require it. Having assigned a position for the stairs of the second story, now generally placed in the centre of the depth of the house, let the *winders* of the other stories be placed perpendicularly over and under them; and be careful to provide for head-room. To ascertain this, when it is doubtful, it is well to draw a vertical section of the whole stairs; but in ordinary cases this is not

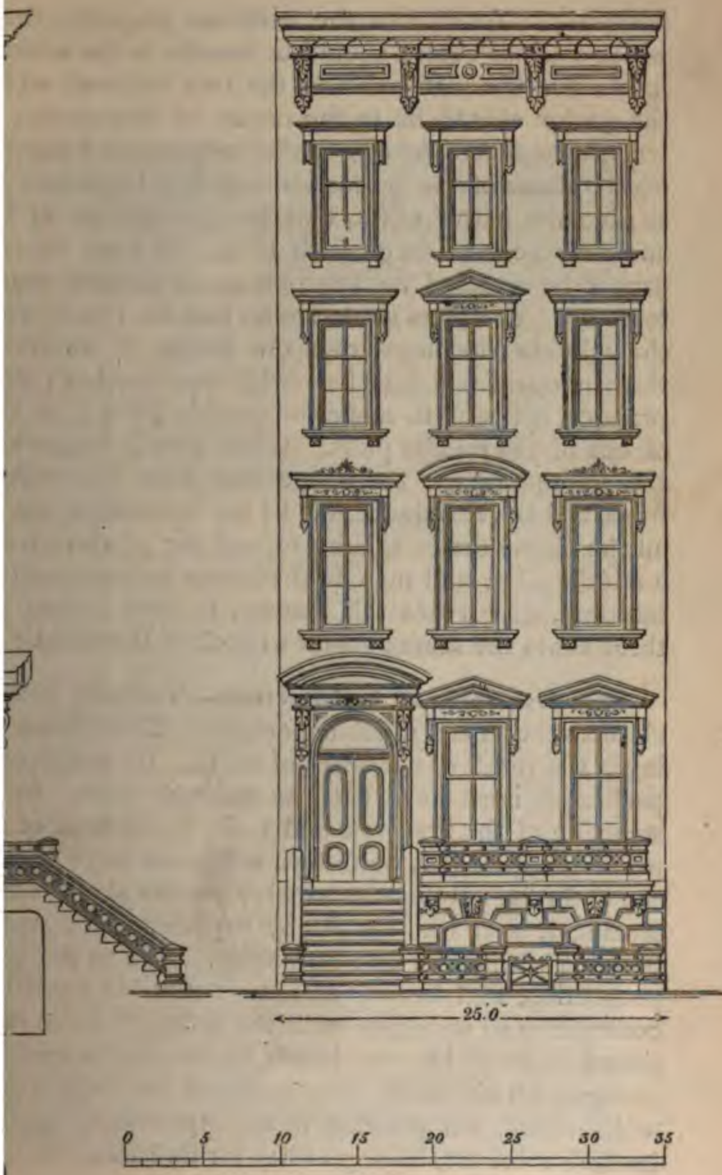


FIG. 21.
ELEVATION.

CITY DWELLING.

necessary. To dispose the windows properly, the middle window of each story should be exactly in the middle of the front; but the pier between the two windows which light the parlor should be in the centre of that room; because when chandeliers or any similar ornaments hang from the centre-pieces of the parlor ceilings, it is important, in order to give the better effect, that the pier-glasses at the front and rear be in a range with them. If both these objects cannot be attained, an approximation to each must be attempted. The piers should in no case be less in width than the window openings, else the blinds or shutters, when thrown open, will interfere with one another; in general practice, it is well to make the outside piers $\frac{2}{3}$ of the width of one of the middle piers. When this is desirable, deduct the amount of the three openings from the width of the front, and the remainder will be the amount of the width of all the piers; divide this by 10, and the product will be $\frac{1}{3}$ of a middle pier; and then, if the parlor arrangements do not interfere, give twice this amount to each corner pier, and three times the same amount to each of the middle piers.

51.—Principles of Architecture.—To build well requires close attention and much experience. The science of building is the result of centuries of study. Its progress towards perfection must have been exceedingly slow. In the construction of the first frail and rude habitations of men, the primary object was, doubtless, utility—a mere shelter from sun and rain. But as successive storms shattered his poor tenement, man was taught by experience the necessity of building with an idea to durability. And as the symmetry, proportion, and beauty of nature met his admiring gaze, contrasting so strangely with the misshapen and disproportioned work of his own hands, he was led to make gradual changes, till his abode was rendered not only commodious and durable, but pleasant in its appearance; and building became a fine art, having utility for its basis.

52.—Arrangement.—In all designs for buildings of importance, utility, durability, and beauty, the first great principles, should be pre-eminent. In order that the edifice be

CHATEAU OF CHAMBORD.

ıseful, commodious, and comfortable, the arrangement of the ıpartments should be such as to fit them for their several lestinations; for publio assemblies, oratory, state, visitors, etiring, eating, reading, sleeping, bathing, dressing, etc.— hese should each have its own peculiar form and situation. Fo accomplish this, and at the same time to make their elative situation agreeable and pleasant, producing regu- arity and harmony, require in some instances much skill ınd sound judgment. Convenience and regularity are very mportant, and each should have due attention; yet when oth cannot be obtained, the latter should in most cases give place to the former. A building that is neither con- venient nor regular, whatever other good qualities it may possess, will be sure of disapprobation.

53.—Ventilation.—Attention should be given to such ırrangements as are calculated to promote health: among hese, *ventilation* is by no means the least. For this pur- ose, the ceilings of the apartments should have a respect- ıble height; and the sky-light, or any part of the roof that an be made movable, should be arranged with cord and oullies, so as to be easily raised and lowered. Small open- ngs near the ceiling, that may be closed at pleasure, should e made in the partitions that separate the rooms from the oassages—especially for those rooms which are used for leeping apartments. All the apartments should be so ar- anged as to secure their being easily kept *dry* and *clean*. n dwellings, suitable apartments should be fitted up for *athing* with all the necessary apparatus for conveying vater.

54.—Stability.—To secure this, an edifice should be de- igned upon well-known geometrical principles: such as cience has demonstrated to be necessary and sufficient for irmness and durability. It is well, also, that it have the *ppearance* of stability as well as the *reality;* for should it eem tottering and unsafe, the sensation of fear, rather than hose of admiration and pleasure, will be excited in the be- ıolder. To secure certainty and accuracy in the applica- ion of those principles, a knowledge of the strength and

other properties of the materials used is indispensable; and in order that the whole design be so made as to be capable of execution, a practical knowledge of the requisite mechanical operations is quite important.

55.—Decoration.—The elegance of a design, although chiefly depending upon a just proportion and harmony of the parts, will be promoted by the introduction of ornaments, provided this be judiciously performed; for enrichments should not only be of a proper character to suit the style of the building, but should also have their true position, and be bestowed in proper quantity. The most common fault, and one which is prominent in Roman architecture, is an excess of enrichment: an error which is carefully to be guarded against. But those who take the Grecian models for their standard will not be liable to go to that extreme. In ornamenting a cornice, or any other assemblage of mouldings, at least every alternate member should be left plain; and those that are near the eye should be more finished than those which are distant. Although the characteristics of good architecture are utility and elegance, in connection with durability, yet some buildings are designed expressly for use, and others again for ornament: in the former, utility, and in the latter, beauty, should be the governing principle.

56.—Elementary Parts of a Building.—The builder should be acquainted with the principles upon which the essential, elementary parts of a building are founded. A scientific knowledge of these will insure certainty and security, and enable the mechanic to erect the most extensive and lofty edifices with confidence. The more important parts are the foundation, the column, the wall, the lintel, the arch, the vault, the dome, and the roof. A separate description of the peculiarities of each would seem to be necessary, and cannot perhaps be better expressed than in the following language of a modern writer on this subject, slightly modified:

THE LOUVRE—FACADE OF THE CLOCK TOWER.

57.—The Foundation: of a building should be begun at a certain depth in the earth, to secure a solid basis, below the reach of frost and common accidents. The most solid basis is rock, or gravel which has not been moved. Next to these are clay and sand, provided no other excavations have been made in the immediate neighborhood. From this basis a stone wall is carried up to the surface of the ground, and constitutes the foundation. Where it is intended that the superstructure shall press unequally, as at its piers, chimneys, or columns, it is sometimes of use to occupy the space between the points of pressure by an inverted arch. This distributes the pressure equally, and prevents the foundation from springing between the different points. In loose or muddy situations, it is always unsafe to build, unless we can reach the solid bottom below. In salt marshes and flats, this is done by depositing timbers, or driving wooden piles into the earth, and raising walls upon them. The preservative quality of the salt will keep these timbers unimpaired for a great length of time, and makes the foundation equally secure with one of brick or stone.

58.—The Column, or Pillar: is the simplest member in any building, though by no means an essential one to all. This is a perpendicular part, commonly of equal breadth and thickness, not intended for the purpose of enclosure, but simply for the support of some part of the superstructure. The principal force which a column has to resist is that of perpendicular pressure. In its shape, the shaft of a column should not be exactly cylindrical, but, since the lower part must support the weight of the superior part, in addition to the weight which presses equally on the whole column, the thickness should gradually decrease from bottom to top. The outline of columns should be a little curved, so as to represent a portion of a very long spheroid, or paraboloid, rather than of a cone. This figure is the joint result of two calculations, independent of beauty of appearance. One of these is, that the form best adapted for stability of base is that of a cone; the other is, that the figure,

which would be of equal strength throughout for supporting a superincumbent weight, would be generated by the revolution of two parabolas round the axis of the column, the vertices of the curves being at its extremities. The swell of the shafts of columns was called the *entasis* by the ancients. It has been lately found that the columns of the Parthenon, at Athens, which have been commonly supposed straight, deviate about an inch from a straight line, and that their greatest swell is at about one third of their height. Columns in the antique orders are usually made to diminish one sixth or one seventh of their diameter, and sometimes even one fourth. The Gothic pillar is commonly of equal thickness throughout.

59.—The Wall: another elementary part of a building, may be considered as the lateral continuation of the column, answering the purpose both of enclosure and support. A wall must diminish as it rises, for the same reasons, and in the same proportion, as the column. It must diminish still more rapidly if it extends through several stories, supporting weights at different heights. A wall, to possess the greatest strength, must also consist of pieces, the upper and lower surfaces of which are horizontal and regular, not rounded nor oblique. The walls of most of the ancient structures which have stood to the present time are constructed in this manner, and frequently have their stones bound together with bolts and clamps of iron. The same method is adopted in such modern structures as are intended to possess great strength and durability, and, in some cases, the stones are even dovetailed together, as in the lighthouses at Eddystone and Bell Rock. But many of our modern stone walls, for the sake of cheapness, have only one face of the stones squared, the inner half of the wall being completed with brick; so that they can, in reality, be considered only as brick walls faced with stone. Such walls are said to be liable to become convex outwardly, from the difference in the shrinking of the cement. *Rubble* walls are made of rough, irregular stones, laid in mortar. The stones should be broken, if possible, so as to produce horizontal

surfaces. The *coffer* walls of the ancient Romans were made by enclosing successive portions of the intended wall in a box, and filling it with stones, sand, and mortar promiscuously. This kind of structure must have been extremely insecure. The Pantheon and various other Roman buildings are surrounded with a double brick wall, having its vacancy filled up with loose bricks and cement. The whole has gradually consolidated into a mass of great firmness.

60.—The Reticulated Walls: of the Romans—composed of bricks with oblique surfaces—would, at the present day, be thought highly unphilosophical. Indeed, they could not long have stood, had it not been for the great strength of their cement. Modern brick walls are laid with great precision, and depend for firmness more upon their position than upon the strength of their cement. The bricks being laid in horizontal courses, and continually overlaying each other, or *breaking joints*, the whole mass is strongly interwoven, and bound together. Wooden walls, composed of timbers covered with boards, are a common but more perishable kind. They require to be constantly covered with a coating of a foreign substance, as paint or plaster, to preserve them from spontaneous decomposition. In some parts of France, and elsewhere, a kind of wall is made of earth, rendered compact by ramming it in moulds or cases. This method is called building in *pisé*, and is much more durable than the nature of the material would lead us to suppose. Walls of all kinds are greatly strengthened by angles and curves, also by projections, such as pilasters, chimneys, and buttresses. These projections serve to increase the breadth of the foundation, and are always to be made use of in large buildings, and in walls of considerable length.

61.—The Lintel, or Beam: extends in a right line over a vacant space, from one column or wall to another. The strength of the lintel will be greater in proportion as its transverse vertical diameter exceeds the horizontal, the strength being always as the square of the depth. The *floor* is the lateral continuation or connection of beams by means of a covering of boards.

62.—The Arch: is a transverse member of a building, answering the same purpose as the lintel, but vastly exceeding it in strength. The arch, unlike the lintel, may consist of any number of constituent pieces, without impairing its strength. It is, however, necessary that all the pieces should possess a uniform shape,—the shape of a portion of a wedge,—and that the joints, formed by the contact of their surfaces, should point towards a common centre. In this case, no one portion of the arch can be displaced or forced inward; and the arch cannot be broken by any force which is not sufficient to crush the materials of which it is made. In arches made of common bricks, the sides of which are parallel, any *one* of the bricks might be forced inward, were it not for the adhesion of the cement. Any *two* of the bricks, however, by the disposition of their mortar, cannot collectively be forced inward. An arch of the proper form, when complete, is rendered stronger, instead of weaker, by the pressure of a considerable weight, provided this pressure be uniform. While building, however, it requires to be supported by a centring of the shape of its internal surface, until it is complete. The upper stone of an arch is called the *keystone*, but is not more essential than any other. In regard to the shape of the arch, its most simple form is that of the semicircle. It is, however, very frequently a smaller arc of a circle, or a portion of an ellipse.

63.—Hooke's Theory of an Arch.—The simplest theory of an arch supporting itself only is that of Dr. Hooke. The arch, when it has only its own weight to bear, may be considered as the inversion of a chain, suspended at each end. The chain hangs in such a form that the weight of each link or portion is held in equilibrium by the result of two forces acting at its extremities; and these forces, or tensions, are produced, the one by the weight of the portion of the chain below the link, the other by the same weight increased by that of the link itself, both of them acting originally in a vertical direction. Now, supposing the chain inverted, so as to constitute an arch of the same form and weight, the relative situations of the forces will be the same,

VIADUCT AT CHAUMONT.

only they will act in contrary directions, so that they are compounded in a similar manner, and balance each other on the same conditions.

The arch thus formed is denominated a *catenary* arch. In common cases, it differs but little from a circular arch of the extent of about one third of a whole circle, and rising from the abutments with an obliquity of about 30 degrees from a perpendicular. But though the catenary arch is the best form for supporting its own weight, and also all additional weight which presses in a vertical direction, it is not the best form to resist lateral pressure, or pressure like that of fluids, acting equally in all directions. Thus the arches of bridges and similar structures, when covered with loose stones and earth, are pressed sideways, as well as vertically, in the same manner as if they supported a weight of fluid. In this case, it is necessary that the arch should arise more perpendicularly from the abutment, and that its general figure should be that of the longitudinal segment of an ellipse. In small arches, in common buildings, where the disturbing force is not great, it is of little consequence what is the shape of the curve. The outlines may even be perfectly straight, as in the tier of bricks which we frequently see over a window. This is, strictly speaking, a real arch, provided the surfaces of the bricks tend toward a common centre. It is the weakest kind of arch, and a part of it is necessarily superfluous, since no greater portion can act in supporting a weight above it than can be included between two curved or arched lines.

64.—Gothic Arches.—Besides these arches, various others are in use. The *acute* or *lancet* arch, much used in Gothic architecture, is described usually from two centres outside the arch. It is a strong arch for supporting vertical pressure. The *rampant* arch is one in which the two ends spring from unequal heights. The *horseshoe* or *Moorish* arch is described from one or more centres placed above the base line. In this arch, the lower parts are in danger of being forced inward. The *ogee* arch is concavo-convex, and therefore fit only for ornament.

65.—Arch: Definitions; Principles.—The upper surface is called the *extrados*, and the inner, the *intrados*. The *spring* is where the intrados meets the abutments. The *span* is the distance between the abutments. The wedge-shaped stones which form an arch are sometimes called *voussoirs*, the uppermost being the keystone. The part of a pier from which an arch springs is called the *impost*, and the curve formed by the under side of the voussoirs, the *archivolt*. It is necessary that the walls, abutments, and piers on which arches are supported should be so firm as to resist the lateral *thrust*, as well as vertical pressure, of the arch. It will at once be seen that the lateral or sideway pressure of an arch is very considerable, when we recollect that every stone, or portion of the arch, is a wedge, a part of whose force acts to separate the abutments. For want of attention to this circumstance, important mistakes have been committed, the strength of buildings materially impaired, and their ruin accelerated. In some cases, the want of lateral firmness in the walls is compensated by a bar of iron stretched across the span of the arch, and connecting the abutments, like the tie-beam of a roof. This is the case in the cathedral of Milan and some other Gothic buildings.

66.—An Arcade: or continuation of arches, needs only that the outer supports of the terminal arches should be strong enough to resist horizontal pressure. In the intermediate arches, the lateral force of each arch is counteracted by the opposing lateral force of the one contiguous to it. In bridges, however, where individual arches are liable to be destroyed by accident, it is desirable that each of the piers should possess sufficient horizontal strength to resist the lateral pressure of the adjoining arches.

67.—The Vault: is the lateral continuation of an arch, serving to cover an area or passage, and bearing the same relation to the arch that the wall does to the column. A simple vault is constructed on the principles of the arch, and distributes its pressure equally along the walls or abutments. A complex or *groined* vault is made by two vaults intersecting each other, in which case the pressure is thrown upon

springing points, and is greatly increased at those points. The groined vault is common in Gothic architecture.

68.—The Dome: sometimes called *cupola*, is a concave covering to a building, or part of it, and may be either a segment of a sphere, of a spheroid, or of any similar figure. When built of stone, it is a very strong kind of structure, even more so than the arch, since the tendency of each part to fall is counteracted, not only by those above and below it, but also by those on each side. It is only necessary that the constituent pieces should have a common form, and that this form should be somewhat like the frustum of a pyramid, so that, when placed in its situation, its four angles may point toward the centre, or axis, of the dome. During the erection of a dome, it is not necessary that it should be supported by a centring, until complete, as is done in the arch. Each circle of stones, when laid, is capable of supporting itself without aid from those above it. It follows that the dome may be left open at top, without a keystone, and yet be perfectly secure in this respect, being the reverse of the arch. The dome of the Pantheon, at Rome, has been always open at top, and yet has stood unimpaired for nearly 2000 years. The upper circle of stones, though apparently the weakest, is nevertheless often made to support the additional weight of a lantern or tower above it. In several of the largest cathedrals, there are two domes, one within the other, which contribute their joint support to the lantern, which rests upon the top. In these buildings, the dome rests upon a circular wall, which is supported, in its turn, by arches upon massive pillars or piers. This construction is called building upon *pendentives*, and gives open space and room for passage beneath the dome. The remarks which have been made in regard to the abutments of the arch apply equally to the walls immediately supporting a dome. They must be of sufficient thickness and solidity to resist the lateral pressure of the dome, which is very great. The walls of the Roman Pantheon are of great depth and solidity. In order that a dome in itself should be perfectly secure, its lower parts must not be too nearly vertical, since,

in this case, they partake of the nature of perpendicular walls, and are acted upon by the spreading force of the parts above them. The dome of St. Paul's Church, in London, and some others of similar construction, are bound with chains or hoops of iron, to prevent them from spreading at bottom. Domes which are made of wood depend, in part, for their strength on their internal carpentry. The Halle du Bled, in Paris, had originally a wooden dome more than 200 feet in diameter, and only one foot in thickness. This has since been replaced by a dome of iron. (See *Art.* 235.)

69.—The Roof: is the most common and cheap method of covering buildings, to protect them from rain and other effects of the weather. It is sometimes flat, but more frequently oblique, in its shape. The flat or platform roof is the least advantageous for shedding rain, and is seldom used in northern countries. The *pent* roof, consisting of two oblique sides meeting at top, is the most common form. These roofs are made steepest in cold climates, where they are liable to be loaded with snow. Where the four sides of the roof are all oblique, it is denominated a *hipped* roof, and where there are two portions to the roof, of different obliquity, it is a *curb*, or *mansard* roof. In modern times, roofs are made almost exclusively of wood, though frequently covered with incombustible materials. The internal structure or carpentry of roofs is a subject of considerable mechanical contrivance. The roof is supported by *rafters*, which abut on the walls on each side, like the extremities of an arch. If no other timbers existed except the rafters, they would exert a strong lateral pressure on the walls, tending to separate and overthrow them. To counteract this lateral force, a *tie-beam*, as it is called, extends across, receiving the ends of the rafters, and protecting the wall from their horizontal thrust. To prevent the tie-beam from *sagging*, or bending downward with its own weight, a *king-post* is erected from this beam, to the upper angle of the rafters, serving to connect the whole, and to suspend the weight of the beam. This is called *trussing*. *Queen-posts*

are sometimes added, parallel to the king-post, in large roofs: also various other connecting timbers. In Gothic buildings, where the vaults do not admit of the use of a tie-beam, the rafters are prevented from spreading, as in an arch, by the strength of the buttresses.

In comparing the lateral pressure of a high roof with that of a low one, the length of the tie-beam being the same, it will be seen that a high roof, from its containing most materials, may produce the greatest pressure, as far as weight is concerned. On the other hand, if the weight of both be equal, then the low roof will exert the greater pressure; and this will increase in proportion to the distance of the point at which perpendiculars, drawn from the end of each rafter, would meet. In roofs, as well as in wooden domes and bridges, the materials are subjected to an internal strain, to resist which the cohesive strength of the material is relied on. On this account, beams should, when possible, be of one piece. Where this cannot be effected, two or more beams are connected together by *splicing*. Spliced beams are never so strong as whole ones, yet they may be made to approach the same strength, by affixing lateral pieces, or by making the ends overlay each other, and connecting them with bolts and straps of iron. The tendency to separate is also resisted, by letting the two pieces into each other by the process called *scarfing*. *Mortices*, intended to *truss* or suspend one piece by another, should be formed upon similar principles.

Roofs in the United States, after being boarded, receive a secondary covering of shingles. When intended to be incombustible, they are covered with slates or earthen tiles, or with sheets of lead, copper, or tinned iron. Slates are preferable to tiles, being lighter, and absorbing less moisture. Metallic sheets are chiefly used for flat roofs, wooden domes, and curved and angular surfaces, which require a flexible material to cover them, or have not a sufficient pitch to shed the rain from slates or shingles. Various artificial compositions are occasionally used to cover roofs, the most common of which are mixtures of tar with lime, and sometimes with sand and gravel.—*Ency. Am.* (See *Art.* 202.)

SECTION II.—CONSTRUCTION.

ART. **70.—Construction Essential.**—Construction is that part of the Science of Building which treats of the Laws of Pressure and the strength of materials. To the architect and builder a knowledge of it is absolutely essential. It deserves a larger place in a volume of this kind than is generally allotted to it. Something, indeed, has been said upon the styles and principles, by which the best arrangements may be ascertained; yet, besides this, there is much to be learned. For however precise or workmanlike the several parts may be made, what will it avail, should the system of framing, from deficient material, or an erroneous position of its timbers, fail to sustain even its own weight? Hence the necessity for a knowledge of the laws of pressure and the strength of materials. These being once understood, we can with confidence determine the best position and dimensions for the several pieces which compose a floor or a roof, a partition or a bridge. As systems of framing are more or less exposed to heavy weights and strains, and, in case of failure, cause not only a loss of labor and material, but frequently that of life itself, it is very important that the materials employed be of the proper quantity and quality to serve their destination. And, on the other hand, any superfluous material is not only useless, but a positive injury, as it is an unnecessary load upon the points of support. It is necessary, therefore, to know the *least* quantity of material that will suffice for strength. Not the least common fault in framing is that of using an excess of material. Economy, at least, would seem to require that this evil be abated.

Before proceeding to consider the principles upon which a system of framing should be constructed, let us attend to a few of the elementary laws in *Mechanics*, which will be found to be of great value in determining those principles.

INTERIOR OF THE CATHEDRAL, SIENNA.

71.—Laws of Pressure.—(1.) A heavy body always exerts a pressure equal to its own weight in a vertical direction. Example: Suppose an iron ball weighing 100 lbs. be supported upon the top of a perpendicular post (*Fig.* 22–*A*); then the pressure exerted upon that post will be equal to the weight of the ball, viz., 100 lbs. (2.) But if two inclined posts (*Fig.* 22–*B*) be substituted for the perpendicular support, the united pressures upon these posts will be more than equal to the weight, and will be in proportion to their position. The farther apart their feet are spread the greater will be the pressure, and *vice versa.* Hence tremendous strains may be exerted by a comparatively small weight. And it follows, therefore, that a piece of timber intended for a strut or post should be so placed that its axis may coincide, as nearly as possible, with the direction of the pressure. The direction of the pressure of the weight W (*Fig.* 22–*B*) is in the vertical line $b\ d$; and the weight W would fall in that line if the two posts were removed; hence the best position for a support for the weight would be in

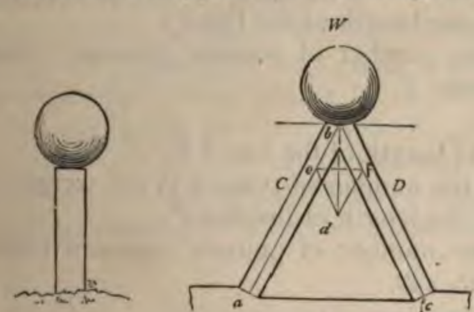


FIG. 22.

that line. But as it rarely occurs in systems of framing that weights can be supported by any single resistance, they requiring generally two or more supports (as in the case of a roof supported by its rafters), it becomes important, therefore, to know the exact amount of pressure any certain weight is capable of exerting upon oblique supports. Now, it has been ascertained that the three lines of a triangle, drawn parallel with the direction of three concurring forces in equilibrium, are in proportion respectively to these

forces. For example, in *Fig.* 22–*B*, we have a representation of three forces concurring in a point, which forces are in equilibrium and at rest; thus, the weight *W* is one force, and the resistances exerted by the two pieces of timber are the other two forces. The direction in which the first force acts is vertical—downwards; the direction of the two other forces is in the axis of each piece of timber respectively. These three forces all tend towards the point *b*.

Draw the axes *a b* and *b c* of the two supports; make *b d* vertical, and from *d* draw *d e* and *d f* parallel with the axes *b c* and *b a* repectively. Then the triangle *b d e* has its lines parallel respectively with the direction of the three forces; thus, *b d* is in the direction of the weight *W*, *d e* parallel with the axis of the timber *D*, and *e b* is in the direction of the timber *C*. In accordance with the principle above stated, the lengths of the sides of the triangle *b d e* are in proportion respectively to the three forces aforesaid; thus—

As the length of the line *b d*
Is to the number of pounds in the weight *W*,
So is the length of the line *b e*
To the number of pounds' pressure resisted by the timber *C*.

Again—

As the length of the line *b d*
Is to the number of pounds in the weight *W*,
So is the length of the line *d e*
To the number of pounds' pressure resisted by the timber *D*.

And again—

As the length of the line *b e*
Is to the pounds' pressure resisted by *C*,
So is the length of the line *d e*
To the pounds' pressure resisted by *D*.

These proportions are more briefly stated thus—

1st. $bd : W :: b\,e : P,$

P being used as a symbol to represent the number of pounds' pressure resisted by the timber *C*.

$$2d. \qquad b\,d : W :: d\,e : Q,$$

Q representing the number of pounds' pressure resisted by the timber D.

$$3d. \qquad b\,e : P :: d\,e : Q.$$

72.—Parallelogram of Forces.—This relation between lines and pressures is applicable in ascertaining the pressures induced by known weights throughout any system of framing. The parallelogram $b\,e\,d\,f$ is called the *Parallelogram of Forces;* the two lines $b\,e$ and $b\,f$ being called the *components,* and the line $b\,d$ the *resultant.* Where it is required to find the *components* from a given resultant (*Fig.* 22–*B*), the fourth line $d\,f$ need not be drawn, for the triangle $b\,d\,e$ gives the desired result. But when the *resultant* is to be ascertained from given components (*Fig.* 28), it is more convenient to draw the fourth line.

73.—The Resolution of Forces: is the finding of two or more forces which, acting in different directions, shall exactly balance the pressure of any given *single* force. To make a practical application of this, let it be required to ascertain the oblique pressure in *Fig.* 22–*B*. In this figure the line $b\,d$ measures half an inch (0·5 inch), and the line $b\,e$ three tenths of an inch (0·3 inch). Now if the weight W be supposed to be 1200 pounds, then the first stated proportion above,

$$b\,d : W :: b\,e : P, \text{ becomes } 0{\cdot}5 : 1200 :: 0{\cdot}3 : P.$$

And since the product of the means divided by one of the extremes gives the other extreme, this proportion may be put in the form of an *equation,* thus—

$$\frac{1200 \times 0{\cdot}3}{0{\cdot}5} = P.$$

Performing the arithmetical operation here indicated—that is, multiplying together the two quantities above the line, and dividing the product by the quantity under the line—the

quotient will be equal to the quantity represented by P, viz., the pressure resisted by the timber C. Thus—

$$\begin{array}{r} 1200 \\ 0\cdot3 \\ \hline 0\cdot5)360\cdot0 \\ \hline 720 = P. \end{array}$$

The strain upon the timber C is, therefore, equal to 720 pounds; and since, in this case (the two timbers being inclined equally from the vertical), the line $e\,d$ is equal to the line $b\,e$, therefore the strain upon the other timber D is also 720 pounds.

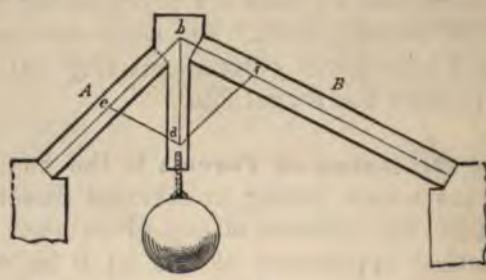


FIG. 23.

74.—Inclination of Supports Unequal.—In *Fig.* 23 the pressures in the two supports are unequal. The supports are also unequal in length. The length of the supports, however, does not alter the amount of pressure from the concentrated load supported; but generally long timbers are not so capable of resistance as shorter ones. For, not being so stiff, they bend more readily, and, since the compression is in proportion to the length, they therefore shorten more. To ascertain the pressures in *Fig.* 23, let the weight suspended from $b\,d$ be equal to two and three quarter tons (2·75 tons). The line $b\,d$ measures five and a half tenths of an inch (0·55 inch), and the line $b\,e$ half an inch (0·5 inch). Therefore, the proportion

$$b\,d : W :: b\,e : P \text{ becomes } 0\cdot55 : 2\cdot75 :: 0\cdot5 : P,$$

and $$\frac{2.75 \times 0.5}{0.55} = P.$$

$$\begin{array}{r} 2.75 \\ 0.5 \\ \hline 0.55)1.375(2.5 = P. \\ 1\ 10 \\ \hline 275 \\ 275 \\ \hline \end{array}$$

The strain upon the timber A is, therefore, equal to two and a half tons.

Again, the line $e\,d$ measures four tenths of an inch (0.4 inch); therefore, the proportion

$b\,d : W :: e\,d : Q$ becomes $0.55 : 2.75 :: 0.4 : Q$,

and $$\frac{2.75 \times 0.4}{0.55} = Q,$$

$$\begin{array}{r} 2.75 \\ 0.4 \\ \hline 0.55)\ 1.100(2 = Q. \\ 1\ 10 \\ \hline \end{array}$$

The strain upon the timber B is, therefore, equal to two tons.

75.—The Strains Exceed the Weights.—Thus the united pressures upon the two inclined supports always exceed the weight. In the last case, $2\frac{3}{4}$ tons exert a pressure of $2\frac{1}{2}$ and 2 tons, equal together to $4\frac{1}{2}$ tons; and in the former case, 1200 pounds exert a pressure of twice 720 pounds, equal to 1440 pounds. The smaller the angle of inclination to the horizontal, the greater will be the pressure upon the supports. So, in the frame of a roof, the strain upon the rafters decreases gradually with the increase of the angle of inclination to the horizon, the length of the rafter remaining the same.

This is true in comparing one system of framing with another; but in a system where the concentrated weight to be supported is not in the middle (see *Fig.* 23), and, in consequence, the supports are not inclined equally, the strain will be *greatest* upon that support which has the greatest inclination to the horizon.

76.—Minimum Thrust of Rafters.—Ordinarily, as in roofs, the load is not concentrated, it being that of the framing itself. Here the *amount* of the load will be in proportion to the length of the rafter, and this will increase with the increase of the angle of inclination, the span remaining the same. So it is seen that in enlarging the angle of inclination to the horizon in order to lessen the oblique thrust, the load is increased in consequence of the elongation of the rafter, thus increasing the oblique thrust. Hence there is an economical angle of inclination. A rafter will have the least oblique thrust when its angle of inclination to the horizon is 35° 16′ nearly. This angle is attained very nearly when the rafter rises $8\frac{1}{2}$ inches per foot, or when the height *B C* (*Fig.* 32) is to the base *A C* as $8\frac{1}{2}$ is to 12, or as 0·7071 is to 1·0.

77.—Practical Method of Determining Strains.—A comparison of pressures in timbers, according to their position, may be readily made by drawing various designs of framing and estimating the several strains in accordance with the parallelogram of forces, always drawing the triangle *b d e* so that the three lines shall be parallel with the three forces or pressures respectively. The *length* of the lines forming this triangle is unimportant, but it will be found more convenient if the line drawn parallel with the *known* force is made to contain as many inches as the known force contains pounds, or as many tenths of an inch as pounds, or as many inches as tons, or tenths of an inch as tons; or, in general, as many divisions of any convenient scale as there are units of weight or pressure in the known force. If drawn in this manner, then the number of divisions of the same scale found in the other two lines of the triangle will equal the units of pressure or weight of the other two forces respect-

ively, and the pressures sought will be ascertained simply by applying the scale to the lines of the triangle.

For example, in *Fig.* 23, the vertical line $b\,d$, of the triangle, measures fifty-five hundredths of an inch (0·55 inch); the line $b\,e$, fifty hundredths (0·50 inch); and the line $e\,d$, forty (0·40 inch). Now, if it be supposed that the vertical pressure, or the weight suspended below $b\,d$, is equal to 55 pounds, then the pressure on A will equal 50 pounds, and that on B will equal 40 pounds; for, by the proportion above stated,

$$b\,d : W :: b\,e : P,$$
$$55 : 55 :: 50 : 50;$$

and so of the other pressure.

If a scale cannot be had of equal proportions with the forces, the arithmetical process will be shortened somewhat by making the line of the triangle that represents the *known* weight equal to unity of a decimally divided scale, then the other lines will be measured in tenths or hundredths; and in the numerical statement of the proportions between the lines and forces, the first term being unity, the fourth term will be ascertained simply by multiplying the second and third terms together.

For example, if the three lines are 1, 0·7, and 1·3, and the known weight is 6 tons, then

$$b\,d : W :: b\,e : P \text{ becomes}$$
$$1 : 6 :: 0{\cdot}7 : P = 4{\cdot}2,$$

equals four and two tenths tons. Again—

$$b\,d : W :: e\,d : Q \text{ becomes}$$
$$1 : 6 :: 1{\cdot}3 : Q = 7{\cdot}8,$$

equals seven and eight tenths tons.

78.—Horizontal Thrust.—In *Fig.* 24, the weight W presses the struts in the direction of their length; their feet, $n\,n$, therefore, tend to move in the direction $n\,o$, and would so move were they not opposed by a sufficient resistance from the blocks, A and A. If a piece of each block be cut off at

the horizontal line, *a n*, the feet of the struts would slide away from each other along that line, in the direction *n a*; but if, instead of these, two pieces were cut off at the vertical line, *n b*, then the struts would descend vertically. To estimate the horizontal and the vertical pressures exerted by the struts, let *n o* be made equal (upon any scale of equal parts) to the number of tons with which the strut is pressed;

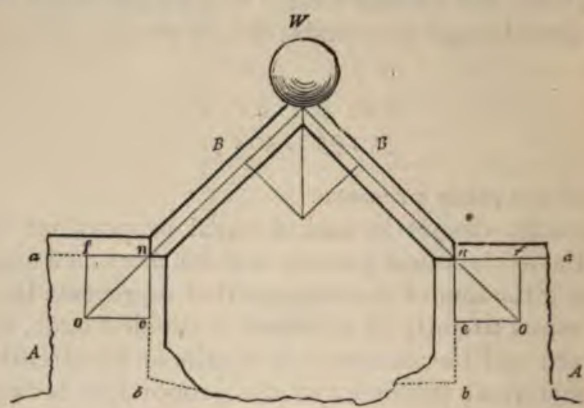


FIG. 24.

construct the parallelogram of forces by drawing *o e* parallel to *a n*, and *o f* parallel to *b n*; then *n f* (by the same scale) shows the number of tons pressure that is exerted by the strut in the direction *n a*, and *n e* shows the amount exerted in the direction *n b*. By constructing designs similar to this, giving various and dissimilar positions to the struts, and then estimating the pressures, it will be found in every case that the horizontal pressure of one strut is exactly equal to that of the other, however much one strut may be inclined more than the other; and also, that the united vertical pressure of the two struts is exactly equal to the weight *W*. (In this calculation the weight of the timbers has not been taken into consideration, simply to avoid complication to the learner. In practice it is requisite to include the weight of the framing with the load upon the framing.)

Suppose that the two struts, *B* and *B* (*Fig.* 24), were rafters of a roof, and that instead of the blocks, *A* and *A*, the walls of a building were the supports: then, to prevent

the walls from being thrown over by the thrust of *B* and *B*, it would be desirable to remove the horizontal pressure. This may be done by uniting the feet of the rafters with a

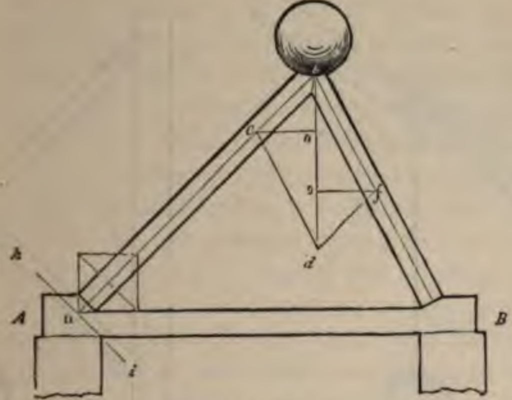


FIG. 25.

rope, iron rod, or piece of timber, as in *Fig.* 25. This figure is similar to the truss of a roof. The horizontal strains on the tie-beam, tending to pull it asunder in the direction of its length, may be measured at the foot of the rafter, as was shown at *Fig.* 24; but it can be more readily and as accurately measured by drawing from *f* and *e* horizontal lines to the vertical line, *b d*, meeting it in *o* and *o*; then *f o* will be the horizontal thrust at *B*, and *e o* at *A*; these will be found to equal one another. When the rafters of a roof are thus connected, all tendency to thrust out the walls horizontally is removed, the only pressure on them is in a vertical direction, being equal to the weight of the roof and whatever it has to support. This pressure is beneficial rather than otherwise, as a roof having trusses thus formed, and the trusses well braced to each other, tends to steady the walls.

79.—Position of Supports.—*Figs.* 26 and 27 exhibit two methods of supporting the equal weights, *W* and *W*. Let it be required to measure and compare the strains produced on the pieces, *A B* and *A C*. Construct the parallelogram of forces, *e b f d*, according to *Art.* 71. Then *b f* will show the

strain on *A B*, and *b e* the strain on *A C*. By comparing the figures, *b d* being equal in each, it will be seen that the strains in *Fig.* 26 are about three times as great as those in

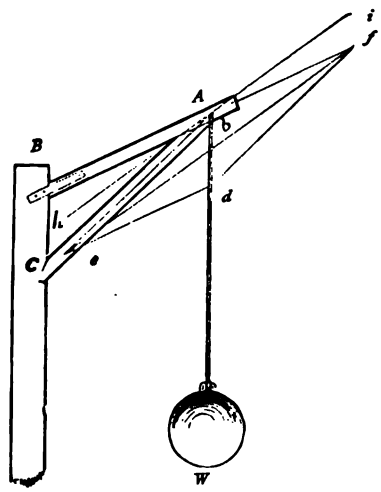


FIG. 26.

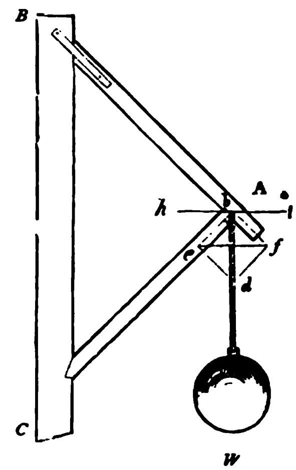


FIG. 27.

Fig. 27; the position of the pieces, *A B* and *A C*, in *Fig.* 27, is therefore far preferable.

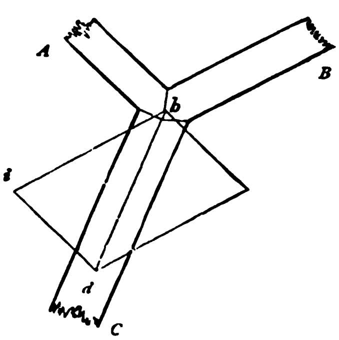


FIG. 28.

80.—The Composition of Forces: consists in ascertaining the direction and amount of *one* force which shall be just capable of balancing *two or more* given forces, acting in different directions. This is only the reverse of the resolu-

tion of forces; and the two are founded on one and the same principle, and may be solved in the same manner. For example, let *A* and *B* (*Fig.* 28) be two pieces of timber, pressed in the direction of their length towards *b*—*A* by a force equal to 6 tons weight, and *B* 9 tons. To find the *direction* and *amount* of pressure they would unitedly exert, draw the lines *b e* and *b f* in a line with the axes of the timbers, and make *b e* equal to the pressure exerted by *B*, viz., 9; also make *b f* equal to the pressure on *A*, viz., 6, and complete the parallelogram of forces *e b f d*; then *b d*, the diagonal of the parallelogram, will be the *direction*, and its length, 9·25, will be the *amount*, of the united pressures of *A* and of *B*. The line *b d* is termed the *resultant* of the two forces *b f* and *b e*. If *A* and *B* are to be supported by one post, *C*, the best position for that post will be in the direction of the diagonal *b d*; and it will require to be sufficiently strong to support the united pressures of *A* and of *B*, which are equal to 9·25 or 9¼ tons.

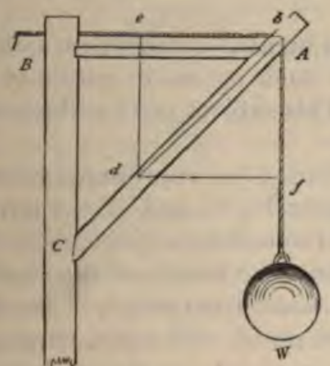


FIG. 29.

81.—Another Example.—Let *Fig.* 29 represent a piece of framing commonly called a crane, which is used for hoisting heavy weights by means of the rope, *B b f*, which passes over a pulley at *b*. This, though similar to *Figs.* 26 and 27, is, however, still materially different. In those figures, the strain is in one direction only, viz., from *b* to *d*; but in this there are two strains, from *A* to *B* and from *A* to *W*. The strain in the direction *A B* is evidently equal to that in the

direction $A\ W$. To ascertain the best position for the strut $A\ C$, make $b\ e$ equal to $b\ f$, and complete the parallelogram of forces $e\ b\ f\ d$; then draw the diagonal $b\ d$, and it will be the position required. Should the foot, C, of the strut be placed either higher or lower, the strain on $A\ C$ would be increased. In constructing cranes, it is advisable, in order that the piece $B\ A$ may be under a gentle pressure, to place the foot of the strut a trifle lower than where the diagonal $b\ d$ would indicate, but never higher.

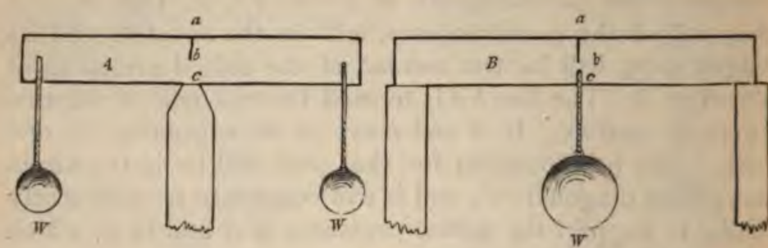


FIG. 30.

82.—Ties and Struts.—Timbers in a state of tension are called *ties*, while such as are in a state of compression are termed *struts*. This subject can be illustrated in the following manner:

Let A and B (*Fig.* 30) represent beams of timber supporting the weights W, W, and W; A having but one support, which is in the middle of its length, and B two, one at each end. To show the nature of the strains, let each beam be sawed in the middle from a to b. The effects are obvious: the cut in the beam A will open, whereas that in B will close. If the weights are heavy enough, the beam A will break at b; while the cut in B will be closed perfectly tight at a, and the beam be very little injured by it. But if, on the other hand, the cuts be made in the bottom edge of the timbers, from c to b, B will be seriously injured, while A will scarcely be affected. By this it appears evident that, in a piece of timber subject to a pressure across the direction of its length, the fibres are exposed to contrary strains. If the timber is supported at both ends, as at B, those from the top edge down to the middle are compressed in the direction

of their length, while those from the middle to the bottom edge are in a state of tension; but if the beam is supported as at *A*, the contrary effect is produced; while the fibres at the middle of either beam are not at all strained. The strains in a framed truss are of the same nature as those in a single beam. The truss for a roof, being supported at each end, has its tie-beam in a state of tension, while its rafters are compressed in the direction of their length. By this, it appears highly important that pieces in a state of tension should be distinguished from such as are compressed, in order that the former may be preserved continuous. A strut may be constructed of two or more pieces; yet, where there are many joints, it will not resist compression so well.

83.—To Distinguish Ties from Struts.—This may be done by the following rule. In *Fig.* 22–*B*, the timbers *C* and *D* are the *sustaining* forces, and the weight *W* is the *straining* force; and if the support be removed, the straining force would move from the point of support *b* towards *d*. Let it be required to ascertain whether the sustaining forces are *stretched* or *pressed* by the straining force. *Rule:* Upon the direction of the straining force *b d*, as a diagonal, construct a parallelogram *e b f d* whose sides shall be parallel with the direction of the sustaining forces *C* and *D*; through the point *b* draw a line parallel to the diagonal *e f*; this may then be called the dividing line between ties and struts. Because all those supports which are on that side of the dividing line which the straining force would occupy if unresisted are compressed, while those on the other side of the dividing line are stretched.

In *Fig.* 22–*B*, the supports are both compressed, being on that side of the dividing line which the straining force would occupy if unresisted. In *Figs.* 26 and 27, in which *A B* and *A C* are the sustaining forces, *A C* is compressed, whereas *A B* is in a state of tension; *A C* being on that side of the line *h i* which the straining force would occupy if unresisted, and *A B* on the opposite side. The place of the latter might be supplied by a chain or rope. In *Fig.* 25, the foot of the rafter at *A* is sustained by two forces, the wall and the tie-

beam, one perpendicular and the other horizontal: the direction of the straining force is indicated by the line *b a*. The dividing line *h i*, ascertained by the rule, shows that the wall is pressed and the tie-beam stretched.

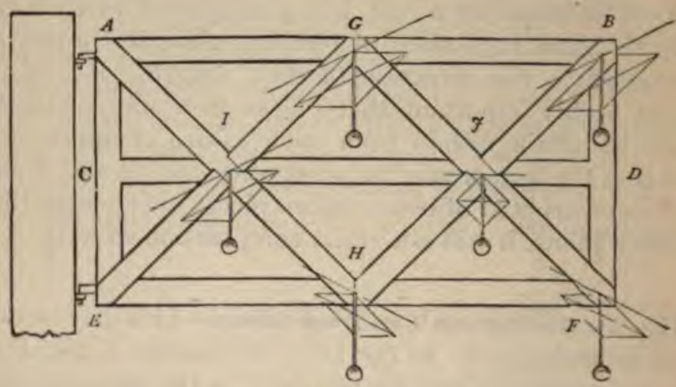


FIG. 31.

84.—Another Example.—Let *E A B F* (*Fig.* 31) represent a gate, supported by hinges at *A* and *E*. In this case, the *straining* force is the weight of the materials, and the direction of course *vertical*. Ascertain the dividing line at the several points, *G*, *B*, *I*, *J*, *H*, and *F*. It will then appear that the force at *G* is sustained by *A G* and *G E*, and the dividing line shows that the former is stretched and the latter compressed. The force at *H* is supported by *A H* and *H E*—the former stretched and the latter compressed. The force at *B* is opposed by *H B* and *A B*, one pressed, the other stretched. The force at *F* is sustained by *G F* and *F E*, *G F* being stretched and *F E* pressed. By this it appears that *A B* is in a state of tension, and *E F* of compression; also, that *A H* and *G F* are stretched, while *B H* and *G E* are compressed: which shows the necessity of having *A H* and *G F* each in one whole length, while *B H* and *G E* may be, as they are shown, each in two pieces. The force at *J* is sustained by *G J* and *J H*, the former stretched and the latter compressed. The piece *C D* is neither stretched nor pressed, and could be dispensed with if the joinings at *J* and *I* could be made

as effectually without it. In case *A B* should fail, then *C D* would be in a state of tension.

85.—Centre of Gravity.—The centre of gravity of a uniform prism or cylinder is in its axis, at the middle of its length; that of a triangle is in a line drawn from one angle to the middle of the opposite side, and at one third of the length of the line from that side; that of a right-angled triangle, at a point distant from the perpendicular equal to one third of the base, and distant from the base equal to one third of the perpendicular; that of a pyramid or cone, in the axis and at one quarter of the height from the base.

The centre of gravity of a trapezoid (a four-sided figure having only two of its sides parallel) is in a line joining the centres of the two parallel sides, and at a distance from the longest of the parallel sides equal to the product of the length in the sum of twice the shorter added to the longer of the parallel sides, divided by three times the sum of the two parallel sides. Algebraically thus:

$$d = \frac{l\,(2\,a + b)}{3\,(a + b)},$$

where d equals the distance from the longest of the parallel sides, l the length of the line joining the two parallel sides, and a the shorter and b the longer of the parallel sides.

Example.—A rafter 25 feet long has the larger end 14 inches wide, and the smaller end 10 inches wide: how far from the larger end is the centre of gravity located?

Here $l = 25$, $a = \frac{10}{12}$, and $b = \frac{14}{12}$,

$$\text{hence } d = \frac{l\,(2\,a + b)}{3\,(a + b)} = \frac{25\,(2 \times \frac{10}{12} + \frac{14}{12})}{3\,(\frac{10}{12} + \frac{14}{12})} = \frac{25 \times \frac{34}{12}}{3 \times \frac{24}{12}} = \frac{25 \times 34}{3 \times 24} =$$

$$\frac{850}{72} = 11 \cdot 8 = 11 \text{ feet } 9\tfrac{5}{8} \text{ inches nearly.}$$

In irregular bodies with plain sides, the centre of gravity may be found by balancing them upon the edge of a prism —upon the edge of a table—in two positions, making a line each time upon the body in a line with the edge of the prism, and the intersection of those lines will indicate the point re-

quired. Or suspend the article by a cord or thread attached to one corner or edge; also from the same point of suspension hang a plumb-line, and mark its position on the face of the article; again, suspend the article from another corner or side (nearly at right angles to its former position), and mark the position of the plumb-line upon its face; then the intersection of the two lines will be the centre of gravity.

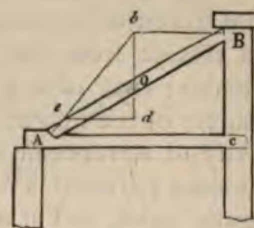


FIG. 32.

86.—Effect of the Weight of Inclined Beams.—An inclined post or strut supporting some heavy pressure applied at its upper end, as at *Fig.* 25, exerts a pressure at its foot in the direction of its length, or nearly so. But when such a beam is loaded uniformly over its whole length, as the rafter of a roof, the pressure at its foot varies considerably from the direction of its length. For example, let $A\ B$ (*Fig.* 32) be a beam leaning against the wall $B\ c$, and supported at its foot by the abutment A, in the beam $A\ c$, and let o be the centre of gravity of the beam. Through o draw the vertical line $b\ d$, and from B draw the horizontal line $B\ b$, cutting $b\ d$ in b; join b and A, and $b\ A$ will be the *direction* of the thrust. To prevent the beam from loosing its footing, the joint at A should be made at right angles to $b\ A$. The *amount* of pressure will be found thus: Let $b\ d$ (by any scale of equal parts) equal the number of tons upon the beam $A\ B$; draw $d\ e$ parallel to $B\ b$; then $b\ e$ (by the same scale) equals the pressure in the direction $b\ A$; and $e\ d$ the pressure against the wall at B—and also the horizontal thrust at A, as these are always equal in a construction of this kind.

The horizontal thrust of an inclined beam (*Fig.* 32)—the effect of its own weight—may be calculated thus:

Rule.—Multiply the weight of the beam in pounds by

its base, $A\ C$, in feet, and by the distance in feet of its centre of gravity, o (see *Art.* 85), from the lower end, at A, and divide this product by the product of the length, $A\ B$, into the height, $B\ C$, and the quotient will be the horizontal thrust in pounds. This may be stated thus: $H = \frac{d\ b\ w}{h\ l}$, where d equals the distance of the centre of gravity, o, from the lower end; b equals the base, $A\ C$; w equals the weight of the beam; h equals the height, $B\ C$; l equals the length of the beam; and H equals the horizontal thrust.

Example.—A beam 20 feet long weighs 300 pounds: its centre of gravity is at 9 feet from its lower end; it is so inclined that its base is 16 feet and its height 12 feet: what is the horizontal thrust?

Here $\frac{d\ b\ w}{h\ l}$ becomes $\frac{9 \times 16 \times 300}{12 \times 20} = \frac{9 \times 4 \times 25}{5} = 9 \times 4 \times 5$ $= 180 = H =$ the horizontal thrust.

This rule is for cases where the centre of gravity does not occur at the middle of the length of the beam, although it is applicable when it *does* occur at the middle; yet a shorter rule will suffice in this case, and it is thus:

Rule.—Multiply the weight of the rafter in pounds by the base, $A\ C$ (*Fig.* 32), in feet, and divide the product by twice the height, $B\ C$, in feet, and the quotient will be the horizontal thrust, when the centre of gravity occurs at the middle of the beam.

If the inclined beam is loaded with an equally distributed load, add this load to the weight of the beam, and use this *total* weight in the rule instead of the weight of the beam. And generally, if the centre of gravity of the combined weights of the beam and load does not occur at the centre of the length of the beam, then the former rule is to be used.

In *Fig.* 33, two equal beams are supported at their feet by the abutments in the tie-beam. This case is similar to the last: for it is obvious that each beam is in precisely the position of the beam in *Fig.* 32. The horizontal pressures at B, being equal and opposite, balance one another; and their horizontal thrusts at the tie-beam are also equal. (See *Art.*

78—*Fig.* 25.) When the height of a roof (*Fig.* 33) is one fourth of the span, or of a shed (*Fig.* 32) is one halt the span, the horizontal thrust of a rafter, whose centre of grav-

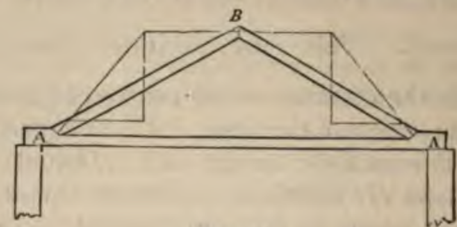


FIG. 33.

ity is at the middle of its length, is exactly equal to the weight distributed uniformly over its surface.

In shed or *lean-to* roofs, as *Fig.* 32, the horizontal pressure will be entirely removed if the bearings of the rafters, as *A* and *B* (*Fig.* 34), are made horizontal—provided, however,

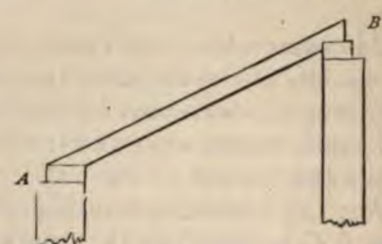


FIG. 34.

that the rafters and other framing do not bend between the points of support. If a beam or rafter have a natural curve, the convex or rounding edge should be laid uppermost.

87.—Effect of Load on Beam.—The strain in a uniformly loaded beam, supported at each end, is greatest at the middle of its length. Hence mortices, large knots, and other defects should be kept as far as possible from that point; and in resting a load upon a beam, as a partition upon a floor-beam, the weight should be so adjusted, if possible, that it will bear at or near the ends.

Twice the weight that will break a beam, acting at the centre of its length, is required to break it when equally

distributed over its length; and precisely the same deflection or *sag* will be produced on a beam by a load equally distributed that five eighths of the load will produce if acting at the centre of its length.

88.—Effect on Bearings.—When a uniformly loaded beam is supported at each end on level bearings (the beam itself being either horizontal or inclined), the amount of pressure caused by the load on each point of support is equal to one half the load; and this is also the ase when the load is concentrated at the middle of the beam, or has its centre of gravity at the middle of the beam; but when the load is unequally distributed, or concentrated so that its centre of gravity occurs at some other point than the middle of the beam, then the amount of pressure caused by the load on one of the points of support is unequal to that on the other. The precise amount on each may be ascertained by the following rule.

Rule.—Multiply the weight W (*Fig.* 35) by its distance, CB, from its nearest point of support, B, and divide the product by the length, AB, of the beam, and the quotient will

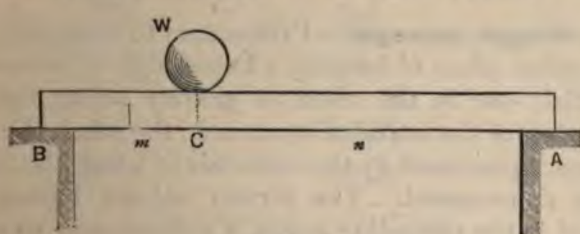


FIG 35.

be the amount of pressure on the *remote* point of support, A. Again, deduct this amount from the weight W, and the remainder will be the amount of pressure on the near point of support, B; or, multiply the weight W by its distance, AC, from the remote point of support, A, and divide the product by the length, AB, and the quotient will be the amount of pressure on the *near* point of support, B.

When l equals the length between the bearings A and B, $n = AC$, $m = CB$, and $W =$ the load; then

$$\frac{W m}{l} = A = \text{the amount of pressure at } A, \text{ and}$$

$$\frac{W n}{l} = B = \text{the amount of pressure at } B.$$

Example.—A beam 20 feet long between the bearings has a load of 100 pounds concentrated at 3 feet from one of the bearings: what is the portion of this weight sustained by each bearing?

Here $W = 100$; n, 17; m, 3; and l, 20.

$$\text{Hence } A = \frac{W\,m}{l} = \frac{100 \times 3}{20} = 15.$$

$$\text{And } B = \frac{W n}{l} = \frac{100 \times 17}{20} = 85.$$

Load on A = 15 pounds.
Load on B = 85 pounds.
Total weight = 100 pounds.

RESISTANCE OF MATERIALS.

89.—Weight—Strength.—Preliminary to designing a roof-truss or other piece of framing, a knowledge of two subjects is essential: one is, the effect of gravity acting upon the various parts of the intended structure; the other, the power of resistance possessed by the materials of which the framing is to be constructed. The former subject having been treated of in the preceding pages, it remains now to call attention to the latter.

90.—Quality of Materials.—Materials used in construction are constituted in their structure either of fibres (threads) or of grains, and are termed, the former fibrous, the latter granular. All woods and wrought metals are fibrous, while cast iron, stone, glass, etc., are granular. The strength of a granular material lies in the power of attraction acting among the grains of matter of which the material is composed, by which it resists any attempt to separate its grains or particles of matter. A fibre of wood or of

wrought metal has a strength by which it resists being compressed or shortened, and finally crushed; also a strength by which it resists being extended or made longer, and finally sundered. There is another kind of strength in a fibrous material: it is the adhesion of one fibre to another along their sides, or the lateral adhesion of the fibres.

91.—Manner of Resisting.—In the strain applied to a post supporting a weight imposed upon it (*Fig.* 36), we have an instance of an essay to shorten the fibres of which the timber is composed. The strength of the timber in this case is termed the *resistance to compression*. In the strain on a piece of timber like a king-post or suspending piece (*A*, *Fig.* 37), we have an instance of an essay to extend or lengthen the fibres of the material. The strength here exhibited is termed the *resistance to tension*. When a piece of timber is strained like a floor-beam or any horizontal piece

FIG. 36.

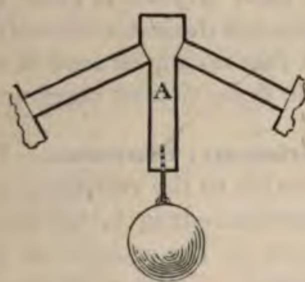


FIG 37.

FIG 38.

carrying a load (*Fig.* 38). we have an instance in which the two strains of compression and tension are both brought into action; the fibres of the upper portion of the beam being compressed, and those of the under part being stretched.

This kind of strength of timber is termed *resistance to cross-strains.* In each of these three kinds of strain to which timber is subjected, the power of resistance is in a measure due to the *lateral* adhesion of the fibres, not so much perhaps in the simple tensile strain, yet to a considerable degree in the compressive and cross strains. But the power of timber, by which it resists a pressure acting compressively in the direction of the length of the fibres, tending to separate the timber by splitting off a part, as in the case of the end of a tie-beam, against which the foot of the rafter presses, is wholly due to the lateral adhesion of the fibres.

92.—Strength and Stiffness.—The *strength* of materials is their power to resist *fracture*, while the *stiffness* of materials is their capability to resist *deflection* or sagging. A knowledge of their *strength* is useful, in order to determine their limits of size to sustain given weights safely; but a knowledge of their *stiffness* is more important, as in almost all constructions it is desirable not only that the load be safely sustained, but that no appearance of weakness be manifested by any sensible deflection or sagging.

93.—Experiments: Constants.—In the investigation of the laws applicable to the resistance of materials, it is found that the dimensions—length, breadth, and thickness—bear certain relations to the weight or pressure to which the piece is subjected. These relations are general; they exist quite independently of the peculiarities of any specific piece of material. These proportions between the dimensions and the load are found to exist alike in wood, metal, stone, and glass, or other material. One law applies alike to all materials; but the capability of materials to resist differs in accordance with the compactness and cohesion of particles, and the tenacity and adhesion of fibres, those qualities upon which depends the superiority of one kind of material over another. The capability of each particular kind of material is ascertained by experiments, made upon several specimens, and an average of the results thus obtained is taken as an index of the capability of that material, and is introduced in the rules as a *constant* number, each specific kind of ma-

terial having its own special *constant*, obtained by experimenting on specimens of that peculiar material. The results of experiments made to test the resistance of various materials useful in construction—their capability to resist the three strains before named—will now be introduced.

94.—Resistance to Compression.—The following table exhibits the results of experiments made to test the resistance to compression of such woods as are in common use in this country for the purposes of construction.

TABLE I.—RESISTANCE TO COMPRESSION.

MATERIAL.	Specific Gravity.	C. To crush fibres longitudinally.	H. To separate fibres by sliding.	To crush fibres transversely $\frac{1}{20}$ inch deep.	P. Value of P in Rules. Sensible Impression.
		Pounds per inch.	Pounds per inch.	Pounds per inch.	
Georgia Pine	0·613	9500	840	2250	900
Locust	0·762	11700	1160	2800	1120
White Oak	0·774	8000	1250	2650	1060
Spruce	0·369	7850	540	650	260
White Pine	0·388	6650	480	800	320
Hemlock	0·423	5700	370	800	320
White Wood	0·397	3400	....	800	320
Chestnut	0·491	6700	....	1250	500
Ash	0·517	5850	....	3100	1240
Maple	0·574	8450	....	2700	1080
Hickory	0·877	13750	....	4100	1640
Cherry	0·494	9050	....	2500	1000
Black Walnut	0·421	7800	....	2100	840
Mahogany (St. Domingo)	0·837	11600	....	5700	2280
" (Bay Wood)	0·439	4900	....	1700	680
Live Oak	0·916	11100	....	6800	2720
Lignum Vitæ	1·282	12100	....	7700	3080

The resistance of timber of the same name varies much; depending as it obviously must on the soil in which it grew on its age before and after cutting, on the time of year when cut, and on the manner in which it has been kept since it was cut. And of wood from the same tree much depends upon its location, whether at the butt or towards the limbs, and whether at the heart or of the sap, or at a point midway from the centre to the circumference of the tree. The

pieces submitted to experiment were of ordinary good quality, such as would be deemed proper to be used in framing. The prisms crushed were generally small, about 2 inches long, and from 1 inch to 1½ inches square; some were wider one way than the other, but all containing in area of cross section from 1 to 2 inches. The weight given in the table is the average weight per superficial inch.

Of the first six woods named, there were nine specimens of each tested; of the others, generally three specimens.

The results for the first six woods named are taken from the author's work on *Transverse Strains*, published by John Wiley & Sons, New York. The results for these six woods, as well as those for all the others named in the table, were obtained by experiments carefully made by the author. The first six woods named were tested in 1874 and 1876, and upon a testing machine, in which the power is transmitted to the pieces tested, by levers acting upon knife-edges. For a description of this machine, see *Transverse Strains*, Art. 704. The woods named in the table, other than the first six, were tested some twenty years since, and upon a hydraulic press, which, owing to friction, gave results too low.

The results, as thus ascertained, were given to the public in the 7th edition of this work, in 1857. In the present edition, the figures in Table I., for these woods, are those which have resulted by adding to the results given by the hydraulic press a certain quantity thought to be requisite to compensate for the loss by friction. Thus corrected, the figures in the table may be taken as sufficiently near approximations for use in the rules,—although not so trustworthy as the results given for the first six woods named, as these were obtained upon a superior testing machine, as above stated.

In the preceding table, the second column contains the specific gravity of the several kinds of wood, showing their comparative density. The weight in pounds of a cubic foot of any kind of wood or other material is equal to its specific gravity multiplied by 62·5, this number being the weight in pounds of a cubic foot of water. The third column

contains the weight in pounds required to crush a prism having a base of one inch square; the pressure applied to the fibres longitudinally. In practice, it is usual never to load material exposed to compression with more than one fourth of the crushing weight, and generally with from one sixth to one tenth only. The fourth column contains the weight in pounds which, applied in line with the length of the fibres, is required to force off a part of the piece, causing the fibres to separate by sliding, the surface separated being one inch square. The fifth column contains the weight in pounds required to crush the piece when the pressure is applied to the fibres transversely, the piece being one inch thick, and the surface crushed being one inch square, and depressed one twentieth of an inch deep. The sixth column contains the value of P in the rules; P being the weight in pounds, applied to the fibres transversely, which is required to make a sensible impression one inch square on the side of the piece, this being the greatest weight that would be proper for a post to be loaded with per inch surface of bearing, resting on the side of the kind of wood set opposite in the table. A greater weight would, in proportion to the excess, crush the side of the wood under the post, and proportionably derange the framing, if not cause a total failure. It will be observed that the measure of this resistance is useful in limiting the load on a post according to the kind of material contained, not in the *post*, but in the *timber upon which* the post presses.

95.—Resistance to Tension.—The resistance of materials to the force of stretching, as exemplified in the case of a rope from which a weight is suspended, is termed the *resistance to tension.* In fibrous materials, this force will be different in the same specimen, in accordance with the *direction* in which the force acts, whether in the direction of the length of the fibres or at right angles to the direction of their length. It has been found that, in hard woods, the resistance in the former direction is about eight to ten times what it is in the latter; and in soft woods, straight grained, such as white pine, the resistance is from sixteen to twenty times. A knowledge of the resistance in the direction of the fibres is the most useful in practice.

In the following table are recorded the results of experiments made to test this resistance in some of the woods in common use, and also in iron, cast and wrought. Each specimen of the woods was turned cylindrical, and about 2 inches diameter, and then the middle part reduced to about ⅝ of an inch diameter, at the middle of the reduced part, and thence gradually increased toward each end, where it was considerably larger at its junction with the enlarged end. The results, in the case of the iron and of the first six woods named, are taken from the author's work, *Transverse Strains*, Table XX. Experiments were made upon the other three woods named by a hydraulic press, some twenty years since, and the results were first published in the 7th edition of this work, in 1857. These results, owing to friction, were too low. Adding to them what is supposed to be the loss by friction of the machine, the results thus corrected are what are given for these three woods in the following table, and may be taken as fair approximations, but are not so trustworthy as the figures given for the other six woods and for the metals.

TABLE II.—RESISTANCE TO TENSION.

MATERIAL.	Specific Gravity.	T. Pounds required to rupture one inch square.
Georgia Pine	0·65	16000
Locust	0·794	24800
White Oak	0·774	19500
Spruce	0·432	19500
White Pine	0·458	12000
Hemlock	0·402	8700
Hickory	0·751	26000
Maple	0·694	20000
Ash	0·608	15000
Cast Iron, American } from	6·944	27000
" English } to	7·584	17000
Wrought Iron, American } from	7·600	60000
" English } to	7·792	50000

The figures in the table denote the *ultimate* capability of a bar one inch square, or the weight in pounds required to

produce rupture. Just what portion of this should be taken as the safe capability will depend upon the nature of the strain to which the material is to be exposed. In practice it is found that, through defects in workmanship, the attachments *may* be so made as to cause the strain to act along *one side* of the piece, instead of through its axis; and that in this case fracture will be produced with one third of the strain that can be sustained through the axis. Due to this and other contingencies, it is usual to subject materials exposed to tensile strain with only from one sixth to one tenth of the breaking weight.

96.—Resistance to Transverse Strains.—In the following table are recorded the results of experiments made to test the capability of the various materials named to resist the effects of transverse strain. The figures are taken from the author's work, *Transverse Strains*, before referred to.

TABLE III.—TRANSVERSE STRAINS.

MATERIAL.	*B.* Resistance to Rupture.	*F.* Resistance to Flexure.	*e.* Extension of Fibres.	*a.* Margin for Safety.
Georgia Pine	850	5900	·00109	1·84
Locust	1200	5050	·0015	2·20
White Oak	650	3100	·00086	3·39
Spruce	550	3500	·00098	2·23
White Pine	500	2900	·0014	1·71
Hemlock	450	2800	·00095	2·35
White Wood	600	3450	·00096	2·52
Chestnut	480	2550	·00103	2·54
Ash	900	4000	·00111	2·82
Maple	1100	5150	·0014	2·12
Hickory	1050	3850	·0013	2·91
Cherry	650	2850	·001563	2·03
Black Walnut	750	3900	·00104	2·57
Mahogany (St. Domingo)	650	3600	·00116	2·16
" (Bay Wood)	850	4750	·00109	2·28
Cast Iron, American	2500	50000		
" English	2100	40000		
Wrought Iron, American	2600	62000		
" English	1900	60000		
Steel, in Bars	6000	70000		
Blue Stone Flagging	200			
Sand Stone	59			
Brick, common	33			
" pressed	37			
Marble, East Chester	147			

The figures in the second column, headed B, denote the weight in pounds required to break a *unit* of the material named when suspended from the middle, the piece being supported at each end. The *unit* of material is a bar one inch square and one foot long between the bearings. The third column, headed F, contains the values of the several materials named as to their resistance to flexure, as explained in Arts. 302–305, *Transverse Strains*. These values of F, as *constants*, are used in the rules. The fourth column, headed e, contains the values of the several materials named, denoting the elasticity of the fibres, as explained in Art. 312, *Transverse Strains*. These values of e, as constants, are used in the rules.

The fifth column, headed a, contains for the several materials named the ratio of the resistance to flexure as compared with that to rupture, and which, as constants used in the rules, indicate the margin of safety to be given for each kind of material. The figures given in each case show the smallest possible value that may be safely given to a, the factor of safety. In practice it is generally taken higher than the amount given in the table. For example, in the table the value of B, the constant for rupture by transverse strain for spruce, is 550.

Now, if the dimensions of a spruce beam to carry a given weight be computed by the rules, using the constant B, at 550, the beam will be of such a size that the given weight will just break it.

But if, in the computation, instead of taking the full value of B, only a part of it be taken, then the beam will not break immediately; and if the part taken be so small that its effect upon the fibres shall not be sufficient to strain them beyond their limit of elasticity, the beam will be capable of sustaining the weight for an indefinite period; in this case the beam will be loaded by what is termed the *safe weight*. Or, since the value of a for spruce is 2·23 in the table, if, instead of taking B at 550, its full value, only the quotient arising from a division of B by a be taken—or 550 divided by 2·23, which equals 246·6—then the beam will be of sufficient size to carry the load safely. Therefore, while the con-

stant B is to be used for a breaking weight, for a safe load the quotient of $\frac{B}{a}$ is to be used. But, again, if a be taken at its value as given in the table, the computed beam will be loaded up to its limit of safety. So loaded that, if the load be increased only in a small degree, the limit of safety will be passed, and the beam liable, in time, to fail by rupture.

Therefore, as the values of a, in the table, are the smallest possible, it is prudent in practice always to take a larger than the table value. For example, the table value of a for spruce is 2·23, but in practice let it be taken at 3 or 4.

97.—Resistance to Compression.—The resistance of materials to the force of compression may be considered in several ways. Posts having their heights less than ten times their least sides will crush before bending; these belong to one class: another class is that which comprises all posts the height of which is equal to ten times their least sides, or more than ten times; these will bend before crushing. Then there remains to be considered the manner in which the pressure is applied: whether in line with the fibres, or transversely to them; and, again, whether the pressure tends to crush the fibres, or simply to force off a part of the piece by splitting. The various pressures may be comprised in the four classes following, namely:

1st.—When the pressure is applied to the fibres transversely.

2d.—When the pressure is applied to the fibres longitudinally, and so as to split off the part pressed against, causing the fibres to separate by sliding.

3d.—When the pressure is applied to the fibres longitudinally, and on short pieces.

4th.—When the pressure is applied to the fibres longitudinally, and on long pieces.

These four classes will now be considered in their regular order.

98.—Compression Transversely to the Fibres.—In this first class of compression, experiment has shown that the resistance is in proportion to the number of fibres pressed, that is, in proportion to the area. For example, if 5000 pounds is required to crush a prism with a base 1 inch square, it will require 20,000 pounds to crush a prism having a base of 2 by 2 inches, equal to 4 inches area; because 4 times 5000 equals 20000.

Therefore, if any given surface pressed be multiplied by the pressure per inch which the kind of material pressed may be safely trusted with, the product will be the total pressure which may with safety be put upon the given surface. Now, the capability for this kind of resistance is given in column *P*, in Table I., for each kind of material named in the table. Therefore, to find the limit of weight, proceed as follows:

99.—The Limit of Weight.—To ascertain what weight a post may be loaded with, so as not to crush the surface against which it presses, we have—

Rule I.—Multiply the area of the post in inches by the value of *P*, Table I., and the product will be the weight required in pounds; or—

$$W = A\,P. \tag{1.}$$

Example.—A post, 8 by 10 inches, stands upon a white-pine girder; the area equals $8 \times 10 = 80$ inches. This being multiplied by 320, the value of *P*, Table I., set opposite white pine, the product, 25600, is the required weight in pounds.

100.—Area of Post.—To ascertain what area a post must have in order to prevent the post, loaded with a given weight, from crushing the surface against which it presses, we have—

Rule II.—Divide the given weight in pounds by the value of *P*, Table I., and the quotient will be the area required in inches; or—

$$A = \frac{W}{P}. \tag{2.}$$

Example.—A post standing on a Georgia-pine girder is loaded with 100,000 pounds: what must be its area? The weight, 100000, divided by 900, the value of P, Table I., set opposite Georgia pine, the quotient, $111 \cdot 11$, is the required area in inches. The post may be 10 by $11\frac{1}{8}$, or 10 by 11 inches; or if square each side will be $10 \cdot 54$ inches, or $11\frac{9}{10}$ inches diameter if round.

101.—Rupture by Sliding.—In this the second class of rupture by compression, it has been ascertained by experiment that the resistance is in proportion to the area of the surface separated without regard to the form of the surface. Now, in Table I., column H, we have the *ultimate* resistance to this strain of the several materials named. But to obtain the *safe* load per inch, the ultimate resistance of the table is to be divided by a *factor of safety*, of such value as circumstances may seem to require. Generally this factor may be taken at 3. Then to obtain the safe load for any given case, we have but to multiply the given surface by the ultimate resistance, and divide by the factor of safety; therefore, proceed as follows:

102.—The Limit of Weight.—To ascertain what weight may be sustained safely by the resistance of a given area of surface, when the weight tends to split off the part pressed against by causing, in case of fracture, one surface to slide on the other, we have—

Rule III.—Multiply the area of the surface by the value of H, in Table I. divide by the factor of safety, and the quotient will be the weight required in pounds; or—

$$W = \frac{A\,H}{a} \qquad (3.)$$

Example.—The foot of a rafter is framed into the end of its tie-beam, so that the uncut substance of the tie-beam is 15 inches long from the end of the tie-beam to the joint of the rafter; the tie-beam is of white pine, and is 6 inches thick: what amount of horizontal thrust will this end of the tie-beam sustain, without danger of having the end of

the tie-beam split off? Here the area of surface that sustains the pressure is 6 by 15 inches, equal to 90 inches. This multiplied by 480, the value of H, set opposite to white pine, Table I., and divided by 3, as a factor of safety, gives a quotient of 14400, and this is the required weight in pounds.

103.—Area of Surface.—To ascertain the area of surface that is required to sustain a given weight safely, when the weight tends to split off the part pressed against, by causing, in case of fracture, one surface to slide on the other, we have—

Rule IV.—Divide the given weight in pounds by the value of H, Table I.; multiply the quotient by the factor of safety, and the product will be the required area in inches; or—

$$A = \frac{W\,a}{H}. \qquad (4.)$$

Example.—The load on a rafter causes a horizontal thrust at its foot of 40,000 pounds, tending to split off the end of the tie-beam: what must be the length of the tie-beam beyond the line where the foot of the rafter is framed into it, the tie-beam being of Georgia pine, and 9 inches thick? The weight, or horizontal thrust, 40000, divided by 840, the value of H, Table I., set opposite Georgia pine, gives a quotient of 47·619, and this multiplied by 3, as a factor of safety, gives a product of 142·857. This, the area of surface in inches, divided by 9, the breadth of the surface strained (equal to the thickness of the tie-beam), the quotient, 15.87, is the length in inches from the end of the tie-beam to the rafter joint, say 16 inches.

104.—Tenons and Splices.—A knowledge of this kind of resistance of materials is useful, also, in ascertaining the length of framed tenons, so as to prevent the pin, or key, with which they are fastened from tearing out; and, also, in cases where tie-beams, or other timber under a tensile strain,

are spliced, this rule gives the length of the joggle at each end of the splice.

105.—Stout Posts.—These comprise the third class of objects subject to compression (*Art.* 97), and include all posts which are less than ten diameters high. The resistance to compression, in this class, is ascertained to be directly in proportion to the area of cross-section of the post.

Now in Table I., column C, the *ultimate* resistance to crushing is given for the several kinds of materials named; from which the safe resistance per inch may be obtained by dividing it by a proper factor of safety. Having the safe resistance per inch, the resistance of any given post may be determined by multiplying it by the area of the cross-section of the post. Therefore, proceed as follows:

106.—The Limit of Weight.—To find the weight that can be safely sustained by a post, when the height of the post is less than ten times the diameter if round, or ten times the thickness if rectangular, and the direction of the pressure coinciding with the axis, we have—

Rule V.—Multiply the area of the cross-section of the post in inches by the value of C, in Table I.; divide the product by the factor of safety, and the quotient will be the required weight in pounds; or—

$$W = \frac{A\,C}{a}. \qquad (5.)$$

Example.—A Georgia-pine post is 6 feet high, and in cross-section 8×12 inches: what weight will it safely sustain? The height of this post, $12 \times 6 = 72$ inches, which is less than 10×8 (the size of the narrowest side) $= 80$ inches; it therefore belongs to the class coming under this rule. The area $= 8 \times 12 = 96$ inches; this multiplied by 9500, the value of C, in the table, set opposite Georgia pine, and divided by 6, as a factor of safety, the quotient, 152000, is the weight in pounds required. It will be observed that the weight would be the same for a Georgia-pine post of any height less than

10 times 8 inches = 80 inches = 6 feet 8 inches, provided its breadth and thickness remain the same, 12 and 8 inches.

107.—Area of Post.—To find the area of the cross-section of a post to sustain a given weight safely, the height of the post being less than ten times the diameter if round, or ten times the least side if rectangular, the pressure coinciding with the axis, we have—

Rule VI.—Divide the given weight in pounds by the value of C, in Table I.; multiply the quotient by the factor of safety, and the product will be the required area in inches; or—

$$A = \frac{W a}{C}. \qquad (6.)$$

Example.—A weight of 40,000 pounds is to be sustained by a white-pine post 4 feet high: what must be its area of section to sustain the weight safely? Here, 40000 divided by 6650, the value of C, in Table I., set opposite white pine, and the quotient multiplied by 6, as a factor of safety, the product is 36; this, therefore, is the required area, and such a post may be 6×6 inches. To find the least side, so that it shall not be less than one tenth of the height, divide the height, reduced to inches, by 10, and make the least side to exceed this quotient. The area divided by the least side so determined will give the wide side. If, however, by this process, the first side found should prove to be the greatest, then the size of the post is to be found by Rule IX., X., or XI.

108.—Area of Round Posts.—In case the post is to be round, its diameter may be found by reference to the Table of Circles in the Appendix, in the column of diameters, opposite to the area of the post to be found in the column of areas, or opposite to the next nearest area. For example, suppose the required area, as just found by the example under Rule VI., is 36: by reference to the column of areas, 35·78 is the nearest to 36, and the diameter set opposite is

6.75, which is a trifle too small. The post may therefore be, say, $6\frac{7}{8}$ inches diameter.

109.—Slender Posts.—When the height of a post is less than ten times its diameter, the resistance of the post to crushing is approximately in proportion to its area of cross-section. But when the height is equal to or more than ten diameters, the resistance per square inch is diminished. The resistance diminishes as the height is increased, the diameter remaining the same (*Transverse Strains*, Art. 643). The strength of a slender post consists in a combination of the resistances of the material to bending and to crushing, and is represented in the following rule:

110.—The Limit of Weight.—To ascertain the weight that can be sustained safely by a post the height of which is at least ten times its least side if rectangular, or ten times its diameter if round, the direction of the pressure coinciding with the axis, we have—

Rule VII.—Divide the height of the post in inches by the diameter, or least side, in inches; multiply the quotient by itself, or take its square; multiply the square by the value of e, in Table III., set opposite the kind of material of which the post is made; to the product add the half of itself; to the sum add unity (or one); multiply this sum by the factor of safety, and reserve the product for use, as below. Now multiply the area of cross-section of the post in inches by the value of C, in Table I., set opposite the material of the post, and divide the product by the above reserved product; the quotient will be the required weight in pounds; or—

$$W = \frac{A\,C}{a(1 + \frac{3}{2}er^2)}. \qquad (7.)$$

Example: A Round Post.—What weight may be safely placed upon a post of Georgia pine 10 inches diameter and 10 feet high, the pressure coinciding with the axis of the post? The height of the post, ($10 \times 12 =$) 120 inches, divided by 10, its diameter, gives a quotient of 12; this multiplied

by itself gives 144, its square; and this by ·00109, the value of e for Georgia pine, in Table III., gives ·15696; to which adding its half, the sum is 0·23544; to which adding unity, the sum is 1·23544; and this multiplied by 7, as a factor of safety, the product is 8·648, the reserved divisor. Now the area of the post is (see Table of Areas of Circles, in the Appendix, opposite its diameter, 10) 78·54; this multiplied by 9500, the value of C for Georgia pine, in Table I., gives a product of 746130; which divided by 8·648, the above reserved divisor, gives a quotient of 86278, the required weight in pounds.

Another Example: A Rectangular Post.—What weight may be safely placed upon a white-pine post 10 × 12 inches, and 15 feet high, the pressure coinciding with the axis of the post? Proceeding according to the rule, we find the height of the post to be 180 inches, which divided by 10, the least side of the post, gives 18; this multiplied by itself gives 324, its square; which multiplied by ·0014, the value of e for white pine, in Table III., gives ·4536; to which adding its half, the sum is ·6804; to which adding unity, the sum is 1·6804; and this multiplied by 8, as a factor of safety, the product is 13·4432, the reserved divisor. Now the area of the post, (10 × 12 =) 120 inches, multiplied by 6650, the value of C for white pine, in Table I., gives a product of 798,000, and this divided by 13·4432, the above reserved divisor, the quotient, 59360, is the required weight in pounds.

III.—Diameter of the Post: when Round.—To ascertain the size of a round post to sustain safely a given weight, when the height of the post is at least ten times the diameter; the direction of the pressure coinciding with the axis of the post; we have—

Rule VIII.—Multiply the given weight by the factor of safety, and divide the product by 1·5708 times the value of C for the material of the post, found in Table I.; reserve the quotient, calling its value G. Now multiply 432 times the value of e for the material of the post, found in Table III., by the square of the height in feet, and by the above quotient G; to the product add the square of G: extract the

square root of the sum, and to it add the value of G; then the square root of this sum will be the required diameter; or—

$$G = \frac{W a}{1.5708\ C}. \qquad (8.)$$

$$d = \sqrt{\sqrt{432\ G e l^2 + G^2} + G}. \qquad (9.)$$

Example.—What should be the diameter of a locust post 10 feet high to sustain safely 40,000 pounds, the pressure coinciding with the axis? Proceeding by the rule, the given weight multiplied by 6, taken as a factor of safety, equals 240000. Dividing this by 1·5708 times 11700, the value of C for locust, in Table I., the quotient, 13·06, is the value of G, the square of which is 170·53. Now, the value of e for locust, in Table III., is ·0015. This multiplied by 432, by 100, the square of the height, and by the above value of G, gives a product of 846·2; which added to 170·53, the above square of G, gives the sum of 1016·73. To 31·89, the square root of this, add the above value of G; then 6·7, the square root of this sum, is the required diameter of the post. The post therefore requires to be 6·7, say $6\frac{3}{4}$ inches diameter.

112.—Side of the Post: when Square.—To ascertain the side of a square post to sustain safely a given weight, when the height of the post is at least ten times the side; the pressure coinciding with the axis; we have—

Rule IX.—Multiply the given weight by the factor of safety, and divide the product by twice the value of C for the material of the post, found in Table I.; reserve the quotient, calling its value G. Now multiply 432 times the value of e for the material of the post, found in Table III., by the square of the height in feet, and by the above quotient G; to the product add the square of G; extract the square root of the sum, and to it add the value of G; then the square root of this sum will be the required side of the post; or—

$$G = \frac{W a}{2\ C}. \qquad (10.)$$

$$S = \sqrt{\sqrt{432\,G\,e\,l^2 + G^2} + G.} \qquad (11.)$$

Example.—What should be the side of a Georgia-pine square post 15 feet high to sustain safely 50,000 pounds, the pressure coinciding with the axis of the post? Proceeding by the rule, 50,000 pounds multiplied by 6, as a factor of safety, gives 300000; this divided by 2×9500 (the value of C) $= 19000$, the quotient, 15·789, is the value of G. The value of e for Georgia pine is ·00109; the square of the height is 225; then, 432 times ·00109 by 225 and by 15·789 (the above value of G) gives a product of 1672·86; the square of G equals 249·31; this added to 1672·86 gives a sum of 1922·17, the square root of which is 43·843; which added to 15·789, the value of G, gives 59·632, the square root of which is 7·722, the required side of the post. The post, therefore, requires to be, say, $7\frac{3}{4}$ inches square.

113.—Thickness of a Rectangular Post.—This may be definitely ascertained when the proportion which the thickness shall bear to the breadth shall have been previously determined. For example, when the proportion is as 6 to 8, then $1\frac{1}{3}$ times 6 equals 8, and the proportion is as 1 to $1\frac{1}{3}$; again, when the proportion is as 8 to 10, then $1\frac{1}{4}$ times 8 equals 10, and in this case the proportion is as 1 to $1\frac{1}{4}$. Let the latter figure of the ratio 1 to $1\frac{1}{4}$, 1 to $1\frac{1}{3}$, etc., be called n, or so that the proportion shall be as 1 to n, then—

To ascertain the thickness of a post to sustain safely a given weight, when the height is at least ten times the thickness; the action of the weight coinciding with the axis; we have—

Rule X.—Multiply the given weight by the factor of safety, and divide the product by twice the value of C for the material of the post, found in Table I., multiplied by n, as above explained; reserve the quotient, calling it G. Now multiply 432 times the value of e for the material of the post, found in Table III., by the square of the height in feet, and by the above quotient G; to the product add the square of G; extract the square root of the sum, and to it add the value

of G; then the square root of this sum will be the required thickness of the post; or—

$$G = \frac{W\,a}{2\,C\,n}. \qquad (12.)$$

$$t = \sqrt{\sqrt{432\,G\,e\,l^2 + G^2} + G}. \qquad (13.)$$

Example.—What should be the thickness of a white-pine rectangular post 20 feet high to sustain safely 30,000 pounds, the pressure coinciding with the axis, and the proportion between the thickness and breadth to be as 10 to 12, or as 1 to 1·2? Proceeding according to the rule, we have the product of 30000, the given weight, by 6, as a factor of safety, equals 180000: this divided by twice $C \times n$, or $2 \times 6650 \times 1 \cdot 2$, ($= 15960$) gives a quotient of $11 \cdot 278$, the value of G. Then, we have $e = \cdot 0014$, the square of the height equals 400; therefore, $432 \times \cdot 0014 \times 400 \times 11 \cdot 278 = 2728 \cdot 43$. To this adding $127 \cdot 2$, the square of G, we have 2855 63, the square root of which is 53·438; and this added to G gives 64·716, the square root of which is 8·045, the required thickness of the post. Now, since the thickness is in proportion to the breadth as 1 to 1·2, therefore $8 \cdot 045 \times 1 \cdot 2 = 9 \cdot 654$, the required width. The post, therefore, may be made $8 \times 9\frac{3}{4}$ inches.

114.—Breadth of a Rectangular Post.—When the thickness of a post is fixed, and the breadth required; then, to ascertain the breadth of a rectangular post to sustain safely a given weight, the direction of the pressure of which coincides with the axis of the post, we have—

Rule XI.—Divide the height in inches by the given thickness, and multiply the quotient by itself, or take its square; multiply this square by the value of e for the material of the post, found in Table III.; to the product add its half, and to the sum add unity; multiply this sum by the given weight, and by the factor of safety; divide the product by the product of the given thickness multiplied by the value of C for

the material of the post, found in Table I., and the quotient will be the required breadth; or—

$$b = \frac{W a \left(1 + \frac{3}{2} e r^2\right)}{C t}. \qquad (14.)$$

Example.—What should be the breadth of a spruce post 18 feet high and 6 inches thick to sustain safely 25,000 pounds, the pressure coinciding with the axis of the post? According to the rule, 216 ($= 12 \times 18$), the height in inches, divided by 6, the given thickness, gives a quotient of 36, the square of which is 1296; the value of e for spruce is ·00098; this multiplied by 1296, the above square, equals 1·27; which increased by ·635, its half, amounts to 1·905; this increased by unity, the sum is 2·905; which multiplied by the given weight, and by the factor of safety, gives a product of 435749; and this divided by 6 (the given thickness) times 7850 (the value of C for spruce) $= 47100$, gives a quotient of 9·2516, the required breadth of the post. The post, therefore, requires to be $6 \times 9\frac{1}{4}$ inches.

Observe that when the breadth obtained by the rule is less than the given thickness, the result shows that the conditions of the case are incompatible with the rule, and that a new computation must be made; taking now for the breadth what was before understood to be the thickness, and proceeding in this case, by Rule X., to find the thickness.

115.—Resistance to Tension.—In *Art.* 95 are recorded the results of experiments made to test the resistance of various materials to tensile strain, showing in each case the capability to such resistance per square inch of sectional area. The action of materials in resisting a tensile strain is quite simple; their resistance is found to be directly as their sectional area. Hence—

116.—The Limit of Weight.—To ascertain the weight or pressure that may be safely applied to a beam or rod as a tensile strain, we have—

Rule XII.—Multiply the area of the cross-section of the beam or rod in inches by the value of T, Table II.; divide

the product by the factor of safety, and the quotient will be the required weight in pounds; or—

$$W = \frac{A\,T}{a}. \qquad (15.)$$

The cross-section here intended is that taken at the smallest part of the beam or rod. A beam, in framing, is usually cut with mortices; the area will probably be smallest at the severest cutting; the area used in the rule must be that of the uncut fibres only.

Example.—The tie-beam of a roof-truss is of white pine, 6 × 10 inches; the cutting for the foot of the rafter reduces the uncut area to 40 inches: what amount of horizontal thrust from the foot of the rafter will this tie-beam safely sustain? Here 40 times 12000, the value of T, equals 480000; this divided by 6, as a factor of safety, gives 80000, the required weight in pounds.

117.—Sectional Area.—To ascertain the sectional area of a beam or rod that will sustain a given weight safely, when applied as a tensile strain, we have—

Rule XIII.—Multiply the given weight in pounds by the factor of safety; divide the product by the value of T, Table II., and the quotient will be the area required in inches; or—

$$A = \frac{W\,a}{T}. \qquad (16.)$$

This is the area of uncut fibres. If the piece is to be cut for mortices, or for any other purpose, then for this an adequate addition is to be made to the result found by the rule.

Example.—A rafter produces a thrust horizontally of 80,000 pounds; the tie-beam is to be of oak: what must be the area of the cross-section of the tie-beam in order to sustain the rafter safely? The given weight, 80000, multiplied by 10, as a factor of safety, gives 800000; this divided by 19500, the value of T, Table II., the quotient, 41, is the area of uncut fibres. This should have usually one half of its amount

added to it as an allowance for cutting; therefore, $41 + 21 = 62$. The tie-beam may be $6 \times 10\frac{1}{2}$ inches.

Another Example. — A tie-rod of American refined wrought iron is required to sustain safely 36,000 pounds: what should be its area of cross-section? Taking 7 as the factor of safety, $7 \times 36000 = 252000$; and this divided by 60000, the value of T, Table II., gives a quotient of 4·2 inches, the required area of the rod.

118.—Weight of the Suspending Piece Included.—Pieces subjected to a tensile strain are frequently suspended vertically. In this case, at the upper end, the strain is due not only to the weight attached at the lower end, but also to the weight of the rod itself. Usually, in timber, this is small in comparison with the load, and may be neglected; although in very long timbers, and where accuracy is decidedly essential, as, also, when the rod is of iron, it may form a part of the rule. Taking the effect of the weight of the beam into account, the relation existing between the weights and the beam requires that the rule for the weight should be as follows:

Rule XIV.—Divide the value of T for the material of the beam or rod, Table II., by the factor of safety; from the quotient subtract 0·434 times the specific gravity of the material in the beam or rod multiplied by the length of the beam or rod in feet; multiply the remainder by the area of cross-section in inches, and the product will be the required weight in pounds; or—

$$W = A\left(\frac{T}{a} - 0{\cdot}434\,l\,s\right). \qquad (17.)$$

N. B.—This rule is based upon the condition that the suspending piece be not cut by mortices or in any other way.

Example.—What weight may be safely sustained by a white-pine rod 4×6 inches, 40 feet long, suspended vertically? For white pine the value of T is 12000; this divided by 8, as a factor of safety, gives 1500; from which subtracting 0·434 times 0·458 (the specific gravity of white pine, Table II.) multiplied by 40, the length in feet, the remainder

is 1492·049; which multiplied by 24 ($= 4 \times 6$, the area of cross-section) equals 35,761 pounds, the required weight to be carried. The weight which the rule would give, neglecting the weight of the rod, would have been 36000; ordinarily, so slight a difference would be quite unimportant.

119.—Area of Suspending Piece.—To ascertain the area of a suspended rod to sustain safely a given weight, when the weight of the suspending piece is regarded, we have—

Rule XV.—Multiply 0·434 times the specific gravity of the suspending piece by the length in feet; deduct the product from the quotient arising from a division of the value of T, Table II., by the factor of safety, and with the remainder divide the given weight in pounds; the quotient will be the required area in inches; or—

$$A = \frac{W}{\frac{T}{a} - 0{\cdot}434\, l\, s}. \qquad (18.)$$

N.B.—This rule is based upon the condition that the rod be not injured in anywise by cutting.

Example.—What should be the area of a bar of English cast iron 20 feet long to sustain safely, suspended from its lower end, a weight of 5000 pounds? Taking the factor of safety at 7·0, and the specific gravity also at 7, and the value of T, Table II., at 17000, we have the product of $0{\cdot}434 \times 7{\cdot}0 \times 20 = 60{\cdot}76$; then 17000 divided by 7 gives a quotient of 2428·57; from which deducting the above 60·76, there remains 2367·81; dividing 5000, the given weight, by this remainder, we have the quotient, 2·11, which is the required area in inches.

RESISTANCE TO TRANSVERSE STRAINS.

120.—Transverse Strains: Rupture.—A load placed upon a beam, laid horizontally or inclined, will bend it, and, if the weight be proportionally large, will break it. The power in the material that resists this bending or breaking is termed the *resistance to cross-strains*, or transverse strains.

While in posts or struts the material is compressed or shortened, and in ties and suspending pieces it is extended or lengthened, in beams subjected to cross-strains the material is both compressed and extended. (See *Art.* 91.) When the beam is bent the fibres on the concave side are compressed, while those on the convex side are extended. The line where these two portions of the beam meet—that is, the portion compressed and the portion extended — the horizontal line of juncture, is termed the *neutral* line or plane. It is so called because at this line the fibres are neither compressed nor extended, and hence are under no strain whatever. The location of this line or plane is not far from the middle of the depth of the beam, when the strain is not sufficient to injure the elasticity of the material; but it removes towards the concave or convex side of the beam as the strain is increased, until, at the period of rupture, its distance from the top of the beam is in proportion to its distance from the bottom of the beam as the tensile strength of the material is to its compressive strength.

121.—Location of Mortices.—In order that the diminution of the strength of a beam by framing be as small as possible, all mortices should be located at or near the middle of the depth. There is a prevalent idea with some, who are aware that the upper fibres of a beam are compressed when subject to cross-strains, that it is not injurious to cut these top fibres, provided that the cutting be for the insertion of another piece of timber—as in the case of *gaining* the ends of beams into the side of a girder. They suppose that the piece filled in will as effectually resist the compression as the part removed would have done, had it not been taken out. Now, besides the effect of shrinkage, which of itself is quite sufficient to prevent the proper resistance to the strain, there is the mechanical difficulty of fitting the joints perfectly throughout; and, also, a great loss in the power of resistance, as the material is so much less capable of resistance when pressed at right angles to the direction of the fibres than when directly with them, as the results of the experiments in the tables show.

122.—Transverse Strains: Relation of Weight to Dimensions.—The strength of various materials, in their resistance to cross-strains, is given in Table III., *Art.* 96. The second column of the table contains the results of experiments made to test their resistance to rupture. In the case of each material, the figures given and represented by B indicate the pounds at the middle required to break a *unit* of the material, or a piece 1 inch square and 1 foot long between the bearings upon which the piece rests. To be able to use these indices of strength, in the computation of the strength of large beams, it is requisite, first, to establish the relation between the unit of material and the larger beam. Now, it may be easily comprehended that the strength of beams will be in proportion to their breadth; that is, when the length and depth remain the same, the strength will be directly as the breadth; for it is evident that a beam 2 inches broad will bear twice as much as one which is only 1 inch broad, or that one which is 6 inches broad will bear three times as much as one which is 2 inches broad. This establishes the relation of the weight to the breadth. With the depth, however, the relation is different; the strength is greater than simply in proportion to the depth. If the boards cut from a squared piece of timber be piled up in the order in which they came from the timber, and be loaded with a heavy weight at the middle, the boards will deflect or sag much more than they would have done in the timber before sawing. The greater strength of the material when in a solid piece of timber is due to the cohesion of the fibres at the line of separation, by which the several boards, before sawing, are prevented from sliding upon each other, and thus the resistance to compression and tension is made to contribute to the strength. This resistance is found to be in proportion to the depth. Thus the strength due to the depth is, first, that which arises from the quantity of the material (the greater the depth, the more the material), which is in proportion to the depth; then, that which ensues from the cohesion of the fibres in such a manner as to prevent sliding; this is also as the depth. Combining the two, we have, as the total result, the resistance in proportion

to the square of the depth. The relation between the weight and the length is such that the longer the beam is, the less it will resist; a beam which is 20 feet long will sustain only half as much as one which is 10 feet long; the breadth and depth each being the same in the two beams. From this it results that the resistance is *inversely* in proportion to the length. To obtain, therefore, the relation between the strength of the unit of material and that of a larger beam, we have these facts, namely: the strength of the unit is the value of B, as recorded in Table III.; and the strength of the larger beam, represented by W, the weight required to break it, is the product of the breadth into the square of the depth, divided by the length; or, while for the unit we have the ratio—

$$B : 1,$$

we have for the larger beam the ratio—

$$W : \frac{b\,d^2}{l}.$$

Therefore, putting these ratios in an expressed proportion, we have—

$$B : 1 :: W : \frac{b\,d^2}{l}.$$

From which (the product of the means equalling the product of the extremes; see *Art.* 373) we have—

$$W = \frac{B\,b\,d^2}{l}. \qquad (19.)$$

In which W represents the pounds required to break a beam, when acting at the middle between the two supports upon which the beam is laid; of which beam b represents the breadth and d the depth, both in inches, and l the length in feet between the supports; and B is from Table III., and represents the pounds required to break a unit of material like that contained in the larger beam.

123.—Safe Weight: Load at Middle.—The relation established, in the last article, between the weight and the dimensions is that which exists at the moment of rupture. The rule (19.) derived therefrom is not, therefore, directly practicable for computing the dimensions of beams for buildings. From it, however, one may readily be deduced which shall be practicable. In the fifth column of Table III. are given the least values of a, the factor of safety, explained in *Art.* 96. Now, if in place of B, the symbol for the breaking weight, the *quotient* of B divided by a be substituted, then the rule at once becomes practicable; the results now being in consonance with the requirements for materials used in buildings. Thus, with this modification, we have—

$$W = \frac{B\,b\,d^2}{a\,l}. \qquad (20.)$$

Therefore, to ascertain the weight which a beam may be safely loaded with at the centre, we have—

Rule XVI.—Multiply the value of B, Table III., for the kind of material in the beam by the breadth and by the square of the depth of the beam in inches; divide the product by the product of the factor of safety into the length of the beam between bearings in feet, and the quotient will be the weight in pounds that the beam will safely sustain at the middle of its length.

Example.—What weight in pounds can be suspended safely from the middle of a Georgia-pine beam 4×10 inches, and 20 feet long between the bearings? For Georgia pine the value of B, in Table III., is 850, and the least value of a is $1 \cdot 84$. For reasons given in *Art.* 96, let a be taken as high as 4; then, in this case, the value of b is 4, and that of d is 10, while that of l is 20. Therefore, proceeding by the rule, $850 \times 4 \times 10^2 = 340000$; this divided by 4×20 ($= 80$) gives a quotient of 4250 pounds, the required weight.

Observe that, had the value of a been taken at 3, instead of 4, the result by the rule would have been a load of 5667 pounds, instead of 4250, and the larger amount would be none too much for a safe load upon such a beam; although,

with it, the deflection would be one third greater than with the lesser load. The value of a should always be assigned higher than the figures of the table, which show it at its *least* value; but just how much higher must depend upon the firmness required and the conditions of each particular case.

124.—Breadth of Beam with Safe Load.—By a simple transposition of the factors in equation (20.), we obtain—

$$b = \frac{W a l}{B d^2}, \qquad (21.)$$

a rule for the breadth of the beam.

Therefore, to ascertain what should be the breadth of a beam of given depth and length to safely sustain at the middle a given weight, we have—

Rule XVII.—Multiply the given weight in pounds by the factor of safety, and by the length in feet, and divide the product by the square of the depth multiplied by the value of B for the material in the beam, in Table III.; the quotient will be the required breadth.

Example.—What should be the breadth of a white-pine beam 8 inches deep and 10 feet long between bearings to sustain safely 2400 pounds at the middle? For white pine the value of B, in Table III., is 500. Taking the value of a at 4, and proceeding by the rule, we have $2400 \times 4 \times 10 = 96000$; this divided by ($8^2 \times 500 =$) 32000 gives a quotient of 3, the required breadth of the beam.

125.—Depth of Beam with Safe Load.—A transposition of the factors in equation (21.), and marking it for extraction of the square root, gives—

$$d = \sqrt{\frac{W a l}{B b}}, \qquad (22.)$$

a rule for the depth of a beam. Therefore, to ascertain what should be the depth of a beam of given breadth and length to safely sustain a given weight at the middle, we have—

Rule XVIII.—Multiply the given weight by the factor of safety, and by the length in feet; divide the product by the product of the breadth into the value of B for the kind of wood, Table III.; then the square root of the quotient will be the required depth.

Example.—What should be the depth of a spruce beam 5 inches broad and 10 feet long between bearings to sustain safely, at middle, 4500 pounds? The value of B from the table is 550; taking a at 4, and proceeding by the rule, we have $4500 \times 4 \times 15 = 270000$; this divided by $(550 \times 5 =)$ 2750 gives a quotient of $98 \cdot 18$, the square root of which is $9 \cdot 909$, the required depth of the beam. The beam should be 5×10 inches.

126.—Safe Load at any Point.—When the load is at the middle of a beam it exerts the greatest possible strain; at any other point the strain would be less. The strain decreases gradually as it approaches one of the bearings, and when arrived at the bearing its effect upon the beam as a cross-strain is zero. The effect of a weight upon a beam is in proportion to its distance from one of the bearings, multiplied by the portion of the load borne by that bearing.

The load upon a beam is divided upon the two bearings, as shown at *Art.* 88. The weight which is required to rupture a beam is in proportion to the breadth and square of the depth, $b\,d^2$, as before shown, and also in proportion to the length divided by 4 times the rectangle of the two parts into which the load divides the length, or $\frac{l}{4\,m\,n}$ (see *Fig.* 35). This, when the load is at the middle, may be put as $\frac{l}{4 \times \frac{1}{2}\,l \times \frac{1}{2}\,l} = \frac{1}{l}$, a result coinciding with the relation before given in *Art.* 122, viz.: "The resistance is inversely in proportion to the length." The total resistance, therefore, putting the two statements together, is in proportion to $\frac{b\,d^2\,l}{4\,m\,n}$. There are, therefore, these two ratios, viz., $W : \frac{b\,d^2\,l}{4\,m\,n}$ and $B : 1$, from which we have the proportion—

$$B : 1 :: W : \frac{b\, d^2\, l}{4\, m\, n},$$

from which we have—

$$W = \frac{B\, b\, d^2\, l}{4\, m\, n}. \qquad (23.)$$

This is the relation at the point of rupture, and when $\frac{B}{a}$ is used instead of B, the expression gives the safe weight. Therefore—

$$W = \frac{B\, b\, d^2\, l}{4\, a\, m\, n} \qquad (24.)$$

is an expression for the safe weight. Now, to ascertain the weight which may be safely borne by a beam at any point in its length, we have—

Rule XIX.—Multiply the breadth by the square of the depth, by the length in feet, and by the value of B for the material of the beam, in Table III.; divide the product by the product of four times the factor of safety into the rectangle of the two parts into which the centre of gravity of the weight divides the beam, and the quotient will be the required weight in pounds.

Example.—What weight may be safely sustained at 3 feet from one end of a Georgia-pine beam which is 4×10 inches, and 20 feet long? The value of B for Georgia pine, in Table III., is 850; therefore, by the rule, $4 \times 10^2 \times 20 \times 850 = 6800000$. Taking the factor of safety at 4, we have $4 \times 4 \times 3 \times 17 = 816$. Using this as a divisor with which to divide the former product, we have as a quotient 8333 pounds, the required weight.

127.—Breadth or Depth: Load at any Point.—By a proper transposition of the factors of (24.) we obtain—

$$b\, d^2 = \frac{4\, W\, a\, m\, n}{B\, l}, \qquad (25.)$$

an expression showing the product of the breadth into the square of the depth; hence, to ascertain the breadth or

depth of a beam to sustain safely a given weight located at any point on the beam, we have—

Rule XX.—Multiply four times the given weight by the factor of safety, and by the rectangle of the two parts into which the load divides the length; divide the product by the product of the length into the value of B for the material of the beam, found in Table III., and the quotient will be equal to the product of the breadth into the square of the depth. Now, to obtain the breadth, divide this product by the square of the depth, and the quotient will be the required breadth. But if, instead of the breadth, the depth be desired, divide the said product by the breadth; then the square root of the quotient will be the required depth.

Example.—What should be the breadth (the depth being 8) of a white-pine beam 12 feet long to safely sustain 3500 pounds at 3 feet from one end? Also, what should be its depth when the breadth is 3 inches? By the rule, taking the factor of safety at 4, $4 \times 3500 \times 4 \times 3 \times 9 = 1512000$. The value of B for white pine, in Table III., is 500; therefore, $500 \times 12 = 6000$; with this as divisor, dividing 1512000, the quotient is 252. Now, to obtain the breadth when the depth is 8, 252 divided by $(8 \times 8 =)$ 64 gives a quotient of 3·9375, the required breadth; or the beam may be, say, 4×8. Again, when the breadth is 3 inches, we have for the quotient of 252 divided by $3 = 84$, and the square root of 84 is 9·165, or $9\frac{1}{8}$ inches. For this case, therefore, the beam should be, say, $3 \times 9\frac{1}{8}$ inches.

128.—Weight Uniformly Distributed.—When the load is spread out uniformly over the length of a beam, the beam will require just twice the weight to break it that would be required if the weight were concentrated at the centre. Therefore, we have $W = \frac{U}{2}$, where U represents the distributed load. Substituting this value of W in equation (20.), we have—

$$\frac{U}{2} = \frac{B b d^2}{a l},$$

or—

$$U = \frac{2 B b d^2}{a l}. \qquad (26.)$$

Therefore, to ascertain the weight which may be safely sustained, when uniformly distributed over the length of a beam, we have—

Rule XXI.—Multiply twice the breadth by the square of the depth, and by the value of B for the material of the beam, in Table III., and divide the product by the product of the length in feet by the factor of safety, and the quotient will be the required weight in pounds.

Example.—What weight uniformly distributed may be safely sustained upon a hemlock beam 4×9 inches, and 20 feet long? The value of B for hemlock, in Table III., is 450; therefore, by the rule, $2 \times 4 \times 9^2 \times 450 = 291600$. Taking the factor of safety at 4, we have $4 \times 20 = 80$, the product by which the former product is to be divided. This division produces a quotient of 3645, the required weight.

129.—Breadth or Depth: Load Uniformly Distributed.—By a proper transposition of factors in (26.), we obtain—

$$b\,d^2 = \frac{U\,a\,l}{2\,B}, \qquad (27.)$$

an expression giving the value of the breadth into the square of the depth. From this, therefore, to ascertain the breadth or the depth of a beam to sustain safely a given weight uniformly distributed over the length of a beam, we have—

Rule XXII.—Multiply the given weight by the factor of safety, and by the length; divide the product by the product of twice the value of B for the material of the beam, in Table III., and the quotient will be equal to the breadth into the square of the depth. Now, to find the breadth, divide the said quotient by the square of the depth; but if, instead of the breadth, the depth be required, then divide said quotient by the breadth, and the square root of this quotient will be the required depth.

Example.—What should be the size of a white-pine beam 20 feet long to sustain safely 10,000 pounds uniformly distributed over its length? The value of B for white pine, in Table III., is 500. Let the factor of safety be taken at 4. Then, by the rule, $10000 \times 4 \times 20 = 800000$; this divided by $(2 \times 500 =)$

1000 gives a quotient of 800. Now, if the depth be fixed at 12, then the said quotient, 800, divided by ($12 \times 12 =$) 144 gives $5\frac{5}{9}$, the required breadth of beam; and the beam may be, say, $5\frac{3}{4} \times 12$. Again, if the breadth is fixed, say, at 6, and the depth is required, then the said quotient, 800, divided by 6 gives $133\frac{1}{3}$, the square root of which, 11·55, is the required depth. The beam in this case should therefore be, say, $6 \times 11\frac{3}{4}$ inches.

130.—Load per Foot Superficial.—When several beams are laid in a tier, placed at equal distances apart, as in a tier of floor-beams, it is desirable to know what should be their size in order to sustain a load equally distributed over the floor.

If the distance apart at which they are placed, measured from the centres of the beams, be multiplied by the length of the beams between bearings, the product will equal the area of the floor sustained by one beam; and if this area be multiplied by the weight upon a superficial foot of the floor, the product will equal the total load uniformly distributed over the length of the beam; or, if c be put to represent the distance apart between the centres of the beams in feet, and l represent the length in feet of the beam between bearings, and f equal the pounds per superficial foot on the floor, then the product of these, or $c f l$, will represent the uniformly distributed load on a beam; but this load was before represented by U (*Art.* 128); therefore, we have $c f l = U$, and they may be substituted for it in (26.) and (27.). Thus we have—

$$b d^2 = \frac{c f l a l}{2 B},$$

or—

$$b d^2 = \frac{a c f l^2}{2 B}. \qquad (28.)$$

Therefore, to ascertain the size of floor-beams to sustain safely a given load per superficial foot, we have—

Rule XXIII.—Multiply the given weight per superficial foot by the factor of safety, by the distance between the

centres of the beams in feet, and by the square of the length in feet; divide the product by twice the value of B for the material of the beams, in Table III., and the quotient will be equal to the breadth into the square of the depth. Now, to obtain the breadth, divide said quotient by the square of the depth, and this quotient will be the required breadth. But if, instead of the breadth, the depth be required, divide the aforesaid quotient by the breadth; then the square root of this quotient will be the required depth.

Example.—What should be the size of white-pine floor-beams 20 feet long, placed 16 inches from centres, to sustain safely 90 pounds per superficial foot, including the weight of the materials of construction—the beams, flooring, plastering, etc.? The value of B for white pine is 500; the factor of safety may be put at 5. Then, by the rule, we have $90 \times 5 \times \frac{16}{12} \times 20^2 = 240000$. This divided by $(2 \times 500 =)$ 1000 gives 240. Now, for the breadth, if the depth be fixed at 9 inches, then 240 divided by $(9^2 =)$ 81 gives a quotient of 2·963. The beams therefore should be, say, 3×9. But if the breadth be fixed, say, at 2·5 inches, then 240 divided by 2·5 gives a quotient of 96, the square root of which is 9·8 nearly. The beams in this case would require therefore to be, say, $2\frac{1}{2} \times 10$ inches.

N. B.—It is well to observe that the question decided by Rule XXII. is simply that of *strength* only. Floor-beams computed by it will be quite *safe* against rupture, but they will in most cases deflect much more than would be consistent with their good appearance. Floor-beams should be computed by the rules which include the effect of deflection. (See *Art.* 152.)

131.—Levers: Load at One End.—The beams so far considered as being exposed to transverse strains have been supposed to be supported at each end. When a piece is held firmly at one end only, and loaded at the other, it is termed a lever; and the load which a piece so held and loaded will sustain is equal to one fourth that which the same piece would sustain if it were supported at each end and loaded at the middle. Or, the strain in a beam sup-

ported at each end caused by a given weight located at the middle is equal to that in a lever of the same breadth and depth, when the length of the latter is equal to one half that of the beam, and the load at its end is equal to one half of that at the middle of the beam. Or, when P represents the load at the end of the lever, and n its length, then $W = 2\,P$, and $l = 2\,n$. Substituting these values of W and l in equation (20.), we have—

$$2\,P = \frac{B\,b\,d^2}{2\,a\,n},$$

from which—

$$P = \frac{B\,b\,d^2}{4\,a\,n}, \qquad (29.)$$

Hence, to ascertain the weight which may be safely sustained at the end of a lever, we have—

Rule XXIV.—Multiply the breadth of the lever by the square of its depth, and by the value of B for the material of the lever, in Table III.; divide the product by the product of four times the length in feet into the factor of safety, and the quotient will be the required weight in pounds.

Example.—What weight can be safely sustained at the end of a maple lever of which the breadth is 2 inches, the depth is 4 inches, and the length is 6 feet? The value of B for maple, in Table III., is 1100; therefore, by the rule, $2 \times 4^2 \times 1100 = 35200$. And, taking the factor of safety at 5, $4 \times 5 \times 6 = 120$, and 35200 divided by 120 gives a quotient of 293·33, or $293\frac{1}{3}$ pounds.

N. B.—When a lever is loaded with a weight uniformly distributed over its length, it will sustain just twice the load which can be sustained at the end.

132.—Levers: Breadth or Depth.—By a proper transposition of the factors in (29.), we obtain—

$$b\,d^2 = \frac{4\,P\,a\,n}{B}. \qquad (30.)$$

Hence, to ascertain the breadth or depth of a lever to sustain safely a given weight, we have—

Rule XXV.—Multiply four times the given weight by the length of the lever, and by the factor of safety; divide the product by the value of B for the material of the lever, in Table III., and the quotient will be equal to the breadth multiplied by the square of the depth. Now, if the breadth be required, divide said quotient by the square of the depth, and this quotient will be the required breadth; but if, instead of the breadth, the depth be required, divide the said quotient by the breadth; then the square root of this quotient will be the required depth.

Example.—What should be the size of a cherry lever 5 feet long to sustain safely 250 pounds at its end? Proceeding by the rule, taking the factor of safety at 5, we have $4 \times 250 \times 5 \times 5 = 25000$. The value of B for cherry, in Table III., is 650; and 25000 divided by 650 gives a quotient of 38·46. Now, if the depth be fixed at 4, then 38·46 divided by $(4 \times 4 =)$ 16 gives a quotient of 2·4, the required breadth. But if the breadth be fixed at 2, then 38·46 divided by 2 gives a quotient of 19·23, the square root of which is 4·38, the required depth. Therefore, the lever may be $2{\cdot}4 \times 4$, or $2 \times 4\frac{3}{8}$ inches.

133.—Deflection: Relation to Weight.—When a load is placed upon a beam supported at each end, the beam bends more or less; the distance that the beam descends under the operation of the load, measured at the middle of its length, is termed its *deflection*. In an investigation of the laws of deflection it has been demonstrated, and experiments have confirmed it, that while the elasticity of the material remains uninjured by the pressure, or is injured in but a small degree, the amount of deflection is directly in proportion to the weight producing it; for example, if 1000 pounds laid upon a beam is found to cause it to deflect or descend at the middle a quarter of an inch, then 2000 pounds will cause it to deflect half an inch, 3000 pounds will deflect it three fourths of an inch, and so on.

134.—Deflection: Relation to Dimensions.—In Table III. are recorded the results of experiments made to test the

resistance of the materials named to deflection. The figures in the third column designated by the letter F (for flexure) show the number of pounds required to deflect a unit of material one inch. This is an extreme state of the case, for in most kinds of material this amount of depression would exceed the limits of elasticity; and hence the rule would here fail to give the correct relation as between the dimensions and pressure. For the law of deflection as above stated (the deflections being in proportion to the weights) is true only while the depressions are small in comparison with the length. Nothing useful is, therefore, derived from this position of the question, except to give an idea of the nature of the quantity represented by the constant F; it being in reality an index of the stiffness of the kind of material used in comparing one material with another. Whatever be the dimensions of the beam, F will always be the same quantity for the same material; but among various materials F will vary according to the flexibility or stiffness of each particular material. For example, F will be much greater for iron than for wood; and again, among the various kinds of wood, it will be larger for the stiff woods than for those that are flexible. The value of F, therefore, is the weight which would deflect the unit of material one inch, upon the supposition that the deflections, from zero to the depth of one inch, continue regularly in proportion to the increments of weight producing the deflections, or, for each deflection—

$$F : 1 :: W : \delta,$$

from which we have—

$$W = F\delta; \text{ or, } F = \frac{W}{\delta},$$

in which δ represents the deflection in inches corresponding to W, the weight producing it. This is for the unit of material. For beams of larger dimensions, investigations have shown (*Transverse Strains*, Chapters XIII. and XIV.) that the power of a beam to resist deflection by a weight at middle is in proportion to its breadth and the cube of its depth, and it is inversely in proportion to the cube of the length;

or, when the resistance of the unit of material is measured, as above, by $\frac{W}{\delta}$, we have the relation between it and a larger beam of—

$$\frac{W}{\delta} : \frac{b d^3}{l^3}.$$

Putting this ratio in a proportion with that of the unit of material, we have—

$$F : 1 :: \frac{W}{\delta} : \frac{b d^3}{l^3},$$

which gives—

$$\frac{W}{\delta} = \frac{F b d^3}{l^3},$$

from which we have—

$$W = \frac{F b d^3 \delta}{l^3}. \qquad (31.)$$

135.—Deflection: Weight when at Middle.—In equation (31.) we have a rule by which to ascertain what weight is required to deflect a given beam to a given depth of deflection; this, in words at length, is—

Rule XXVI.—Multiply the breadth of the beam by the cube of its depth, and by the given deflection, all in inches, and by the value of F for the material of the beam, in Table III.; divide the product by the cube of the length in feet, and the quotient will be the required weight in pounds.

Example.—What weight is required at the middle of a 4×12 inch Georgia-pine beam 20 feet long to deflect it three quarters of an inch? The value of F for Georgia pine, in Table III., is 5900; therefore, by the rule, we have $4 \times 12^3 \times 0{\cdot}75 \times 5900 = 30585600$, which divided by $(20 \times 20 \times 20 =)$ 8000 gives a quotient of $3823{\cdot}2$, the required weight in pounds.

136.—Deflection: Breadth or Depth, Weight at Middle.—By a transposition of equation (31.), we obtain—

$$b d^3 = \frac{W l^3}{F \delta}, \qquad (32.)$$

ule by which may be found the breadth or depth of a ım, with a given load at middle and with a given deflec- n; this, in words at length, is—

Rule XXVII.—Multiply the given load by the cube of : length in feet, and divide the product by the product of : deflection into the value of F for the material of the ım, in Table III.; then the quotient will be equal to the ·adth of the beam multiplied by the cube of its depth, th in inches.

Now, to obtain the breadth, divide the said quotient by : cube of the depth, and this quotient will be the required ·adth. But if, instead of the breadth, the depth be re- ired, then divide the said quotient by the breadth, and : cube root of this quotient will be the required depth. t if neither breadth nor depth be previously fixed, but it required that they bear a certain proportion to each ıer; such that $d : b :: 1 : r$, r being a decimal, then $b = r d$, l $b d^3 = r d^4$; then, to find the depth, divide the aforesaid ıtient by the decimal r, and the fourth root (or the square ıt of the square root) will be the required depth, and this ltiplied by the decimal r will give the breadth.

Example.—What should be the size of a spruce beam 20 : long between bearings, sustaining 2000 pounds at the ldle, with a deflection of one inch? By the rule, the ght into the cube of the length is $2000 \times 8000 = 16000000$. : value of F for spruce, in Table III., is 3500; this by the ection $= 1$ gives 3500, which used as a divisor in divid- the above 16000000 gives a quotient of 4571·43. Now, ıe breadth be required, the depth being fixed, say, at 10, ı 4571·43 divided by $(10 \times 10 \times 10 =)$ 1000 gives 4·57, the ıired breadth. The beam should be, say, $4\frac{5}{8}$ by 10 inches. if the depth be required, the breadth being fixed, say, at ıen 4571·43 divided by 4 gives 1142·86, the cube root vhich is 10·46; so in this case, therefore, the beam is ıired to be $4 \times 10\frac{1}{2}$ inches. Again, if the breadth is to r a certain proportion to the depth, or that the ratio be- en them is to be, say, 0·6 to 1, then let $r = 0·6$, and then $·43 = 0·6 d^4$, and dividing by 0·6, we have 7619·05 ^{4}. This equals $d^2 \times d^2$; therefore the square root of 7619

is 87·29, and the square root of this is 9·343, the required depth in inches. Now 9·343 × 0·6 equals the breadth, or 9·343 × 0·6 = 5·6; therefore the beam is required to be 5·6 × 9·34 inches, or, say, $5\frac{5}{8} \times 9\frac{1}{8}$ inches.

137.—Deflection: when Weight is at Middle.—By a transposition of the factors in (32.), we obtain—

$$\delta = \frac{W l^3}{F b d^3}, \qquad (33.)$$

a rule by which the deflection of any given beam may be ascertained, and which, in words at length, is—

Rule XXVIII.—Multiply the given weight by the cube of the length in feet; divide the product by the product of the breadth into the cube of the depth in inches, multiplied by the value of F for the material of the beam, in Table III., and the quotient will be the required deflection in inches.

Example.—To what depth will 1000 pounds deflect a 3 × 10 inch white-pine beam 20 feet long, the weight being at the middle of the beam? By the rule, we have $1000 \times 20^3 = 8000000$; then, since the value of F for white pine, in Table III., is 2900, we have $3 \times 10^3 \times 2900 = 8700000$; using this product as a divisor and by it dividing the former product, we obtain a quotient of 0.9195, the required deflection in inches.

138.—Deflection: Load Uniformly Distributed.—In two beams of equal capacity, suppose the one loaded at the middle, and the other with its load uniformly distributed over its length, and so loaded that the deflection in one beam shall equal that in the other; then the weight at the middle of the former beam will be equal to five eighths of that on the latter. This proportion between the two has been demonstrated by writers on the strength of materials. (See p. 484, *Mechanics of Eng. and Arch.*, by Prof. Mosely, Am. ed. by Prof. Mahan, 1856.) Hence, when U is put to represent the uniformly distributed load, we have—

$$W = \tfrac{5}{8} U;$$

or, when an equally distributed load deflects a beam to a certain depth, five eighths of that load, if concentrated at the middle, would cause an equal deflection. This value of W may therefore be substituted for it in equation (31.), and give—

$$\tfrac{5}{8}\,U = \frac{F b d^3 \delta}{l^3},$$

from which we obtain—

$$U = \frac{1\cdot6\, F b d^3 \delta}{l^3}, \tag{34.}$$

a rule for a uniformly distributed load.

139.—Deflection: Weight when Uniformly Distributed.—In equation (34.) we have a rule by which we may ascertain what weight is required to deflect to a given depth any given beam. This, in words at length, is—

Rule XXIX.—Multiply 1·6 times the deflection by the breadth of the beam, and by the cube of its depth, all in inches, and by the value of F for the material of the beam, in Table III.; divide the product by the cube of the length in feet, and the quotient will be the required weight in pounds.

Example.—What weight, uniformly distributed over the length of a spruce beam, will be required to deflect it to the depth of 0·5 of an inch, the beam being 3×10 inches and 10 feet long? The value of F for spruce, in Table III., is 3500. Therefore, by the rule, we have $1\cdot6 \times 0\cdot5 \times 3 \times 10^3 \times 3500 =$ 8400000, and this divided by $(10 \times 10 \times 10 =)$ 1000 gives 8400, the required weight in pounds.

140.—Deflection: Breadth or Depth, Load Uniformly Distributed.—By transposition of the factors in equation (34.), we obtain—

$$b d^3 = \frac{U l^3}{1\cdot6\, F \delta}, \tag{35.}$$

a rule for the dimensions, which, in words at length, is—

Rule XXX.—Multiply the given weight by the cube of the length of the beam; divide the product by 1·6 times the given deflection in inches, multiplied by the value of F for the material of the beam, in Table III., and the quotient will equal the breadth into the cube of the depth. Now, to obtain the breadth, divide this quotient by the cube of the depth, and the resulting quotient will be the required breadth in inches. But if, instead of the breadth, the depth be required, then divide the aforesaid quotient by the breadth, and the cube root of the resulting quotient will be the required depth in inches. Again, if neither breadth nor depth be previously determined, but to be in proportion to each other at a given ratio, as r to 1, r being a decimal fixed at pleasure, then divide the aforesaid quotient by the value of r, and take the square root of the quotient; then the square root of this square root will be the required depth in inches. The breadth will equal the depth multiplied by the value of the decimal r.

Example.—What should be the size of a locust beam 10 feet long which is to be loaded with 6000 pounds equally distributed over the length, and with which the beam is to be deflected $\frac{3}{4}$ of an inch? The value of F for locust, in Table III., is 5050. By the rule, we have $6000 \times (10 \times 10 \times 10 =) 1000 = 6000000$, which is to be divided by $(1{\cdot}6 \times 0{\cdot}75 \times 5050 =) 6060$, giving a quotient of 990·1. Now, if the depth be, say, 6 inches, then 990·1 divided by $(6 \times 6 \times 6 =) 216$ gives a quotient of 4·584, the required breadth in inches, say $4\frac{5}{8}$. But if the breadth be assumed at 4 inches, then 990·1 divided by 4 gives a quotient of 247·525, the cube root of which is 6·279, the required depth in inches, or, say, $6\frac{1}{4}$. And, again, if the ratio between the breadth and depth be as 0·7 to 1, then 990·1 divided by 0·7 gives a quotient of 1414·43, the square root of which is 37·609, of which the square root is 6·1326, the required depth in inches, or, say, $6\frac{1}{8}$; and then $6{\cdot}1326 \times 0{\cdot}7 = 4{\cdot}293$, the required breadth in inches; or, the beam should be $4\frac{3}{16} \times 6\frac{1}{8}$ inches.

141.—Deflection: when Weight is Uniformly Distributed. —By a transposition of the factors of equation (35.), we obtain—

$$\delta = \frac{U l^3}{1 \cdot 6 F b d^3}, \qquad (36.)$$

a result nearly the same as that in equation (33.), which is a rule for deflection by a weight at middle, and which by slight modifications may be used for deflection by an equally distributed load. Thus by—

Rule XXXI.—Proceed as directed in Rule XXVIII. (*Art.* 137), using the equally distributed weight instead of a concentrated weight, and then divide the result there obtained for deflection by 1·6; then the quotient will be the required deflection in inches.

Example.—Taking the example given under *Rule* XXVIII., in *Art.* 137, and assuming that the 1000 pounds load with which the beam is loaded be equally distributed, then 0·9195, the result for deflection as there found, divided by 1·6, as by the above rule, gives 0·5747, the required deflection. This result is just five eighths of 0·9195, the deflection by the load at middle.

N.B.—The deflection by a uniformly distributed load is just five eighths of that produced by the same load when concentrated at the middle of the beam; therefore, five eighths of the deflection obtained by Rule XXVIII. will be the deflection of the same beam when the same weight is uniformly distributed.

142.—Deflection of Levers.—The deflection of a lever is the same as that of a beam of the same breadth and depth, but of twice the length, and loaded at the middle with a load equal to twice that which is at the end of the lever. Therefore, if P represents the weight at the end of a lever, and n the length of the lever in feet, then $2\,P = W$ and $2\,n = l$, and if these values of W and l be substituted for those in equation (33.), we obtain—

$$\delta = \frac{2\,P \times \overline{2\,n}^3}{F b d^3},$$

which reduces to—

$$\delta = \frac{16\,P n^3}{F b d^3}, \qquad (37.)$$

a result 16 times that in equation (33.), which is the deflection in a beam. Therefore, when a beam and a lever equal in sectional area and in length be loaded by equal weights, the one at the middle, the other at one end, the deflection of the lever will be 16 times that of the beam. This proportion is based upon the condition that neither the beam nor the lever shall be deflected beyond the limits of elasticity.

143.—Deflection of a Lever: Load at End.—Equation (37.), in words at length, is—

Rule XXXII.—Multiply 16 times the given weight by the cube of the length in feet; divide the product by the product of the breadth into the cube of the depth multiplied by the value of F for the material of the lever, in Table III., and the quotient will be the required deflection.

Example.—What would be the deflection of a bar of American wrought iron one inch broad, two inches deep, loaded with 150 pounds at a point 5 feet distant from the wall in which the bar is imbedded? The value of F for American wrought iron, in Table III., is 62000. Therefore, by the rule, $16 \times 150 \times 5^3 = 300000$. This divided by $(1 \times 2^3 \times 62000 =)$ 496000 gives 0·6048, the required deflection—nearly $\frac{5}{8}$ of an inch.

144.—Deflection of a Lever: Weight when at End.—By a transposition of the factors in equation (37.), we obtain—

$$P = \frac{F b d^3 \delta}{16 n^3}. \qquad (38.)$$

This result is equal to one sixteenth of that shown in equation (31.), a rule for the weight at the middle. Therefore, for—

Rule XXXIII.—Proceed as directed in Rule XXVII.; divide the quotient there obtained by 16, and the resulting quotient will be the required weight in pounds.

Example.—What weight is required at the end of a 4×12 inch Georgia-pine lever 20 feet long to deflect it three quarters of an inch? Proceeding by Rule XXVII., we obtain a quotient of 3823·2; this divided by 16 gives 238·95, say 239, the required weight in pounds.

145.—Deflection of a Lever: Breadth or Depth, Load at End.—A transposition of the factors of equation (38.) gives—

$$b\,d^3 = \frac{16\,P\,n^3}{F\,\delta}, \qquad (39.)$$

a rule by which to obtain the sectional area of the lever. By comparison with equation (32.) it is seen that the result in (39.) is 16 times that found by (32.). Therefore, the dimensions for a lever loaded at the end may be found by—

Rule XXXIV.—Multiply by 16 the first quotient found by Rule XXVII., and then proceed as farther directed in Rule XXVII., using the product of 16 times the quotient, instead of the said quotient.

Example.—What should be the size of a spruce lever 20 feet long, between weight and wall, to sustain 2000 pounds at the end with a deflection of 1 inch? Proceeding by Rule XXVII., we obtain a first quotient of 4571·43. By Rule XXXIV., 4571·43 × 16 = 73144·88. Now, if the depth be fixed, say, at 20, then 73144·88 divided by (20 × 20 × 20 =) 8000 gives 9·143, the required breadth. But to obtain the depth, fixing the breadth, say, at 9, we have for 73144·88 divided by 9 = 8127·21, the cube root of which is 20·1055, the required depth. Again, if the breadth and depth are to be in proportion, say, as 0·7 to 1·0, then 73144·88 divided by 0·7 gives 104492·7, the square root of which is 323·254, of which the square root is 17·98, the required depth in inches; and 17·98 × 0·7 = 12·586, the required breadth in inches. The lever, therefore, should be, say, $12\frac{5}{8}$ × 18 inches.

146.—Deflection of Levers: Weight Uniformly Distributed.—A comparison of the effects of loads upon levers shows (*Transverse Strains*, Art. 347) that the deflection by a uniformly distributed load is equal to that which would be produced by three eighths of that load if suspended from the end of the lever. Or, $P = \frac{3}{8}\,U$. Substituting this value of P, in equation (37.), gives—

$$\delta = \frac{16 \times \frac{3}{8}\,U\,n^3}{F\,b\,d^3},$$

which reduces to—

$$\delta = \frac{6\,U\,n^3}{F\,b\,d^3}, \qquad (40.)$$

a rule for the deflection of levers loaded with an equally distributed load.

147.—Deflection of Levers with Uniformly Distributed Load.—The deflection shown in equation (40.) is just six times that shown in equation (33.). The result by (33.) multiplied by 6 will equal the result by (40.); therefore, we have—

Rule XXXV.—Proceed as directed in Rule XXVIII.; the result thereby obtained multiplied by 6 will give the required deflection.

Example.—To what depth will 500 pounds deflect a 3×10 inch white-pine lever 10 feet long, the weight uniformly distributed over the lever? Here, by Rule XXVIII., we obtain the result 0·05747; this multiplied by 6 gives 0·3448, the required deflection.

148.—Deflection of Levers: Weight when Uniformly Distributed.— By a transposition of factors in (40.), we obtain—

$$U = \frac{F\,b\,d^3\,\delta}{6\,n^3}. \qquad (41.)$$

This is equal to one sixth that of equation (31.); therefore, we have—

Rule XXXVI.—Proceed as directed in Rule XXVI.; the quotient thereby obtained divide by 6, and the quotient thus obtained will be the required weight.

Example.—What weight will be required to deflect a 4×5 inch spruce lever 1 inch, the weight uniformly distributed over its length? Proceeding as directed in Rule XXVI., the result thereby obtained is 1750; this divided by 6 gives $291\frac{2}{3}$, the required weight in pounds.

149.—Deflection of Levers: Breadth or Depth, Load Uniformly Distributed.—A transposition of factors in equation (41.) gives—

$$b\,d^3 = \frac{6\,U\,n^3}{F\,\delta}. \qquad (42.)$$

This result is just six times that of equation (32.); we, therefore, have—

Rule XXXVII.—Proceed as directed in Rule XXVII.; multiply the first quotient thereby obtained by 6; then in the subsequent directions use this multiplied quotient instead of the said first quotient, to obtain the required breadth and depth.

Example.—What should be the size of a spruce lever 10 feet long, sustaining $2666\frac{2}{3}$ pounds, uniformly distributed over its length, with a deflection of 1 inch? Proceeding by Rule XXVII., the first quotient obtained is 761·905; this multiplied by 6 gives 4571·43, the multiplied quotient which is to be used in place of the said first quotient. Now, to obtain the breadth, the depth being fixed, say, at 10; 4571·43 divided by (cube of 10 =) 1000, the quotient, 4·57, is the required breadth. But if the breadth be fixed, say, at 4, then, to obtain the depth, 4571·43 divided by 4 gives 1142·86, the cube root of which is 10·46, the required depth. Again, if the breadth and depth are to be in proportion, say, as 0·6 to 1·0, then 4571·43 divided by 0·6 gives 7619·05, the square root of which is 87·27, of which the square root is 9·343, the required depth in inches; and $9·343 \times 0·6$ equals 5·6, the required breadth in inches; or, the lever may be, say, $5\frac{5}{8} \times 9\frac{1}{3}$ inches.

CONSTRUCTION IN GENERAL.

150.—Construction: Object Clearly Defined.—In the various parts of timber construction, known as floors, partitions, roofs, bridges, etc., each has a specific object, and in all designs for such constructions this object should be kept clearly in view, the various parts being so disposed as to serve the design with the least quantity of material. The simplest form is the best, not only because it is the most economical, but for many other reasons. The great number of joints, in a complex design, render the construction liable to derangement by multiplied compressions, shrinkage, and, in consequence, highly increased oblique strains; by which its stability and durability are greatly lessened.

FLOORS.

151.—Floors Described.—Floors are most generally constructed *single;* that is, simply a series of parallel beams, each

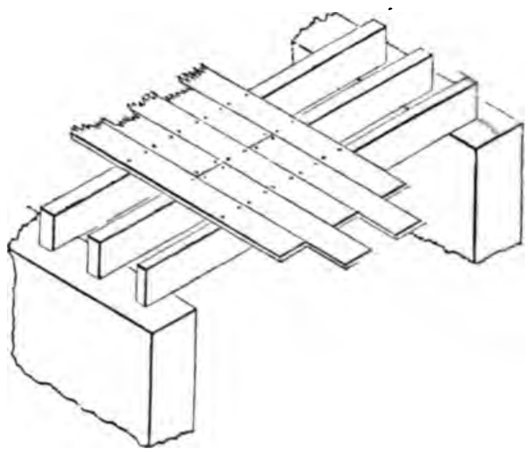


FIG. 39.

spanning the width of the building, as seen at *Fig.* 39. Oc-

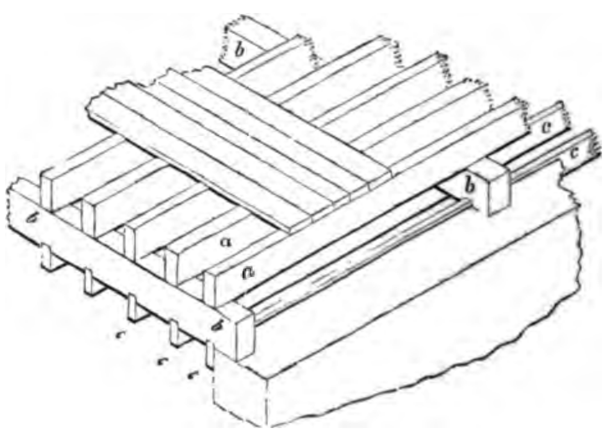


FIG. 40.

casionally floors are constructed *double*, as at *Fig.* 40; and sometimes *framed*, as at *Fig.* 41; but these methods are

seldom practised, inasmuch as either of these requires more timber than the single floor. Where lathing and plastering is attached to the floor-beams to form a ceiling below, the springing of the beams, by customary use, is liable to crack the plastering. To obviate this in good dwellings, the double and framed floors have been resorted to, but more in former times than now, as the *cross-furring* (a series of narrow strips of board or plank nailed transversely to the underside of

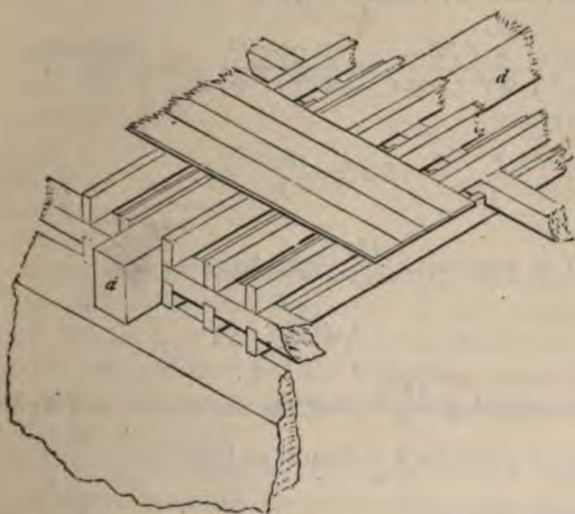


FIG. 41.

the beams to receive the lathing for the plastering) serves a like purpose very nearly as well.

152.—Floor-Beams.—The size of floor-beams can be ascertained by the preceding rules for the *stiffness* of materials. These rules give the required dimensions for the various kinds of material in common use. The rules may be somewhat abridged for ordinary use, if some of the quantities represented in the formula be made constant within certain limits. For example, if the load per foot superficial upon the floor be fixed, and the deflection, then these, together with the constant represented by F, may be reduced to one

constant. For dwellings, the load per foot may be taken at 70 pounds, the weight proper to be allowed for a crowd of people on their feet. (*Transverse Strains*, Art. 114.) To this add 20 for the weight of the material of which the floor is composed, and the sum, 90, is the value of f, or the weight per foot superficial for dwellings. Then $c f l = U$ (*Art.* 130). The rate of deflection allowable for this load may be fixed at 0·03 inch per foot of the length, or $\delta = 0\cdot03\ l$. Substituting these values in equation (35.), we obtain—

$$b\ d^3 = \frac{c\ f\ l\ l^3}{1\cdot6\ F \times \cdot03\ l} = \frac{90\ c\ l^3}{1\cdot6 \times \cdot03\ F} = \frac{1875\ c\ l^3}{F},$$

or—

$$b\ d^3 = \frac{1875}{F}\ c\ l^3. \qquad (43.)$$

Putting j to represent $\frac{1875}{F}$, we have—

$$b\ d^3 = j\ c\ l^3. \qquad (44.)$$

Now, by reducing $\frac{1875}{F}$, for the six woods in common use, the value of j for each is found as follows:

Georgia Pine	$j = 0\cdot32$
Locust	$j = 0\cdot37$
White Oak............................	$j = 0\cdot6$
Spruce	$j = 0\cdot54$
White Pine...........................	$j = 0\cdot65$
Hemlock..............................	$j = 0\cdot67$

Equation (44.) is a rule for the floor-beams of dwellings; it may be used also to obtain the dimensions of beams for stores for all ordinary business· for it will require from 3 to 5 times the weight used in this rule, or from 200 to 400 (average 300) pounds to increase the deflection to the limit of elasticity in beams of the usual depths and lengths. For light stores, therefore, loaded, say, to 150 pounds per foot, the beams would be safe, but the deflection would be in-

creased to 0·06 per foot. When so great a deflection as this would not be objectionable to the eye, then this rule (44.) will serve for the beams of light stores. But for first-class stores, taking the rate of deflection at ·04 per foot, and fixing the weight per superficial foot at 275 pounds, including the weight of the material of which the floor is constructed, and letting k represent the constant, then—

$$b\,d^3 = k\,c\,l^3, \qquad (45.)$$

and for—

Georgia Pine	$k = 0·73$
Locust	$k = 0·85$
White Oak	$k = 1·38$
Spruce	$k = 1·48$
White Pine	$k = 1·23$
Hemlock	$k = 1·53$

153.—Floor-Beams for Dwellings.—To find the dimensions of floor-beams for *dwellings*, when the rate of deflection is 0·03 inch per foot, or for *ordinary stores* when the load is about 150 pounds per foot, and the deflection caused by this weight is within the limits of the elasticity of the material, we have the following rule:

Rule XXXVIII.—Multiply the cube of the length by the distance apart between the beams (from centres), both in feet, and multiply the product by the value of j (*Art.* 152) for the material of the beam, and the product will equal the product of the breadth into the cube of the depth. Now, to find the breadth, divide this product by the cube of the depth in inches, and the quotient will be the breadth in inches. But if the depth is sought, divide the said product by the breadth in inches, and the cube root of the quotient will be the depth in inches; or if the breadth and depth are to be in proportion as r is to unity, r representing any required decimal, then divide the aforesaid product by the value of r, and extract the square root of the quotient, and the square root of this square root will be the depth required in inches, and the depth multiplied by the value of r will be the breadth in inches.

Example.—In a dwelling or ordinary store, what must be the breadth of the beams, when placed 15 inches from centres, to support a floor covering a span of 16 feet, the depth being 11 inches, the beams of white oak? By the rule, 4096, the cube of the length, by $1\frac{1}{4}$, the distance from centres, and by 0·6, the value of j for white oak, equals 3072. This divided by 1331, the cube of the depth, equals 2·31 inches, or $2\frac{5}{16}$ inches, the required breadth. But if, instead of the breadth, the depth be required, the breadth being fixed at 3 inches, then the product, 3072, as above, divided by 3, the breadth, equals 1024; the cube root of this is 10·08, or, say, 10 inches nearly. But if the breadth and depth are to be in proportion, say, as 0·3 to 1·0, then the aforesaid product, 3072, divided by 0·3, the value of r, equals 10240, the square root of which is 101·2, and the square root of this is 10·06, the required depth. This multiplied by 0·3, the value of r, equals 3·02, the required breadth; the beam is therefore to be, say, 3 × 10 inches.

154.—Floor-Beams for First-Class Stores.—To find the breadth and depth of the beams for a floor of a first-class store sufficient to sustain 250 pounds per foot superficial (exclusive of the weight of the material in the floor), with a deflection of 0·04 inch per foot of the length, we have—

Rule XXXIX.—The same as XXXVIII., with the exception that the value of k (*Art.* 152) is to be used instead of the value of j.

Example.—The beams of the floor of a first-class store are to be of Georgia pine, with a clear bearing between the walls of 18 feet, and placed 14 inches from centres: what must be the breadth when the depth is 11 inches? By the rule, 5832, the cube of the length, and $1\frac{1}{6}$, the distance from centres, and 0·73, the value of k for Georgia pine, all multiplied together equal 4966·92; and this product divided by 1331, the cube of the depth, equals 3·732, the required breadth, or $3\frac{3}{4}$ inches.

But if, instead of the breadth, the depth be required: what must be the depth when the breadth is 3 inches?

The said product, 4966·92, divided by 3, the breadth, equals 1655·64, and the cube root of this, 11·83, or, say, 12 inches, is the depth required.

But if the breadth and depth are to be in a given proportion, say 0·35 to 1·0, the 4966·92 aforesaid divided by 0·35, the value of r, equals 14191, the square root of which is 119·13, and the square root of this square root is 10·91, or, say, 11 inches, the required depth. And 10·91 multiplied by 0·35, the value of r, equals 3·82, the required breadth, say $3\frac{7}{8}$ inches.

155.—Floor-Beams: Distance from Centres.—It is sometimes desirable, when the breadth and depth of the beams are fixed, or when the beams have been sawed and are now ready for use, to know the distance from centres at which such beams should be placed in order that the floor be sufficiently stiff. By a transposition of the factors in equation (44.), we obtain—

$$c = \frac{b d^3}{j l^3}. \qquad (46.)$$

In like manner, equation (45.) produces—

$$c = \frac{b d^3}{k l^3}. \qquad (47.)$$

These, in words at length, are as follows:

Rule XL.—Multiply the cube of the depth by the breadth, both in inches, and divide the product by the cube of the length in feet multiplied by the value of j, for dwellings and for ordinary stores, or by k, for first-class stores, and the quotient will be the distance apart from centres in feet.

Example.—A span of 17 feet, in a dwelling, is to be covered by white-pine beams 3×12 inches: at what distance apart from centres should they be placed? By the rule, 1728, the cube of the depth, multiplied by 3, the breadth, equals 5184. The cube of 17 is 4913; this by 0·65, the value of j for white pine, equals 3193·45. The aforesaid 5184 divided by this 3193·45 equals 1·6233 feet, or, say, 20 inches.

156.—Framed Openings for Chimneys and Stairs.—Where chimneys, flues, stairs, etc., occur to interrupt the bearing, the beams are framed into a piece, *b* (*Fig.* 42), called a *header*. The beams, *a a*, into which the header is framed are called *trimmers* or *carriage-beams*. These framed beams require to be made thicker than the common beams. The header must be strong enough to sustain one half of the weight that is sustained upon the *tail*-beams, *c c* (the wall at the opposite end or another header there sustaining the other half), and the trimmers must each sustain one half of the weight sustained by the header in *addition* to the weight it supports as a common beam. It is usual in practice to make

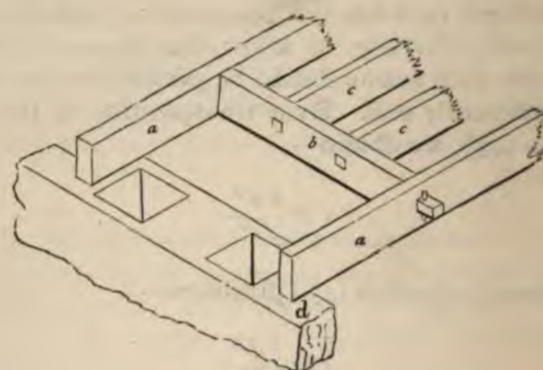


Fig. 42.

these framed beams one inch thicker than the common beams for dwellings, and two inches thicker for heavy stores. This practice in ordinary cases answers very well, but in extreme cases these dimensions are not proper. Rules applicable generally must be deduced from the conditions of the case—the load to be sustained and the strength of the material.

157.—Breadth of Headers.—The load sustained by a header is equally distributed, and is equal to the superficial area of the floor supported by the header multiplied by the load on every superficial foot of the floor. This is equal to the length of the header multiplied by half the length of the tail-beams, and by the load per superficial foot. Putting *g*

for the length of the header, n for the length of the tail-beams, and f for the load per superficial foot; U, the uniformly distributed load carried by the header, will equal $\frac{1}{2} f n g$. By substituting for U, in equation (35.), this value of it, we obtain—

$$b d^3 = \frac{\frac{1}{2} f n g l^3}{1 \cdot 6 F \delta}.$$

The symbols g and l here both represent the same thing, the length of the header; combining these, and for δ putting its value $g r$, we obtain—

$$b d^3 = \frac{f n g^3}{3 \cdot 2 F r}.$$

To allow for the weakening of the header by the mortices for the tail-beams (which should be cut as near the middle of the depth of the header as practicable), the depth should be taken at, say, one inch less than the actual depth. With this modification, we obtain—

$$b = \frac{f n g^3}{3 \cdot 2 F r (d - 1)^3}. \qquad (48.)$$

If f be taken at 90, and r at 0·03, we have, by reducing—

$$b = \frac{937 \cdot 5\, n g^3}{F (d - 1)^3}, \qquad (49.)$$

which is a rule for the breadth of headers for dwellings and for ordinary stores. This, in words, is as follows:

Rule XLI.—Multiply 937·5 times the length of the tail-beams by the cube of the length of the header, both in feet. The product divided by the cube of one less than the depth multiplied by the value of F, Table III., will equal the breadth of the header in inches for *dwellings* or *ordinary stores.*

Example.—A header of white pine, for a dwelling, is 10 feet long, and sustains tail-beams 20 feet long; its depth is 12 inches: what must be its breadth? By the rule, $937 \cdot 5 \times 20 \times 10^3 = 18750000$. This divided by $(12 - 1)^3 \times 2900 =$

3859900, equals 4·858, say 5 inches, the required breadth. For *first-class stores*, f should be taken at 275, and r at 0·04. With these values the constants in equation (48.) reduce to 2148·4375, or, say, 2150. This gives—

$$b = \frac{2150\, n\, g^3}{F(d-1)^3}, \qquad (50.)$$

a rule for the breadth of a header for first-class stores. It is the same as that for dwellings, except that the constant 2150 is to be used in place of 937·5. Taking the same example, and using the constant 2150 instead of 937·5, we obtain 11·14 as the required breadth of the header for a first-class store. Modifying the question by using Georgia pine instead of white pine, we obtain 5·476 as the required thickness, say $5\frac{1}{2}$ inches.

158.—Breadth of Carriage-Beams.—A carriage-beam or trimmer, in *addition* to its load as a common beam, carries one half of the load on the header, which, as has been seen in the last article, is equal to one half of the superficial area of the floor supported by the tail-beams multiplied by the weight per superficial foot of the load upon the floor; therefore, when the length of the header in feet is represented by g, and the length of the tail-beams by n, w equals $\frac{g}{2} \times \frac{n}{2} \times f$, equals $\frac{1}{4} f\, g\, n$.*

For a load not at middle, we have (25.)—

$$b\, d^2 = \frac{4\, W a\, m\, n}{B\, l}.$$

*The load from the header, instead of being $\frac{1}{4} f\, g\, n$, is, more accurately, $\frac{1}{4} f\, n\, (g - c)$: because the surface of floor carried by the header is only that which occurs between the surfaces carried by the carriage-beams, each of which carries so much of the floor as extends half way to the first tail-beam from it, or the distance $\frac{c}{2}$; therefore, the width of the surface carried equals the length of the header less $\left(2 \times \frac{c}{2} = \right) c$, or $g - c$. When, however, it is considered that the carriage-beam is liable to receive some weight from a stairs or other article in the well-hole, the small additional load above referred to is not only not objectionable, but is really quite necessary to be included in the calculation.

This is a rule based upon resistance to rupture. By substituting for a, the factor of safety, $\frac{B\,l}{F\,d\,r}$, its value in terms of resistance to flexure (*Transverse Strains*, (154.)), we have—

$$b\,d^2 = \frac{4\,W\,B\,l\,m\,n}{B\,l\,F\,d\,r} = \frac{4\,W\,m\,n}{F\,d\,r};$$

or—

$$b\,d^3 = \frac{4\,W\,m\,n}{F\,r}.$$

In this expression, W is a concentrated weight at the distances m and n from the two ends of the beam. Taking the load upon a carriage-beam due to the load from the header, as above found, and substituting it for W, we obtain—

$$b\,d^3 = \frac{4 \times \frac{1}{4} f\,g\,n\,m\,n}{F\,r} = \frac{f\,g\,m\,n^2}{F\,r}.$$

This is the expression required for the concentrated load. To this is to be added the uniformly distributed load upon the carriage-beam; this is given in equation (35.). Substituting for U of this equation its value, $f\,c\,l$, gives—

$$b\,d^3 = \frac{f\,c\,l^4}{1{\cdot}6\,F\,\delta} = \frac{\frac{5}{8} f\,c\,l^3}{F\,r}.$$

Combining these two equations, we have for the total load—

$$b\,d^3 = \frac{f(g\,m\,n^2 + \frac{5}{8}\,c\,l^3)}{F\,r}. \qquad (51.)$$

If, in this equation, f be taken at 90, and r at 0·03, these reduce to 3000; therefore, with this value of $\frac{f}{r}$, we have—

$$b = \frac{3000\,(g\,m\,n^2 + \frac{5}{8}\,c\,l^3)}{F\,d^3}. \qquad (52.)$$

This rule for the breadth of carriage-beams with one header, for *dwellings* and for *ordinary stores*, is put in words as follows:

Rule XLII.—Multiply the length of the framed opening by its breadth, and by the square of the length of the tail-beams; to this product add $\frac{5}{8}$ of the cube of the length into the distance of the common beams from centres—all in feet: divide 3000 times the sum by the cube of the depth in inches multiplied by the value of F for the material of the beam, in Table III., and the quotient will be the breadth in inches.

Example.—In a tier of 3×10 inch beams, placed 14 inches from centres, what should be the breadth of a Georgia-pine carriage-beam 20 feet long, carrying a header 12 feet long, having tail-beams 15 feet long? Here the framed opening is 5×12 feet. Therefore, according to the rule, $12 \times 5 \times 15^2 = 13500$; to which add $(\frac{5}{8} \times 20^3 \times \frac{14}{12} =) 5833\frac{1}{3}$; the sum is $19333\frac{1}{3}$, and this by $3000 = 58000000$. The value of F for Georgia pine, in Table III., is 5900; the cube of the depth is 1000; the product of these two is 5900000; therefore, dividing the above 58000000 by 5900000 gives a quotient of 9.83, the required breadth in inches. If, in equation (51.), f be taken at 275, and r at 0·04, then $\frac{f}{r}$ becomes 6875, and the equation becomes—

$$b = \frac{6875\,(g\,m\,n^2 + \frac{5}{8}\,c\,l^3)}{F d^3}, \tag{53.}$$

a rule for the breadth of carriage-beams for *first-class stores*; the same as that for dwellings, except that the constant is 6875 instead of 3000.

159.—Breadth of Carriage-Beams Carrying Two Sets of Tail-Beams.—A rule for this is the same as that for a carriage-beam carrying one set of tail-beams, if to it there be added the effect of the second set of tail-beams. Equation (51.) with the addition named becomes—

$$b = \frac{f\,[g\,n\,(m\,n + s^2) + \frac{5}{8}\,c\,l^3]}{F d^3 r}, \tag{54.}$$

in which n is the length of one set of tail-beams, and s the length of the other set; and $m + n = l$.

If f be taken at 90, and r at 0·03, these two reduce to 3000, and we have—

$$b = \frac{3000\,[g\,n\,(m\,n + s^2) + \frac{5}{8}\,c\,l^3]}{F\,d^3}, \qquad (55.)$$

a rule for the breadth of a carriage-beam carrying two sets of headers, for dwellings and for ordinary stores. It may be stated in words as follows:

Rule XLIII.—Multiply the length of the longer set of tail-beams by the difference between this length and the length of the carriage-beam, and to the product add the square of the length of the shorter set of tail-beams; multiply the sum by the length of the longer set of tail-beams, and by the length of the header; to this product add $\frac{5}{8}$ of the product of the cube of the length of the carriage-beam into the distance apart from centres of the common beams; multiply this sum by 3000; divide this product by the product of the cube of the depth in inches into the value of F for the material of the carriage-beam, in Table III., and the quotient will be the required breadth.

Example.—In a tier of 3×12 inch beams, placed 14 inches from centres, what should be the breadth of a spruce carriage-beam 20 feet long in the clear of the bearings, carrying two sets of tail-beams, one of them 9 feet long, the other 5 feet; the headers being 15 feet long? The difference between the longer set of tail-beams and the carriage-beam is $(20 - 9 =)$ 11 feet. Therefore, by the rule, $9 \times 11 + 5^2 = 124$; then $(124 \times 9 \times 15 =)\ 16740 + (\frac{5}{8} \times 20^3 \times \frac{14}{12} =)\ 5833\frac{1}{3} = 22573\frac{1}{3}$; then $22573\frac{1}{3} \times 3000 = 67720000$. Now the value of F for spruce, Table III., is 3500; this by 12^3, the cube of the depth, equals 6048000; by this dividing the aforesaid 67720000, we obtain a quotient of 11·197, the required breadth of the carriage-beam. If, in equation (54.), f be taken at 275, and r at 0·04, these reduce to 6875, and we obtain—

$$b = \frac{6875\,[g\,n\,(m\,n + s^2) + \frac{5}{8}\,c\,l^3]}{F\,d^3}, \qquad (56.)$$

a rule for the breadth of carriage-beams carrying two sets

of tail-beams, in the floors of ***first-class stores***. This is like the rule for dwellings, except that the constant is 6875 instead of 3000.

160.—Breadth of Carriage-Beam with Well-Hole at Middle.—When the framed opening between the two sets of tail-beams occurs at the middle, or when the lengths of the two sets of tail-beams are equal, then equation (54.) reduces to

$$b = \frac{fl(gn^2 + \frac{5}{8}cl^2)}{Fd^3r}; \qquad (57.)$$

and if f be taken at 90, and r at 0·03, these reduce to 3000, and we have—

$$b = \frac{3000\, l(gn^2 + \frac{5}{8}cl^2)}{Fd^3}, \qquad (58.)$$

a rule for the breadth of a carriage-beam carrying two sets of tail-beams of equal length, in the floor of a *dwelling* or of an *ordinary store;* and which in words is as follows:

Rule XLIV.—Multiply the length of the header by the square of the length of the tail-beams, and to the product add $\frac{5}{8}$ of the product of the square of the length of the carriage-beam by the distance apart from centres of the common beams; multiply the sum by 3000 times the length of the carriage-beam; divide the product by the product of the cube of the depth into the value of F for the material of the carriage-beam, in Table III., and the quotient will be the required breadth.

Example.—In a tier of 3 × 12 inch beams, placed 12 inches from centres, what must be the thickness of a hemlock carriage-beam 20 feet long, carrying two sets of tail-beams, each 8 feet long, with headers 10 feet long? By the rule, $\overline{10 \times 8^2} + \frac{5}{8} \times \overline{1 \times 20^2} = 890$; $890 \times 3000 \times 20 = 53400000$. Now, the value of F, in Table III., for hemlock is 2800; this by the cube of the depth, 1728, equals 4838400; by this dividing the former product, 53400000, and the quotient, 11·0367, is the required breadth of the carriage-beam.

If, in equation (57.), f be taken at 275, and r at 0.04, these will reduce to 6875, and we shall have—

$$b = \frac{6875\ l\,(g\ n^2 + \frac{5}{8}\,c\ l^2)}{F\ d^3}, \qquad (59.)$$

a result the same as in equation (58.), except that the constant is 6875 instead of 3000. Equation (59.) is a rule for the breadth of carriage-beams carrying two sets of tail-beams of equal length, in the floor of a first-class store. In words at length, it is the same as Rule XLIV., except that the constant 6875 is to be used in place of 3000.

161.—Cross-Bridging, or Herring-Bone Bridging.—The diagonal struts set between floor-beams, as in *Fig.* 43, are known as cross-bridging, or herring-bone bridging. By connecting the beams thus at intervals, say, of from 5 to 8 feet, the stiffness of the floor is greatly increased. The absolute strength of a tier of beams to resist a weight uniformly distributed over the whole tier is augmented but little by cross-bridging; but the power of any one beam in the tier to resist a concentrated load upon it, as a heavy article of furniture or an iron safe, is greatly increased by the cross-bridging; for this device, by connecting the loaded beam with the adjacent beams on each side, causes these beams to *assist* in carrying the load. To secure the full benefit of the diagonal struts, it is very important that the beams be well secured from separating laterally, by having strips, such as *cross-furring*, firmly nailed to the under edges of the beams. The tie thus made, together with that of the floor-plank on the top edges, will prevent the thrust of the struts from separating the beams.

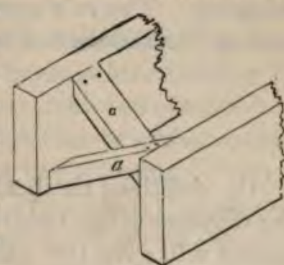


FIG. 43.

162.—Bridging: Value to Resist Concentrated Loads.—A rule for determining the additional load which any one beam connected by bridging will be capable of sustaining, by the assistance derived from the other beams, through the

bridging, may be found in Chapter XVIII., *Transverse Strains.* This rule may be stated thus:

$$R = \frac{5\,c^3 f\,l}{4\,d^2}(1 + 2^2 + 3^2 + 4^2 + etc.)\,; \qquad (60.)$$

in which R is the increased resistance, equal to the additional load which may be put upon the loaded beam; c is the distance from centres in feet at which the beams in the tier are placed: f is the load in pounds per superficial foot upon the floor; l is the length of the beams in feet; and d is the depth of the beams in inches. The squares within the bracket are to be extended to as many places as there are beams on each side which contribute assistance through the bridging. The rule given in the work referred to, for ascertaining the number of spaces between the beams, is—

$$n = \frac{d}{c^2}\,; \qquad (61.)$$

or, the depth of the beam in inches divided by the square of the distance from centres, in feet, at which the beams are placed will give the number of spaces between the beams which contribute on each side in sustaining the concentrated load. The nearest whole number, minus unity, will equal the required number of beams.

The value of c for beams in floors of dwellings is given in equation (46.), and for those in first-class stores in equation (47.). By a modification of equation (34.), putting $c\,f\,l$ for U, we have—

$$c\,f\,l = \frac{1{\cdot}6\,F\,b\,d^3\,\delta}{l^3},$$

and—

$$c = \frac{1{\cdot}6\,F\,b\,d^3\,\delta}{f\,l^4}, \qquad (62.)$$

or—

$$c = \frac{1{\cdot}6\,F\,b\,d^2\,r}{f\,l^3}. \qquad (63.)$$

These equations give general rules for the value of c.

Now, the rule, in words at length, for the resistance offered by the adjoining beams to a weight concentrated upon one of the beams sustained by cross-bridging to the others, is—

Rule XLV.—Divide the depth of the beam in inches by the square of the distance apart from centres in feet at which the floor-beams are placed; from the quotient deduct unity, and call the whole number nearest to the remainder the First Result. Take the sum of the squares of the consecutive numbers from unity to as many places as shall equal the above first result; multiply this sum by 5 times the length in feet, by the load per foot superficial upon the floor, and by the fifth power of the distance apart from centres in feet at which the beams are placed; divide the product by 4 times the square of the depth in inches, and the quotient will be the weight in pounds required.

Example.—In a tier of 3×12 inch floor-beams 20 feet long, placed in a dwelling 16 inches from centres and well bridged, what load may be uniformly distributed upon one of the beams, additional to the load which that beam is capable of sustaining safely when unassisted by bridging? Here, according to the rule, 12 divided by $(1\frac{1}{3} + 1\frac{1}{3} =)$ $1\frac{7}{9}$ equals $6\frac{3}{4}$; $6\frac{3}{4} - 1 = 5\frac{3}{4}$, the nearest whole number to which is 6, the first result. The sum of the square of the first 6 numbers equals $(1 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 =)$ $1 + 4 + 9 + 16 + 25 + 36 = 91$. Therefore, $91 \times 5 \times 20 \times 90 \times (\frac{4}{3})^5 = 3451266$.* The square of the depth $(12 \times 12 =)$ $144 \times 4 = 576$; by this dividing the above 3451266, we have the quotient 5991·78, say 5992 pounds, the required weight. This is the *additional* load which may be placed upon the beam. At 90 pounds per superficial foot, the common load on each beam, we have

---

* The value of *c*, 16 inches, equals $\frac{4}{3}$ feet. The fifth power of this, or $(\frac{4}{3})^5$, is obtained by involving both numerator and denominator to the fifth power, and dividing the fifth power of the former by the fifth power of the latter; for $(\frac{4}{3})^5 = \frac{4^5}{3^5}$. For the numerator we have $4 \times 4 \times 4 \times 4 \times 4 = 1024$, and for the denominator $3 \times 3 \times 3 \times 3 \times 3 = 243$. The former divided by the latter gives as a quotient 4·214, the value of $(\frac{4}{3})^5$. The process of involving a number to a high power, or the reverse operation of extracting high roots, may be performed by logarithms with great facility. (See *Art.* 427.)

$90 \times 20 \times \frac{4}{3} = 2400$ as the common load. To this add 5992, the load sustained through the bridging by the other beams, and the sum, 8392 pounds, will be the total load which may be safely sustained, uniformly distributed, upon one beam—nearly $3\frac{1}{2}$ times the common load.

163.—Girders.—When the distance between the walls of a building is greater than that which would be the limit for the length of ordinary single beams, it becomes requisite to introduce one or more additional supports. Where supports are needed for a floor and partitions are not desirable, it is usual to use a large piece of timber called a girder, sustained by posts set at intervals of from 8 to 15 feet; or, when posts are objectionable, a framed construction called a framed girder (*Art.* 196); or an iron box called a tubular iron girder (*Art.* 182). When a simple timber girder is used it is advisable, if it be large, to divide it vertically from end to end and reverse the two pieces, exposing the heart of the timber to the air in order that it may dry quickly, and also to detect decay at the heart. When the halves are bolted together, thin slips of wood should be inserted between them at the several points at which they are bolted, in order to leave sufficient space for the air to circulate freely in the space thus formed between them. This tends to prevent decay, which will be found first at such parts as are not exactly tight, nor yet far enough apart to permit the escape of moisture. When girders are required for a long bearing, it is usual to truss them; that is, to insert between the halves two pieces of oak which are inclined towards each other, and which meet at the centre of the length of the girder like the rafters of a roof-truss, though nearly if not quite concealed within the girder. This and many similar methods, though extensively practised, are generally worse than useless; since it has been ascertained that, in nearly all such cases, the operation has positively *weakened* the girder.

A girder may be strengthened by mechanical contrivance, when its depth is required to be greater than any one piece of timber will allow. *Fig.* 44 shows a very simple yet invaluable method of doing this. The two pieces of which the gir-

der is composed are bolted or pinned together, having keys inserted between to prevent the pieces from sliding. The keys should be of hard wood, well seasoned. The two pieces should be about equal in depth, in order that the joint between them may be in the neutral line. (See *Arts.* 120, 121.) The thickness of the keys should be about half their breadth, and the amount of their united thickness should be equal to a trifle over the depth and one third of the depth of the girder. Instead of bolts or pins, iron hoops are sometimes used; and when they can be procured, they are far preferable. In this case, the girder is diminished at the ends, and the hoops driven from each end towards the middle. A girder may be spliced if timber of a sufficient length cannot be obtained; though not at or near the mid-

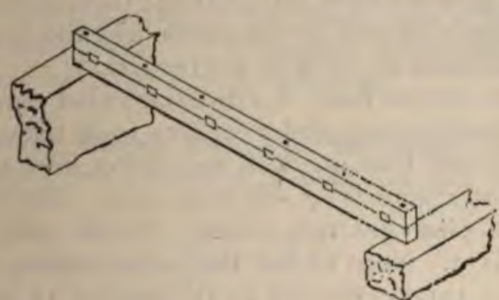


FIG. 44.

dle, if it can be avoided. (See *Art.* 87.) Girders should rest from 9 to 12 inches on each wall, and a space should be left for the air to circulate around the ends, that the dampness may evaporate.

164.—Girders: Dimensions.—The size of a girder, for any special case, may be determined by equations (21.), (22.), (25.), (27.), and (28.), to resist rupture; and to resist deflection, by equations (32.) and (35.). For girders in dwellings, equation (44.) may be used. In this case, the value of c is to be taken equal to the width of floor supported by the girder, which is equal to the sum of the distances half way to the wall or next bearing on each side. When there is but one

girder between the two walls, the value of c is equal to half the distance between the walls. The rule for girders for *dwellings*, in words, is—

Rule XLVI.—Multiply the cube of the length of the girder by the sum of the distances from the girder half way to the next bearing on each side, and by the value of j for the material of the girder, in *Art.* 152; the product will equal the product of the breadth of the girder into the cube of the depth. To obtain the breadth, divide this product by the cube of the depth; the quotient will be the breadth. To obtain the depth, divide the said product by the breadth; the cube root of the quotient will be the depth. If the breadth and depth are to be in a given proportion, say as $r : 1 \cdot 0$, then divide the aforesaid quotient by the value of r; take the square root of the quotient; then the square root of this square root will be the depth, and the depth multiplied by the value of r will be the breadth.

Example.—In the floor of a dwelling, what should be the size of a Georgia-pine girder 14 feet long between posts, placed at 10 feet from one wall and 20 feet from the other? The value of c here is $\frac{10}{2} + \frac{20}{2} = \frac{30}{2} = 15$. The value of j for Georgia pine (*Art.* 152) is $0 \cdot 32$. By the rule, $14^3 \times 15 \times 0 \cdot 32 = 13171 \cdot 2$. Now, to find the breadth when the depth is 12 inches; $13171 \cdot 2$ divided by the cube of 12, or by 1728, gives a quotient of $7 \cdot 622$, or $7\frac{5}{8}$, the required breadth. Again, to find the depth, when the breadth is 8 inches: $13171 \cdot 2$ divided by 8 gives $1646 \cdot 4$, the cube root of which is $11 \cdot 808$, or, say, $11\frac{7}{8}$ inches, the required depth. But if neither breadth nor depth have been previously determined, except as to their proportion, say as $0 \cdot 7$ to $1 \cdot 0$, then $13171 \cdot 2$ divided by $0 \cdot 7$ gives 18816, of which the square root is $137 \cdot 171$, and of this the square root is $11 \cdot 712$, or, say, $11\frac{3}{4}$ inches, the required depth. For the breadth, we have $11 \cdot 712$ by $0 \cdot 7$ equals $8 \cdot 198$, or, say, $8\frac{1}{4}$, the required breadth. Thus the girder is required to be $7\frac{5}{8} \times 12$, $8 \times 11\frac{7}{8}$, or $8\frac{1}{4} \times 11\frac{3}{4}$ inches. This example is one in a dwelling or ordinary store; for *first-class stores* the rule for girders is the same as the last, except that the value of k is to be taken instead of j, in *Art.* 152.

165.—Solid Timber Floors.—Floors constructed with rolled-iron beams and brick arches are proof against fire only to a limited degree; for experience has shown that the heat, in an extensive conflagration, is sufficiently intense to deprive the iron of its rigidity, and consequently of its strength. Singular as it may seem, it is nevertheless true that wood, under certain circumstances, has a greater fire-resisting quality than iron. Floors of timber constructed, as is usual, with the beams set apart, have but little power to resist fire, but if the spaces between the beams be filled up solid with other beams, which thus close the openings against the passage of the flames, and the under surface be coated with plastering mortar containing a large portion of plaster of Paris, and finished smooth, then this wooden floor will resist the action of fire longer than a floor of iron beams and brick arches. The wooden beams should be secured to each other by dowels or spikes.

166.—Solid Timber Floors for Dwellings and Assembly-Rooms.—From *Transverse Strains*, Art. 702, we have—

$$d^3 = \frac{(82 + y\,d)\,l^3}{0 \cdot 576\,F},$$

which may be modified so as to take this form:

$$d = \sqrt[3]{\frac{(82 + y\,h\,l)\,l^2}{0 \cdot 576\,F}}, \qquad (64.)$$

which is a rule for the depth or thickness* of solid timber floors for dwellings, assembly-rooms, or office buildings, and in which y and h are constants depending upon the material; thus, for—

Georgia Pine.................$y = 4$, and $h = 0 \cdot 314$
Spruce ....$y = 2\frac{1}{2}$, " $h = 0 \cdot 365$
White Pine...................$y = 2\frac{1}{8}$, " $h = 0 \cdot 389$
Hemlock......................$y = 2$, " $h = 0 \cdot 39$

The rule may be stated in words thus:

Rule XLVII.—Multiply the length by the value of y,

and by the value of h, as above given; to the product add 82; multiply the sum by the cube of the length; divide this product by 0·576 times the value of F, in Table III.; then the cube root of the quotient will be the required depth in inches.

Example.—What depth is required for a solid Georgia-pine floor to cover a span of 20 feet? For Georgia pine $F = 5900$; y, as above given, equals 4, and h equals 0·314; therefore, by the rule—

$$d = \sqrt[3]{\frac{(82 + 4 \times 0 \cdot 314 \times 20)\, l^3}{0 \cdot 576 \times 5900}} = \sqrt[3]{\frac{856960}{3398 \cdot 4}} = 6 \cdot 318;$$

or, the depth required is, say, 6·32 or $6\frac{5}{16}$ inches.

167.—Solid Timber Floors for First-Class Stores.—The equation given for first-class stores, in *Transverse Strains*, Art. 702, is—

$$d^3 = \frac{(263 + y\, d)\, l^3}{\cdot 768\, F}$$

which may be changed to this form:

$$d = \sqrt[3]{\frac{(263 + y\, k\, l)\, l^3}{0 \cdot 768\, F}}, \qquad (65.)$$

in which y is as before, and k for—

Georgia Pine equals	0·4
Spruce equals	0·472
White Pine equals	0·502
Hemlock equals	0·506

This rule may be put in words the same as Rule XLVII., except as to the constants, which require that 263 be used in place of 82, that k be used in place of h, and that 0·768 be used in place of 0·576. Table XXI. of *Transverse Strains* contains the results of computation showing the depths of solid timber floors for dwellings and assembly-rooms and for first-class stores, in floors of spans varying from 8 to 30 feet, and for the four kinds of timber before named.

168. — Rolled - Iron Beams.—The dimensions of iron beams, whether wrought or cast, are to be ascertained by the rules already given, when the beams are of rectangular form in their cross-section; these rules are applicable alike to wood and iron (*Art.* 93), and may be used for any material, provided the constant appropriate to the given material be used. But when the form of cross-section is such as that which is usual for rolled-iron beams (*Fig.* 45), the rules need modifying. Without attempting to explain these modifications (referring for this to *Transverse Strains*, Art. 457 and following article), it may be remarked that the elements of resistance to flexure in a beam constitute what is termed the *Moment* of *Inertia*. This, in a beam of rectangular cross-section, is equal to $\frac{1}{12}$ of the breadth into the cube of the depth; or—

FIG. 45.

$$I = \tfrac{1}{12}\, b\, d^3. \tag{66.}$$

This would be appropriate to rolled-iron beams if the hollow on each side were filled with metal, so as to complete the form of cross-section into a rectangle. The proper expression for them may be obtained by taking first the moment for the beam as if it were a solid rectangle, and from this deducting the moment for the part which on each side is wanting, or for the rectangles of the hollows. In accordance with this view of the case, we have—

$$I = \tfrac{1}{12}\,(b\, d^3 - b_{,}\, d_{,}^3)\,; \tag{67.}$$

in which b is the breadth of the beam or width of the flanges; $b_{,}$ is the breadth of the two hollows, or is equal to b less the thickness of the *web* or *stem*; d is the depth including top and bottom flanges; and $d_{,}$ is the depth in the clear between the top and bottom flanges.

Now, if equation (32.) be divided by 12, we shall have—

$$\tfrac{1}{12}\, b\, d^3 = \frac{W l^3}{12\, F \delta}\,;$$

and since $\frac{1}{12}\, b\, d^3$ represents the moment of inertia, we have—

$$I = \frac{W\, l^3}{12\, F\, \delta}. \qquad (68.)$$

This gives the value of I for a beam of any form in cross-section loaded at the middle. By this equation the values of I have been computed for rolled-iron beams of many sizes, and the results recorded in Table XVII., *Transverse Strains*. A few of these are included in Table IV., as follows:

TABLE IV.—ROLLED-IRON BEAMS.

NAME.	Depth.	Weight per yard.	$I =$	NAME.	Depth.	Weight per yard.	$I =$
Trenton.....	4	30	7·84	Buffalo.......	9	90	109·117
Paterson....	5	30	12·082	Phœnix......	9	150	190·63
Phœnix....	5	36	14·317	Buffalo.	$10\frac{1}{2}$	90	151·436
Trenton.....	6	40	23·761	Buffalo.......	$10\frac{1}{2}$	105	175·645
Phœnix.....	7	55	42·43	Trenton......	$10\frac{1}{8}$	135	241·473
Trenton.....	7	60	46·012	Buffalo	$12\frac{1}{4}$	125	286·019
Buffalo......	8	65	64·526	Paterson.....	$12\frac{1}{4}$	125	292·05
Paterson....	8	80	84·735	Paterson.	$12\frac{1}{4}$	170	398·936
Phœnix	9	70	92·207	Buffalo.......	$12\frac{1}{4}$	180	418·945
Phœnix	9	84	107·793	Trenton......	$15\frac{1}{16}$	150	528·223

169. — Rolled-Iron Beams: Dimensions; Weight at Middle.—If, in equation (68.), there be substituted for F its value for wrought iron, as in Table III., we shall have—

$$I = \frac{W\, l^3}{12 \times 62000\, \delta};$$

or—

$$I = \frac{W\, l^3}{744000\, \delta}. \qquad (69.)$$

This is a rule by which to ascertain the size of a rolled-iron beam to sustain a given weight at middle with a given deflection, and, in words at length, is as follows:

Rule XLVIII.—Multiply the weight in pounds by the cube of the length in feet; divide the product by 744000 times the deflection in inches, and the quotient will be the

moment of inertia of the required beam, and may be found, or the next nearest number, in Table IV. in column headed I. Opposite to the number thus found, to the left, will be found the name, depth, and weight per yard of the required beam.

Example.—Which of the beams of Table IV. would be proper to carry 10,000 pounds at the middle with a deflection of one inch, the length between bearings being 20 feet? Here we have, substituting for the symbols their values—

$$I = \frac{W\,l^3}{744000\,\delta} = \frac{10000 \times 20^3}{744000 \times 1} = \frac{80000000}{744000} = 107 \cdot 527;$$

or, the moment of inertia of the required beam is 107·527, the nearest to which, in the table, is 107·793, pertaining to the Phœnix 9-inch, 84-pound beam. This, then, is the required beam.

170.—Rolled-Iron Beams: Deflection when Weight is at Middle.—By a transposition of symbols in equation (69.), we have—

$$\delta = \frac{W\,l^3}{744000\,I}, \qquad (70.)$$

or a rule for the deflection of rolled-iron beams when the weight is at the middle. This, in words, is—

Rule XLIX.—Multiply the weight in pounds by the cube of the length in feet; divide the product by 744000 times the value of I for the given beam, and the quotient will be the required deflection in inches.

Example.—What will be the deflection of a Phœnix 9-inch, 70-pound beam 20 feet long, loaded at the middle with 7500 pounds? The value of I for this beam, in Table IV., is 92·207; therefore, substituting for the symbols their values, and proceeding by the rule, we have—

$$\delta = \frac{W\,l^3}{744000\,I} = \frac{7500 \times 20^3}{744000 \times 92 \cdot 207} = 0.87461;$$

or, the deflection will be, say, $\frac{7}{8}$ of an inch.

171.—Rolled-Iron Beams: Weight when at Middle.—A transposition of factors in equation (70.) gives—

$$W = \frac{744000\, I\, \delta}{l^3}. \qquad (71.)$$

This is a rule for the weight at middle, and, in words, is—

Rule L.—Multiply 744000 times the value of I by the deflection in inches; divide the product by the cube of the length, and the quotient will be the required weight in pounds.

Example.—What weight at the middle of a Buffalo 9-inch, 90-pound beam will deflect it one inch, the length between bearings being 20 feet? The value of I for this beam, in Table IV., is 109·117; therefore—

$$W = \frac{744000\, I\, \delta}{l^3} = \frac{744000 \times 109 \cdot 117 \times 1}{20^3} = 10147 \cdot 88;$$

or, the required weight is, say, 10,148 pounds.

172.—Rolled-Iron Beams: Weight at any Point.—The equation for a load at any point is (*Transverse Strains*, Art. 485)—

$$W = \frac{186000\, I\, \delta}{l\, m\, n}; \qquad (72.)$$

in which m and n represent the two parts in feet into which the point where the load rests divides the length. This, in words, is as follows:

Rule LI.—Multiply 186000 times the value of I by the deflection in inches; divide the product by the product of the length into the rectangle formed by the two parts into which the point where the load rests divides the length; the quotient will be the required weight in pounds.

Example.—What weight is required, located at 10 feet from one end, to deflect $1\frac{1}{2}$ inches a Paterson $12\frac{1}{4}$-inch, 125-pound beam 25 feet long between bearings? The value of I for this beam, in Table IV., is 292·05; $m = 10$, and $n = l - m = 25 - 10 = 15$; therefore—

$$W = \frac{186000\, I\,\delta}{l\,m\,n} = \frac{186000 \times 292 \cdot 05 \times 1 \cdot 5}{25 \times 10 \times 15} = 21728 \cdot 52;$$

or, the required weight is, say, 21,730 pounds.

173.—**Rolled-Iron Beams: Dimensions; Weight at any Point.**—By transposition of factors in equation (72.), we obtain—

$$I = \frac{W\,l\,m\,n}{186000\,\delta}. \qquad (73.)$$

This may be expressed in words as follows:

Rule LII.—Multiply the weight by the length, and by the rectangle of the two parts into which the point where the weight rests divides the length; divide the product by 186000 times the deflection, and the quotient will be the value of I, which (or its next nearest number) may be found in Table IV., opposite to which will be found the required beam.

Example.—What beam 10 feet long will be required to carry 5000 pounds at 3 feet from one end with a deflection of 0·4 inch? Here we have m equal 3, and n equal 7; therefore—

$$I = \frac{W\,l\,m\,n}{186000\,\delta} = \frac{5000 \times 10 \times 3 \times 7}{186000 \times 0 \cdot 4} = 14 \cdot 113.$$

The value of I is 14·113, the nearest number to which in the table, is 14·317, the moment of inertia of the Phœnix 5-inch, 36-pound beam; this, therefore, is the beam required.

174.—**Rolled-Iron Beams: Dimensions; Weight Uniformly Distributed.**—Since $\frac{5}{8}\,U = W$ (*Art.* 138), equation (69.) may be modified by the substitution of this value of W, when we obtain—

$$I = \frac{\frac{5}{8}\,U\,l^3}{744000\,\delta},$$

which reduces to—

$$I = \frac{U\,l^3}{1190400\,\delta}, \qquad (74.)$$

a rule for the dimensions of a beam for a uniformly distributed load, which, in words, is as follows:

Rule LIII.—Multiply the uniformly distributed load by the cube of the length; divide the product by 1190400 times the deflection, and the quotient will be the value of I, corresponding to which, or to its next nearest number will be found in Table IV. the required beam.

Example.—What beam 10 feet long is required to sustain an equally distributed load of 14,000 pounds with a deflection of half an inch? For this we have—

$$I = \frac{14000 \times 10^3}{1190400 \times 0 \cdot 5} = 23 \cdot 52.$$

This is the moment of inertia of the required beam; nearly the same as 23·761, in Table IV., the value of I for a Trenton 6-inch, 40-pound beam, which will serve as the required beam.

175.—Rolled-Iron Beams: Deflection; Weight Uniformly Distributed.—A transposition of the factors in equation (74.) gives—

$$\delta = \frac{U l^3}{1190400 \, I}, \qquad (75.)$$

a rule for the deflection of a uniformly loaded beam, and which may be put in these words, namely:

Rule LIV.—Multiply the uniformly distributed load by the cube of the length; divide the product by 1190400 times the value of I, Table IV., and the quotient will be the required deflection.

Example.—To what depth will 14,000 pounds, uniformly distributed, deflect a Buffalo 10½-inch, 90-pound beam 20 feet long? The value of I for this beam, as per the table, is 151·436; therefore—

$$\delta = \frac{14000 \times 20^3}{1190400 \times 151 \cdot 436} = 0 \cdot 6213;$$

or, the required deflection is, say, ⅝ of an inch.

176.—Rolled-Iron Beams: Weight when Uniformly Distributed.—Equation (75.), by a transposition of factors, gives—

$$U = \frac{1190400\ I\ \delta}{l^3}, \tag{76.}$$

a rule for the weight uniformly distributed, and which may be worded thus:

Rule LV.—Multiply 1190400 times the value of I, Table IV., by the deflection; divide the product by the cube of the length, and the quotient will be the required weight.

Example.—What weight uniformly distributed upon a Buffalo $10\frac{1}{2}$-inch, 105-pound beam 25 feet long between bearings will deflect it $\frac{3}{4}$ of an inch?

The value of I for this beam, as per Table IV., is 175·645; therefore—

$$U = \frac{1190400 \times 175 \cdot 645 \times \frac{3}{4}}{25^3} = 10036 \cdot 21;$$

or, the required weight is, say, 10,036 pounds.

177.—Rolled-Iron Beams: Floors of Dwellings or Assembly-Rooms.—From *Transverse Strains*, Art. 500, we have—

$$c = \frac{255\ I}{l^3} - \frac{y}{420}, \tag{77.}$$

a rule for the distance from centres of rolled-iron beams in floors of dwellings, assembly-rooms, or offices, where the spaces between the beams are filled in with brick arches and concrete. In the equation, c is the distance apart from centres in feet, and y is the weight per yard of the beam. This, in words, is thus expressed:

Rule LVI.—Divide 255 times the value of I by the cube of the length; from the quotient deduct one 420th part of the weight of the beam per yard, and the remainder will be the required distance apart from centres.

Example.—What should be the distance apart from cen-

tres of Buffalo 12¼-inch, 125-pound beams 25 feet long between bearings, in the floor of an assembly-room? For these beams, in Table IV., I equals 286·019, and $y = 125$; therefore—

$$c = \frac{255 \times 286 \cdot 019}{25^3} - \frac{125}{420};$$

$$c = \frac{72934 \cdot 8}{15625} - \frac{125}{420} = 4 \cdot 668 - \cdot 298 = 4 \cdot 37;$$

or, the required distance from centres is, say, 4 feet 4½ inches.

178.—Rolled-Iron Beams: Floors of First-Class Stores. —From *Transverse Strains*, Art. 504, we have—

$$c = \frac{148 \cdot 8\, I}{l^3} - \frac{y}{960}, \tag{78.}$$

a rule for the distance from centres of rolled-iron beams in the floor of a first-class store; the spaces between the beams being filled with brick arches and concrete. This rule may be put in words as follows:

Rule LVII.—Divide 148·8 times the value of I by the cube of the length; from the quotient deduct one 960th part of the weight of the beam per yard, and the remainder will be the distance apart of the beams from centres in feet.

Example.—What should be the distance apart from centres of Buffalo 12¼-inch, 180-pound beams 20 feet long between bearings, in the floor of a first-class store? For these beams the value of I, Table IV., is 418·945, and the value of y is 180; therefore—

$$c = \frac{148 \cdot 8 \times 418 \cdot 945}{20^3} - \frac{180}{960} = 7\ 60;$$

or, the required distance from centres is, say, 7 feet 7¼ inches.

179.—Floor-Arches: General Considerations.—In filling the spaces between the iron beams of a floor, the arches should be constructed with hard whole brick of good shape, laid upon the supporting centre in contact with each other, and the joints thoroughly filled with cement grout, and keyed with slate. Made in this manner, the arches need not be over four inches thick at the crown for spans extending to 7 or 8 feet, and 8 inches thick at the springing, where they should be started upon a proper skew-back. The rise of the arch should not be less than $1\frac{1}{2}$ inches for each foot of the span.

180.—Floor-Arches; Tie-Rods: Dwellings.—From *Transverse Strains*, Art. 507, we have—

$$d = \sqrt{0\cdot0198\, c\, s}, \qquad (79.)$$

which is a rule for the diameter in inches of a tie-rod for an arch in the floor of a bank, office building, or assembly-room; in which d is the diameter in inches of the rod, s is the span of the arch, and c is the distance apart between the rods (s and c both in feet). This rule requires that the arch rise $1\frac{1}{2}$ inches per foot of the span, and that the brick-work and the superimposed load each weigh 70 pounds, or together 140 pounds. This rule, in words, is as follows:

Rule LVIII.—Multiply the span of the arch by the distance apart at which the rods are placed, and by the decimal 0·0198; the square root of the product will be the diameter of the required rod.

Example.—What should be the diameter of the wrought-iron ties of brick arches of 5 feet span, in a bank or hall of assembly, where the ties are 8 feet apart? For this we have—

$$d = \sqrt{0\cdot0198 \times 8 \times 5} = \sqrt{\cdot792} = 0\cdot89;$$

or, the diameter of the required rods should be, say, $\frac{7}{8}$ of an inch.

181.—Floor-Arches; Tie-Rods: First-Class Stores.—From the same source as in last article, we have—

$$d = \sqrt{0\cdot04527\, c\, s}, \qquad (80.)$$

which is a rule for the size of tie-rods for the brick arches of the floors of first-class stores, where the arches have a rise of $1\frac{1}{2}$ inches for each foot of the span, and where the weight of the brick arch and concrete is not over 70 pounds per superficial foot of the floor, and the loading does not exceed 250 pounds per superficial foot. As the rule is the same as the one in the preceding article, except the decimal, a recital of the rule, in words, is not here needed. To obtain the required diameter, proceed as directed in Rule LVIII., using the decimal 0·04527 instead of the one there given.

TUBULAR IRON GIRDERS.

182.—Tubular Iron Girders: Description.—The use of wooden beams for floors is limited to spans of about 25 feet. When greater spans than this are to be covered, some expedient must be resorted to by which intermediate bearings for the floor-beams may be provided. Wooden girders may be used, but these need to be supported by posts at intervals of from 10 to 15 feet, unless the girders are trussed, or made up of top and bottom chords, struts, and ties. And even this is objectionable, owing to the height such a piece of framing requires, and which encumbers the otherwise free space of the hall. A substitute for the framed girder has been found in the tubular iron girder, as in *Fig.* 46, made of rolled plate iron and angle irons, riveted. They require to be stiffened by an occasional upright *T* iron along eachside, and a cross-head at least at each bearing.

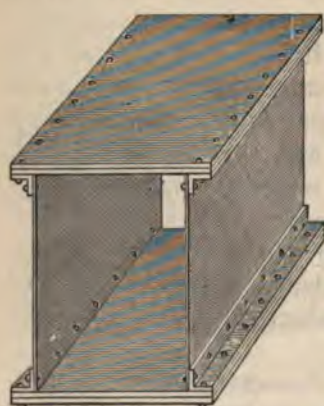


FIG. 46.

183.—Tubular Iron Girders: Area of Flanges; Load at Middle.—In wrought-iron tubular girders it is usual to make the top and bottom flanges of equal thickness. From *Transverse Strains*, Art. 551, we have—

$$a' = \frac{W\,l}{4\,d\,k}, \tag{81.}$$

a rule for the area of the bottom flange; in which a' equals the area of the flange in inches, W the weight in pounds at the middle, l the length and d the depth of the girder, both in feet, and k the safe load in pounds per inch with which the metal may be loaded, and which is usually taken at 9000. The rule may be stated thus:

Rule LIX.—Multiply the weight by the length; divide the product by 4 times the depth into the value of k, and the quotient will be the required area of the bottom flange.

Example.—In a girder 40 feet long and 3 feet high, to carry 75,000 pounds at the middle, what area of metal is required in the bottom flange, putting k at 9000? For this we have, by the rule—

$$a' = \frac{W\,l}{4\,d\,k} = \frac{75000 \times 40}{4 \times 3 \times 9000} = 27 \cdot 77\,;$$

or, the area required is $27\frac{7}{9}$ inches. This is the amount of uncut metal. An allowance is required for that which will be cut by rivet-holes. This is usually an addition of one sixth.

184.—Tubular Iron Girders: Area of Flanges; Load at any Point.—The equation suitable for this (*Transverse Strains*, Art. 553) is—

$$a' = W\frac{m\,n}{d\,k\,l}\,; \tag{82.}$$

in which m and n are the distances respectively from the location of the load to the two ends of the girder. The other symbols are the same as in the last article. This rule may be thus stated:

Rule LX.—Multiply the weight by the values of m and of n; divide the product by the product of the depth into the length and into the value of k, and the quotient will be the required area of the bottom flange.

Example.—In a girder 50 feet long between bearings and

$3\frac{1}{2}$ feet high, what area of metal is required in the bottom flange to sustain 50,000 pounds at 20 feet from one end, when k equals 9000? By the rule, we have—

$$a' = W\frac{m\ n}{d\,k\,l} = \frac{50000 \times 20 \times 30}{3\frac{1}{2} \times 9000 \times 50} = 19 \cdot 05\,;$$

or, each flange requires 19 inches of solid metal uncut for rivets.

185.—Tubular Iron Girders: Area of Flanges; Load Uniformly Distributed.—The equation appropriate here is (*Transverse Strains*, Art. 555)—

$$a' = U\frac{m\ n}{2\,d\,k\,l}. \qquad (83.)$$

This is a rule by which to obtain the area of cross-section of the bottom flange at any point in the length of the girder, the load uniformly distributed; m and n being the respective distances from the point measured to the two ends of the girder, and U representing the uniformly distributed load in pounds. This, in words, is described as follows:

Rule LXI.—Divide the weight by the product of twice the depth into the length and into the value of k; then the quotient multiplied by the values of m and of n will be the required area of the bottom flange at the point measured, the distance of which from the ends equals m and n.

Example.—In a girder 50 feet long and $3\frac{1}{2}$ feet high, to carry a uniformly distributed load of 120,000 pounds, what area of cross-section is required in the bottom flange, at the middle and at intervals of 5 feet thence, to each support; k being taken at 9000? Here we have, first—

$$a' = U\frac{m\ n}{2\,d\,k\,l} = \frac{120000\ m\ n}{2 \times 3\frac{1}{2} \times 9000 \times 50} = 0 \cdot 038095\ m\ n.$$

Now, when $m = n = 25$, we have the middle point; then—

$$a' = 0 \cdot 038095\ m\ n = 0 \cdot 038095 \times 25 \times 25 = 23 \cdot 81\,;$$

or, the area of the bottom flange at mid-length is 23·81 inches.

When $m = 20$, then $n = 30$, and—

$$a' = 0 \cdot 038095 \times 20 \times 30 = 22 \cdot 86;$$

or, the required area, at 5 feet either way from the middle, is $22\frac{7}{8}$ inches.

When $m = 15$, then $n = 35$, and—

$$a' = 0 \cdot 038095 \times 15 \times 35 = 20 \cdot 0;$$

or, at 10 feet either way from the middle, the required area is 20 inches.

When $m = 10$, then $n = 40$, and—

$$a' = 0 \cdot 038095 \times 10 \times 40 = 15 \cdot 24;$$

or, at 15 feet either way from the middle, the required area is $15\frac{1}{4}$ inches.

When $m = 5$, then $n = 45$, and—

$$a' = 0 \cdot 038095 \times 5 \times 45 = 8 \cdot 57;$$

or, at 20 feet each side of the middle, the required area is $8\frac{5}{8}$ inches.

The area of cross-section found in every case is that of the uncut fibres; to this is to be added as much as will be cut by the rivets. This is usually about one sixth of the area given by the rule. The top flange is to be made equal in area to the bottom flange. The flanges are unvarying in width from end to end, the variation of area being obtained by varying the thickness of the flanges, and this being attained by building the flange in lamina, or plates; but these should not be less than a quarter of an inch thick. There should be added to the length of the girder, in the clear, about one tenth of its length for supports on the walls: thus, a girder 30 feet long requires 3 feet added for supports, or 18 inches on each wall.

186.—Tubular Iron Girders: Shearing Strain.—The top and bottom flanges are provided of sufficient size to resist

the transverse strain: the two upright plates, technically termed the *web*, need, therefore, to be thick enough to resist only the shearing strain. This, upon a beam uniformly loaded, is at the middle theoretically nothing, but from thence it increases regularly towards each support, where it equals half the whole weight. For example, the girder of *Art.* 185, 50 feet long between supports, carries 120,000 pounds uniformly distributed over its length. In this case the shearing strain at the wall at each end is the half of 120,000 pounds, or 60,000 pounds; at 5 feet from the wall it is $\frac{5}{25}$ or $\frac{1}{5}$ less, or 48,000 pounds; at 10 feet from the wall it is $\frac{2}{5}$ less, or 36,000 pounds; at 15 feet it is 24,000; at 20 feet it is 12,000; and at 25 feet or the middle, it is nothing.

187.—Tubular Iron Girders: Thickness of Web.—The equation appropriate for this is—

$$t = \frac{G}{d\,k'}\,; \qquad (84.)$$

in which t is the thickness of the web (equal to the sum of the thicknesses of the two side plates), d is the height of the plate (t and d both in inches), G is the shearing strain, and k' is the effective resistance of wrought iron to shearing per inch of cross-section. This may be put in words as follows:

Rule LXII.—Divide the shearing strain by the product of the depth in inches into the value of k', and the quotient will be the thickness of the web, or of the two side plates taken together.

Example.—What is the required thickness of web in a girder 50 feet between bearings, side plates 38 inches high between top and bottom flanges, and to carry 120,000 pounds, uniformly distributed? Here, putting the shearing resistance of the plates at 7000 pounds per inch, we have—

$$t = \frac{G}{d\,k'} = \frac{G}{38 \times 7000} = \frac{G}{266000}.$$

The shearing strain at the supports, as in last article, is 60000; therefore, we have for this point—

$$t = \frac{60000}{266000} = 0 \cdot 225.$$

When $G = 48000$, then—

$$t = \frac{48000}{266000} = 0 \cdot 18;$$

and when $G = 36000$, then—

$$t = \frac{36000}{266000} = 0 \cdot 135.$$

Those nearer the middle of the girder are still less than these; and these are all below the practicable thickness, which is half an inch for the two plates. The plates ought not in practice ever to be made less than a quarter of an inch thick.

188.—Tubular Iron Girders, for Floors of Dwellings, Assembly-Rooms, and Office Buildings.—When the floors of these buildings are constructed with rolled-iron beams and brick arches, then the following (Art. 568, *Transverse Strains*) is the appropriate equation for the area of cross-section of the bottom flange of the girder:

$$a' = \left(140 + \frac{y}{3c}\right) \frac{700}{700 - l} \times \frac{c' m n}{2 d k}; \qquad (85.)$$

in which a' is in inches, and c, c', d, l, m, and n are in feet. Also, a' is the area required; y is the weight per yard of the rolled-iron beam of the floor; c, their distances from centres; c', the distance from centres at which the tubular girders are placed, or the breadth of floor carried by one girder; d, the depth of the girder; k, the effective resistance of the metal per inch in the flanges of the girder; and m and n are the distances respectively from the two ends of the girder to the point at which the area of cross-section of the bottom flange is required. The rule may be thus described:

Rule LXIII.—Divide the weight per yard of the rolled-iron beams by 3 times their distance from centres; to the quotient add 140 and reserve the sum; deduct the length in feet from 700, and with the remainder as a divisor divide 700; multiply the quotient by the above reserved sum, and

by the value of c'; divide the product by the product of twice the depth into the value of k, and the quotient multiplied by the values of m and of n will be the required area of cross-section of the bottom flange at the point in the length distant from the two ends equal to m and n respectively.

Example.—In a floor of 9-inch, 70-pound beams, 4 feet from centres, what ought to be the area of the bottom flange of a tubular girder 40 feet long between bearings, 2 feet 8 inches deep, and placed 17 feet from the walls or from other girders; the area of the flange to be ascertained at every 5 feet of the length; the value of k to be put at 9000? Here $y = 70$, $c = 4$, $c' = 17$, $l = 40$, and $d = 2\frac{2}{3}$. Therefore, by the rule—

$$a' = \left(140 + \frac{70}{3 \times 4}\right) \frac{700}{700 - 40} \times \frac{17}{2 \times 2\frac{2}{3} \times 9000} \times m\,n;$$

$$a' = 145 \cdot 8\tfrac{1}{3} \times 1 \cdot 0606 \times 0 \cdot 0003541\tfrac{2}{3} \times m\,n;$$

$$a' = 0 \cdot 05478\ m\,n.$$

The values of m and n are—

At the middle........................ $m = 20$; $n = 20$
5 feet from middle.................... $m = 15$; $n = 25$
10 " " " .... $m = 10$; $n = 30$
15 " " " .... $m = 5$; $n = 35$

These give—

At the middle........... $a' = 0 \cdot 05478 \times 20 \times 20 = 21 \cdot 91$
" 5 feet from middle.... $a' = 0 \cdot 05478 \times 15 \times 25 = 20 \cdot 54$
" 10 " " " .... $a' = 0 \cdot 05478 \times 10 \times 30 = 16 \cdot 43$
" 15 " " " .... $a' = 0 \cdot 05478 \times 5 \times 35 = 9 \cdot 59$

These are the areas of uncut fibres at the points named, in the lower flange; the upper flange requires the same sizes.

189.—Tubular Iron Girders, for Floors of First-Class Stores.—The equation proper for this is (*Transverse Strains*, Art. 570)—

$$a' = \left(320 + \frac{y}{3c}\right) \frac{700}{700 - l} \times \frac{c' m n}{2 d k}, \qquad (86.)$$

a rule the same in form as that of the previous article; hence it needs no particular exemplification.

Rule LXIII. of last article may be used for this case, simply by using the constant 320 in place of that of 140.

CAST-IRON GIRDERS.

190.—Cast-Iron Girders: Inferior.—Rolled-iron beams have been so extensively introduced within a few years as to have superseded almost entirely the formerly much used cast-iron beam or girder. The tensile strength of cast iron is far inferior to that of wrought iron. This inferiority and the contingencies to which the metal is subject in casting render it very untrustworthy; it should not be used where rolled-iron beams can be procured. A very substantial girder to carry a brick wall is made by placing two or more rolled-iron beams side by side, and securing them together by bolts at mid-height of the web; placing thimbles or separators at each bolt. As there may be cases, however, in which cast-iron girders will be used, a few rules for them will here be given.

191.—Cast-Iron Girder: Load at Middle.—The form of cross-section given to this girder usually is as shown in *Fig.* 47.

FIG. 47.

In the cross-section, the bottom flange is made to contain in area four times as much as the top flange. The strength will be in proportion to the area of the bottom flange, and to the height or depth of the girder at middle. Hence, to obtain the greater strength from a given amount of material, it is requisite to make the upright part, or the web, rather thin; yet, in order to prevent injurious strains in the casting while it is cooling, the parts should be nearly equal in thickness. The thickness of the three parts—web,

top flange, and bottom flange—may be made in proportion as 5, 6, and 8.

For a weight at middle, the form of the web should be that of a triangle; the top flange forming two straight lines declining from the centre each way to the bottom flange at the ends, like the rafters of a roof to its tie-beam. From *Transverse Strains*, Art. 583, we have—

$$a' = \frac{W a l}{4850 d}, \qquad (87.)$$

which is a rule for the area in inches of the bottom flange, for a load at middle; the area of the top flange is to be equal to one fourth of that of the bottom flange. To secure this, make the width of the top flange equal to one third of the width of the bottom flange; the thickness of the former, as before directed, being made equal to $\frac{5}{8}$ or $\frac{3}{4}$ of the latter. The weight W is in pounds; the length l is in feet; and the depth d is in inches. The factor of safety a should be taken at not less than 3; better at 4 or 5.

The equation in words may be as follows:

Rule LXIV.—Multiply the weight by the length, and by the factor of safety; divide the product by 4850 times the depth at middle, and the quotient will be the area in inches of the bottom flange; divide this area by the width of the bottom flange, and the quotient will be its thickness. Of the top flange make its width equal one third that of the bottom flange, and its thickness equal to three quarters that of the latter. Make the thickness of the web equal to $\frac{5}{8}$ that of the bottom flange.

Example.—What should be the dimensions of the cross-section of a cast-iron girder 20 feet long between bearings, and 24 inches high at middle, where 30,000 pounds is to be carried; the factor of safety being put at 5?

Here we have $W = 30000$; $a = 5$; $l = 20$; and $d = 24$; therefore, by the rule—

$$a' = \frac{30000 \times 5 \times 20}{4850 \times 24} = 25 \cdot 773.$$

This is the area of the bottom flange. If the width of this flange be 12 inches, then 25·773 divided by 12 gives 2·15, or $2\frac{1}{8}$ full, as the thickness. One third of 12 equals 4, equals the width of the top flange; and $\frac{3}{4}$ of 2·15 equals 1·61, or $1\frac{5}{8}$—its thickness. The thickness of the web equals $\frac{5}{8} \times 2{\cdot}15 =$ 1·34 or $1\frac{1}{8}$ inches.

192.—Cast-Iron Girder: Load Uniformly Distributed.—The equation suitable to this is—

$$a' = \frac{U\,a\,l}{9700\,d'}, \qquad (88.)$$

a rule of like form with that of the last article; therefore, Rule LXIV. may be used for this case, simply by substituting 9700 for 4850.

193.—Cast-Iron Bowstring Girder.—An arched girder, such as that in *Fig.* 48, is technically termed a "bowstring girder." The curved part is a cast-iron beam of T form in section, and the horizontal line is a wrought-iron tie-rod attached to the ends of the arch. This girder has but little to commend it, and is by no means worthy the confidence placed in it by builders, with many of whom it is quite popular. The *brick* arch usually turned over it is adequate to sustain the entire compressive force induced from the load (the brick wall built above it), and it thereby supersedes the necessity for the *iron* arch, which is a useless expense. The tie-rod is the only useful part of the bowstring girder, but it is usually made too small, and not infrequently is seriously injured by the needless strain to which it is subjected when it is "shrunk in" to the sockets in the ends of the arch. The bowstring girder, therefore, should never be used.

FIG. 48.

194.—Substitute for the Bowstring Girder.—As the cast-iron arch of a bowstring girder serves only to resist com-

pression, its place can as well be filled by an arch of brick, footed on a pair of cast-iron skew-backs; and these held in position by a pair of wrought-iron tie-rods, as shown in *Fig.* 49. This system of construction is preferable to the bowstring girder, in that the tie-rods are not liable to injury by "shrinking in," and the cost is less. From *Transverse Strains*, Art. 596, we have—

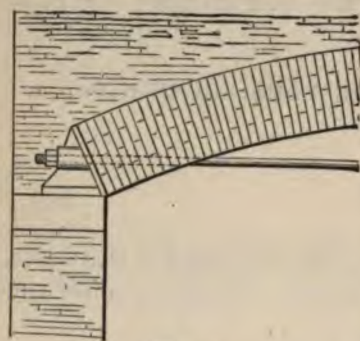


FIG. 49.

$$D = \sqrt{\frac{U\,l}{9425\,d}}, \qquad (89.)$$

an equation in which D is the diameter in inches of each of the two tie-rods of the brick arch; U is the load in pounds uniformly distributed over the arch; l is the span of the arch in feet; and d, in inches, is its versed sine, or its height measured from the centre of the tie-rod to the centre of the thickness or height of the arch at middle.

This equation may be put in words as follows:

Rule LXV.—Multiply the weight by the length; divide the product by 9425 times the depth, and the square root of the quotient will be the diameter of each rod.

Example.—What should be the diameter of each of the pair of tie-rods required to sustain a brick arch 20 feet span from centres, with a versed sine or height at middle of 30 inches, to carry a brick wall 12 inches thick and 30 feet high, weighing 100 pounds per cubic foot? The load upon this arch will be for so much of the wall as will occur over the opening, which will be about one foot less than the span of the arch, or $20 - 1 = 19$ feet. Therefore, the load will equal $19 \times 30 \times 1 \times 100 = 57{,}000$ pounds; and hence, $U = 57000$, $l = 20$, $d = 30$, and, by the rule—

$$D = \sqrt{\frac{57000 \times 20}{9425 \times 30}} = \sqrt{4 \cdot 0318} = 2 \cdot 008;$$

or, the diameter of each rod is required to be 2 inches.

FRAMED GIRDERS.

195.—Graphic Representation of Strains.—In the first part of this section, commencing at *Art.* 71, the method was developed of ascertaining the strains in the various parts of a frame by the parallelogram or triangle of forces. The method, so far as there explained, is adequate to solve simple cases; but when more than three pieces of a frame converge in one point, the task by that method becomes difficult. This difficulty, however, disappears when recourse is had to the method known as that of "Reciprocal Figures, Frames, and Diagrams of Forces," proposed by Professor I. Clerk Maxwell in 1867. This is an extension of the method by the triangle of forces, and may be illustrated as follows:

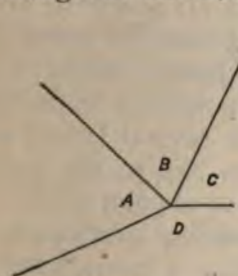


FIG. 50.

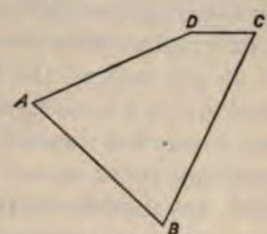


FIG. 51.

Let the lines in *Fig.* 50 represent, in direction and amount, four converging forces in equilibrium in any frame, as, for example, the truss of a roof; let the lines in *Fig.* 51 be drawn parallel to those in *Fig.* 50, in the manner following, namely: Let the line $A\ B$ be drawn parallel with the line of *Fig.* 50 which is between the corresponding letters A and B, and let it be of corresponding length; from B draw the line $B\ C$ parallel with the line of *Fig.* 50 which is between the letters B and C, and of corresponding length; then from C draw $C\ D$, and from A draw $A\ D$, respectively parallel with the lines of *Fig.* 50 designated by the corresponding letters, and extend them till they intersect at D. The lengths of these two lines, the last two drawn, are determined by the point D where they intersect; their lengths, therefore, need not be previously known. The lengths of the lines in *Fig.* 51 are respectively in proportion to the

several strains in *Fig.* 50, provided these strains are in equilibrium. *Fig.* 51 is termed a closed polygon of forces. A system of such polygons, one for each point, in the frame where forces converge, so constructed that no line representing a force shall be repeated, is termed a *diagram of forces*. This diagram of forces is a reciprocal of the frame from which it is drawn, its lines and angles being the same. The facility of tracing the forces in the diagram of forces depends materially upon the system of lettering here shown, and which was proposed by Mr. Bow, in his excellent work on the *Economics of Construction*. In this system each line of the frame is designated by the two letters which it separates; thus the line between A and B is called line AB; that between C and D is called line CD; and so of others; and in the diagram the corresponding lines are called by the same letters, but here the letters designating the line are, as usual, at the ends of the line. Any point in a frame where forces converge is designated by the several letters which cluster around it; as, for example, in *Fig.* 50, the point of convergence there shown is designated as point $A\ B\ C\ D$.

This invaluable method of defining graphically the strains in the various pieces composing a frame, such as a girder or roof-truss, is remarkably simple, and is of general application. Its utility will now be exemplified in its application to framed girders, and afterwards to roof-trusses.

196.—Framed Girders.—Girders of solid timber are useful for the support of floors only where posts are admissible as supports, at intervals of from 8 to 15 feet. For unobstructed long spans it becomes requisite to construct a frame to serve as a girder (*Arts.* 163, 182). A frame of this kind requires two horizontal pieces, a top and a bottom chord, and a system of struts and suspension-pieces by which the top and bottom chords are held in position, and the strains from the load are transmitted to the bearings at the ends of the girders. Various methods of arranging these struts and ties have been proposed. One of the most simple and effective is shown in *Fig.* 52, forming a series of isosceles triangles. The proportion between the length and height of a girder is important as an element of economy

both of space and cost. When circumstances do not control in limiting the height, it may be determined by this equation from *Transverse Strains*, Art. 624—

$$d = \frac{(175 + l)\,l}{2400}; \qquad (90.)$$

in which d is the depth or height between the axes of the top and bottom chords, and l is the length between the centres of bearings at the supports (d and l both in feet). This equation in words is as follows:

Rule LXVI.—To the length add 175; multiply the sum by the length; divide the product by 2400, and the quotient will be the required height between the axes of the top and bottom chords.

Example.—What should be the depth of a girder which is 40 feet long between the centres of action at the supports? For this the rule gives—

$$d = \frac{(175 + 40) \times 40}{2400} = 3 \cdot 58\tfrac{1}{3};$$

or, the proper depth for economy of material is 3 feet and 7 inches.

The number of bays, panels, or triangles into which the bottom chord may be divided is a matter of some consideration. Usually girders from—

20	to	59	feet	long	should	have	5	bays.
59	"	85	"	"	"	"	6	"
85	"	107	"	"	"	"	7	"
107	"	127	"	"	"	"	8	"
127	"	146	"	"	"	"	9	"

197.—Framed Girder and Diagram of Forces.—Let *Fig.* 52 represent a framed girder of six bays of, say, 11 feet each, or of a total length of 66 feet.

The lines shown are the axial lines, or the imaginary lines passing through the axes of the several pieces composing the frame. The six arrows indicate the six pressures into which the equally distributed load is supposed to be divided. Each of these is at the apex of a triangle, the base of which lies along the lower chord.

The spaces between the arrows are lettered; so, also, the space between the last arrow at either end and the point of support has a letter, and so has each triangle, and there is one for the space beneath the lower chord. These letters are to be used in describing the diagram of forces, as was explained in *Art.* 195. The diagram of forces (*Fig.* 53) for this girder-frame is drawn as follows, namely: Upon a verti-

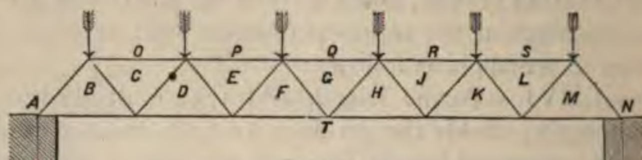


FIG. 52.

cal line *A N* mark the points *A*, *O*, *P*, *Q*, *R*, *S*, and *N*, at equal distances, to represent the six equal vertical pressures indicated by the arrows in *Fig.* 52. The equal distances *A O*, *O P*, etc., may be made of any convenient size; but it will

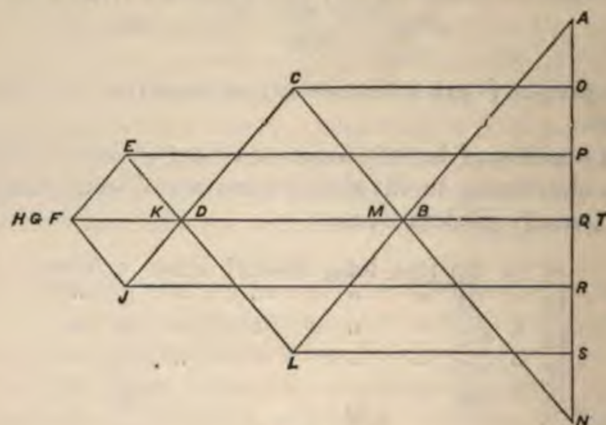


FIG. 53.

serve to facilitate the measurement of the forces in the diagram if they are made by a scale of equal parts, and the number of parts given to each division be made equal to the number of tons of 2000 pounds each which is contained in the pressure indicated by each arrow. On this vertical line the distance *A O* represents the load at the apex of the triangle *B*, or the point *A O C B* (*Art.* 195); the distance *O P*

represents the weight at the second arrow, or at the point $O P E D C$, and so of the rest. If the weights upon the points in the upper chord had been unequal, then the division of the vertical line $A N$ would have had to be correspondingly unequal, each division being laid off by the scale, to accord with the weight represented by each. The line of loads, $A N$, being adjusted, the other lines are drawn from it (*Art.* 195), so as to make a closed polygon for the forces converging at each point of the frame, *Fig.* 52—commencing with the point $A B T$, *Fig.* 52, where there are three forces, namely, the force acting through the inclined strut $A B$, the horizontal force in $B T$, and the vertical reaction $A T$ at the point of support. This last is equal to half the entire load, or equal to the pressure indicated by the three arrows, $A O$, $O P$, and $P Q$, and is represented in *Fig.* 53 by $A Q$ or $A T$. From the point Q draw a horizontal line $Q B$; this is parallel with the force $B T$ of *Fig.* 52, in the lower chord. From the point A draw $A B$ parallel with the strut $A B$ of *Fig.* 52. This line intersects the line $B T$ in B and closes the polygon $A B T A$; the point B defines the length of the lines $A B$ and $B T$, and these lines measured by the scale by which the line of loads was constructed give the required pressures in the corresponding lines, $A B$ and $B T$, of *Fig.* 52.

Taking next the point $A B C O$, where four forces meet, of which we already have two, namely, the force in the strut $A B$ and the load $A O$—from the point O draw the horizontal line $O C$; this is parallel to the horizontal force $O C$ of *Fig.* 52. Now from B draw $B C$ parallel with the suspension-piece $B C$ of *Fig.* 52. This line intersects $O C$ in C, and the point C limits the lines $O C$ and $B C$ and closes the polygon $A B C O A$, the four sides of which are respectively in proportion to the four forces converging at the point $A B C O$ of *Fig.* 52, and when measured by the scale by which the line of loads was constructed give the required strains respectively in each. Taking next the point $B C D T$, where four forces converge, of which we already have two, $B C$ and $B T$—from B extend the horizontal line $T B$ to D; from C draw $C D$ parallel with $C D$ of *Fig.* 52, and extend it to intersect $T D$ in D, and thus close the polygon $T B C D T$.

The lines in a part of this polygon coincide—those from B to T; this is because the two strains $B\ T$ and $D\ T$, *Fig.* 52, lie in the same horizontal line. Again, taking the point $O\ C\ D\ E\ P$, where five forces meet, three of which, $O\ P$, $O\ C$, and $C\ D$, we already have—draw from D the line $D\ E$ parallel with $D\ E$ of *Fig.* 52, and from P the line $P\ E$ horizontally or parallel with $P\ E$ of *Fig.* 52. These two lines intersect at E and close the polygon $P\ O\ C\ D\ E\ P$, the sides of which measure the forces converging in the point $P\ O\ C\ D\ E$, *Fig.* 52. Next in order is the point $D\ E\ F\ T$, *Fig.* 52, where four forces meet, two of which, $T\ D$ and $D\ E$, are known. From E draw $E\ F$ parallel with $E\ F$ in *Fig.* 52; and from T, $T\ F$ parallel with $T\ F$ in *Fig.* 52; these two lines meet in F and close the polygon $T\ D\ E\ F\ T$, the sides of which measure the required strains in the lines converging at the point $D\ E\ F\ T$, *Fig.* 52. Taking next the point $P\ E\ F\ G\ Q$, *Fig.* 52, where five forces meet, of which we already have three, $Q\ P$, $P\ E$, and $E\ F$—from F draw a line parallel with $F\ G$ of *Fig.* 52, and from Q a line parallel with $Q\ G$ of *Fig.* 52. These two intersect at G and complete the polygon $Q\ P\ E\ F\ G\ Q$, the lines of which measure the forces converging at $P\ E\ F\ G\ Q$ in *Fig.* 52.

In this last polygon, a peculiarity seems to indicate an error: the line $F\ G$ has no length; it begins and ends at the same point; or, rather, the polygon is complete without it. This is easily understood when it is considered that the two lines $F\ G$ and $G\ H$ do not contribute any strength towards sustaining the loads $P\ Q$ and $Q\ R$, and in so far as these weights are concerned they might be dispensed with, and the space occupied by the three triangles F, G, and H left free, and be designated by only one letter instead of three. Thus it appears that there are only four instead of five forces at the point $P\ E\ F\ G\ Q$, and that the four are represented by the lines of the polygon $Q\ P\ E\ F\ Q$.

The peculiarity above explained arises from considering loads only on the top chord: the analysis of the case is correct as worked from the premises given; but in practice there is always more or less load on the bottom chord at the middle, which should be considered. This will be included in a case proposed in the next article. One half of the dia-

gram of forces is now complete. The other half being exactly the same, except that it is in reversed order, need not here be drawn.

198.—Framed Girders: Load on Both Chords.—Let *Fig.* 54 represent the axial lines of a girder carrying an

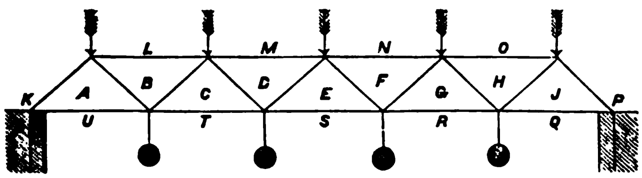


FIG. 54.

equally distributed load on each chord, represented by the arrows and balls shown in the figure. Let each bay measure 10 feet, or the length of the girder be 50 feet, and its height

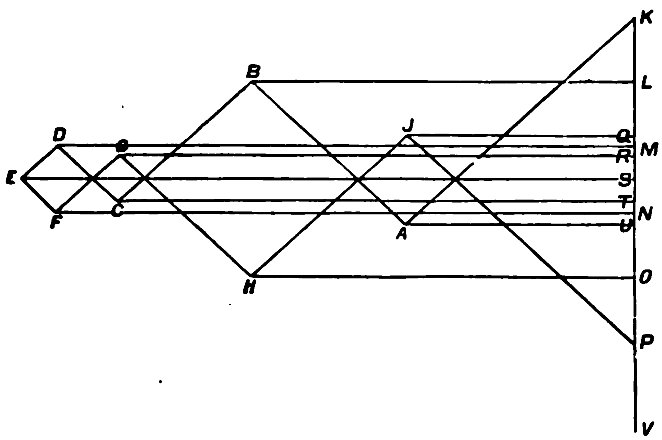


FIG. 55.

be $4\frac{1}{2}$ feet. The diagram of forces (*Fig.* 55) for this girder is obtained thus:

The plan of the girder, *Fig.* 54, requires to be lettered as shown; having one letter within each panel and outside the frame, and one between every two weights or strains. Then, in *Fig.* 55, mark the vertical line *KV* at *L, M, N, O,*

and P, dividing it by scale into equal parts, corresponding with the weights on the top chord represented by the arrows. For example, if the load at each arrow equals 6½ tons, make $K L$, $L M$, $M N$, etc., each equal to 6½ parts of the scale. Then $K P$ will equal the total load on the top flange. Make the distance $P V$ equal to the sum of the loads on the bottom chord. Then $K V$ equals the total load on the girder. Bisect $K V$ in U; then $K U$ or $U V$ equals half the total load; consequently, equals the reaction of the bearing at K or P of *Fig.* 54.

Now, to obtain the polygon of forces converging at $K A U$, *Fig.* 54, we have one of these forces, $K U$, or the reaction of the bearing at $K A U$, equal to $K U$, *Fig.* 55. From U draw $U A$ parallel with $U A$ of *Fig.* 54, and from K draw $K A$ parallel with the strut $K A$, *Fig.* 54, and intersecting the line $U A$ at A, a point which marks the limit of $K A$ and $U A$, and closes the polygon $K A U K$, the sides of which are in proportion respectively to the three strains which converge at the point $A U K$, *Fig.* 54. For example, since the line $K U$ by scale measures the vertical reaction, $K U$, of the bearing at $A U K$, *Fig.* 54, therefore the line $K A$ of the diagram of forces by the same scale measures the strain in the strut $K A$, *Fig.* 54, and the line $A U$ of the diagram by the same scale measures the strain in the bottom chord at $A U$, *Fig.* 54. For the strains converging at $K A B L$, *Fig.* 54, of which two, $K A$ and $K L$, are already known, we draw from A the line $A B$ parallel with the line $A B$, *Fig.* 54, and from L draw $L B$ parallel with $L B$, *Fig.* 54, meeting $A B$ at B, a point which limits the two lines and closes the polygon $K A B L K$, the lines of which are in proportion respectively to the strains converging at the point $K A B L$, *Fig.* 54, as before explained. Of the five strains converging at $U A B C T$, we already have three—$T U$, $U A$, and $A B$; to obtain the other two, make $U Q$ equal to $P V$, equal to the total load upon the lower flange; divide $U Q$ into four equal parts, $Q R$, $R S$, $S T$, and $T U$, corresponding with the four weights on the lower chord, and represented by the four balls, *Fig.* 54. Now, from T, the point marking the first of these divisions, draw $T C$ parallel with $T C$, *Fig.* 54, and from B draw $B C$ paral-

lel with the strut BC, *Fig.* 54, meeting TC in C, a point which limits the lines BC and TC and closes the polygon $TUABCT$, the sides of which are in proportion respectively to the strains converging in the point $TUABCT$, *Fig.* 54. Of the five forces converging at $MLBCD$, we already have three—ML, LB, and BC; to obtain the other two, from M draw MD parallel with MD, *Fig.* 54, and from C draw CD parallel with CD, *Fig.* 54, meeting MD at D, a point limiting the lines MD and CD and closing the polygon $MLBCDM$, the sides of which are in proportion to the strains converging at the point $MLBCD$, *Fig.* 54. Of the five forces converging at the point $STCDE$, three—ST, TC, and CD—are known; to obtain the other two, from S draw SE parallel with SE, *Fig.* 54, and from D draw DE, parallel with the strut DE, *Fig.* 54, meeting the line SE in E, a point limiting the two lines SE and DE and closing the polygon $STCDES$, the sides of which are in proportion to the strains converging at $STCDE$, *Fig.* 54. One half of the strains in *Fig.* 54 are now shown in its diagram of forces, *Fig.* 55; and since the two halves of the girder are symmetrical, the forces in one half corresponding to those in the other, hence the lines of the diagram for one half of the forces may be used for the corresponding forces of the other half.

199.—Framed Girders: Dimensions of Parts.—The parts of a framed girder are the two horizontal chords (top and bottom) and the diagonals—the struts and ties. The top chord is in a state of compression, while the bottom chord experiences a tensile strain. Those of the diagonal pieces which have a direction from the top to the bottom chord, and from the middle towards one of the bearings of the girder, as KA, BC, or DE, *Fig.* 54, are *struts*, and are subjected to compression. The diagonal pieces which have a direction from the bottom to the top chord, and from the middle towards one of the supports, as AB or CD, *Fig.* 54, are *ties*, and are subjected to extension, (*Art.* 83). The amount of strain in each piece in a framed girder having been ascertained in a diagram of forces, as shown in *Arts.* 197 and 198, the dimensions of each piece may be obtained

by rules already given. The dimensions of the pieces in a state of compression are to be ascertained by the rules for posts in *Arts.* 107 to 114, and those in a state of tension by *Arts.* 117 to 119 (see *Arts.* 226 to 229). Care is required, in obtaining the size of the lower chord, to allow for the joints which necessarily occur in long ties, for the reason that timber is not readily obtained sufficiently long without splicing. Usually, in cases where the length of the girder is too great to obtain a bottom chord in one piece, the chord is made up of vertical lamina, and in as long lengths as practicable, and secured with bolts. A chord thus made will usually require about twice the material: or, its sectional area of cross-section will require to be twice the size of a chord which is in one whole piece; and in this chord it is usual to put the factor of safety at from 8 to 10.

The diagonal ties are usually made of wrought iron, and it is well to secure the struts, especially the end ones, with iron stirrups and bolts. And, to prevent the evil effects of shrinkage, it is well to provide iron bearings extending through the depth of each chord, so shaped that the struts and rods may have their bearings upon it, instead of upon the wood.

PARTITIONS.

200.—Partitions.—Such partitions as are required for the divisions in ordinary houses are usually formed by timber of small size, termed *studs* or *joists*. These are placed upright at 12 or 16 inches from centres, and well nailed. Upon these studs lath are nailed, and these are covered with plastering. The strength of the plastering depends in a great measure upon the clinch formed by the mortar which has been pressed through between the lath. That this clinch may be interfered with in the least possible degree, it is proper that the edges of the partition-joists which are presented to receive the lath should be as narrow as practicable; those which are necessarily large should be reduced by chamfering the corners. The derangements in floors, plastering, and doors which too frequently disfigure the interior of pretentious houses with gaping cracks in the

astering and in the door-casings are due in nearly all cases defective partitions, and to the shrinkage of floor-timbers. plastered partition is too heavy to be trusted upon an ordiary tier of beams, unless so braced as to prevent its weight rom pressing upon the beams. This precaution becomes especially important when, in addition to its own weight, the partition serves as a girder to carry the weight of the floor-beams next above it. In order to reduce to the smallest practicable degree the derangements named, it is important that the studs in a partition should be *trussed* or braced so as to throw the weight upon firmly sustained points in the construction beneath, and that the timber in both partitions and floors should be well seasoned and carefully framed. To avoid the settlement due to the shrinkage of a tier of beams, it is important, in a partition standing over one in the story below or over a girder, that the studs pass between the beams to the plate of the lower partition, or to the girder; and, to be able to do this, it is also important to arrange the partitions of the several stories vertically over each other. All principal partitions should be of brick, especially such as are required to assist in sustaining the floors of the building.

FIG. 56.

201.—Examples of Partitions.—*Fig.* 56 represents a partition having a door in the middle. Its construction is simple but effective. *Fig.* 57 shows the manner of constructing a

partition having doors near the ends. The truss is formed above the door-heads, and the lower parts are suspended from it. The posts *a* and *b* are halved, and nailed to the tie *c d* and the sill *e f*. The braces in a trussed partition

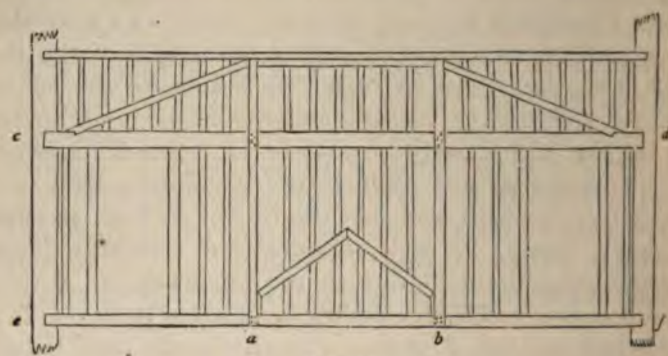


FIG. 57.

should be placed so as to form, as near as possible, an angle of 40 degrees with the horizon. The braces in a partition should be so placed as to discharge the weight upon the

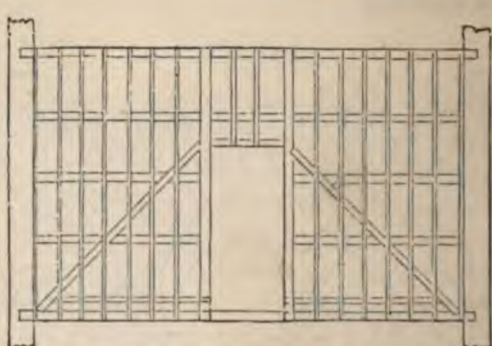


FIG. 58.

points of support. All oblique pieces that fail to do this should be omitted.

When the principal timbers of a partition require to be large for the purpose of greater strength, it is a good plan

omit the upright filling-in pieces, and in their stead to lace a few horizontal pieces, as in *Fig.* 58, in order that upon hese and the principal timbers upright battens may be nailed at the proper distances for lathing. A partition thus constructed requires a little more space than others; but it has the advantage of insuring greater stability to the plastering, and also of preventing to a good degree the conversation of one room from being overheard in the adjoining one. Ordinary partitions are constructed with 3×4, 3×5, or 4×6 inch joists, for the principal pieces, and with 2×4, 2×5, or 2×6 filling-in studs, well strutted at intervals of about 5 feet. When a partition is required to support, in addition to its own weight, that of a floor or some other burden resting upon it, the dimensions of the timbers should be ascertained, by applying the principles which regulate the laws of pressure and those of the resistance of timber, as explained in the first part of this section, and in *Arts.* 196 to 199 for framed girders. The following data may assist in calculating the amount of pressure upon partitions:

White-pine timber weighs from 22 to 32 pounds per cubic foot, varying in accordance with the amount of seasoning it has had. Assuming it to weigh 30 pounds, the weight of the beams and floor-plank in every superficial foot of the flooring will be—

6 pounds	when	the beams	are	3×8 inches,	and	placed	20	inches	from centres.
7½	"	"	"	3×10	"	"	18	"	"
9	"	"	"	3×12	"	"	16	"	"
11	"	"	"	3×12	"	"	12	"	"
13	"	"	"	4×12	"	"	12	"	"
13	"	"	"	4×14	"	"	14	"	"

In addition to the beams and plank, there is generally the *plastering* of the ceiling of the apartments beneath, and sometimes the *deafening*. Plastering may be assumed to weigh 9 pounds per superficial foot, and deafening 11 pounds.

Hemlock weighs about the same as white pine. A partition of 3×4 joists of hemlock, set 12 inches from centres, therefore, will weigh about 2½ pounds per foot superficial and when plastered on both sides, 20½ pounds.

ROOFS.

202.—Roofs.—In ancient Norman and Gothic buildings, the walls and buttresses were erected so massive and firm that it was customary to construct their roofs without a tie-beam, the walls being abundantly capable of resisting the lateral pressure exerted by the rafters. But in modern buildings, usually the walls are so slightly built as to be incapable of resisting much if any oblique pressure; hence the necessity of care in constructing the roof so as to avoid oblique and lateral strains. The roof so constructed, instead of tending to separate the walls, will bind and steady them.

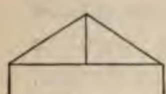


FIG. 59.


FIG. 60.

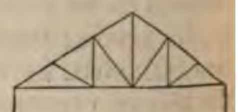


FIG. 61.


FIG. 62.

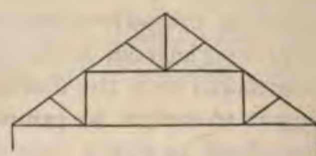


FIG. 63.


FIG. 64.


FIG. 65.


FIG. 66.


FIG. 67.

203.—Comparison of Roof-Trusses.—Designs for roof-trusses, illustrating various principles of roof construction, are herewith presented.

The designs at *Figs.* 59 to 63 are distinguished from those at *Figs.* 64 to 67 by having a horizontal tie-beam. In the latter group, and in all designs similarly destitute of the horizontal tie at the foot of the rafters, the strains are much greater than in those having the tie, unless the truss be pro-

tected by exterior resistance, such as may be afforded by competent buttresses.

To the uninitiated it may appear preferable, in *Fig.* 64, to extend the inclined ties to the rafters, as shown by the dotted lines. But this would not be beneficial; on the contrary, it would be injurious. The point of the rafter where the tie would be attached is near the middle of its length, and consequently is a point the least capable of resisting transverse strains. The weight of the roofing itself tends to bend the rafter; and the inclined tie, were it attached to the rafter, would, by its tension, have a tendency to increase this bending. As a necessary consequence, the feet of the rafters would separate, and the ridge descend.

In *Fig.* 65 the inclined ties are extended to the rafters; but here the horizontal strut or straining beam, located at the points of contact between the ties and rafters, counteracts the bending tendency of the rafters and renders these points stable. In this design, therefore, and only in such designs, it is permissible to extend the ties through to the rafters. Even here it is not advisable to do so, because of the increased strain produced. (See *Figs.* 77 and 79.) The design in *Fig.* 64, 66, or 67 is to be preferred to that in *Fig.* 65.

204.—Force Diagram: Load upon Each Support.—By a comparison of the force diagrams hereinafter given, of each of the foregoing designs, we may see that the strains in the trusses without horizontal tie-beams at the feet of the rafters are greatly in excess of those having the tie. In constructing these diagrams, the first step is to ascertain the reaction of, or load carried by, each of the supports at the ends of the truss. In symmetrically loaded trusses, the weight upon each support is always just one half of the whole load.

205.—Force Diagram for Truss in *Fig.* 59.—To obtain the force diagram appropriate to the design in *Fig.* 59, first letter the figure as directed in *Art.* 195, and as in *Fig.* 68. Then draw a vertical line, $E\ F$ (*Fig.* 69), equal to the weight W at the apex of roof; or (which is the same thing in effect) equal to the sum of the two loads of the roof, one extending on each side of W half-way to the foot of the rafter. Di-

vide EF into two equal parts at G. Make GC and GD each equal to one half of the weight N. Now, since EG is equal to one half of the upper load, and GD to one half of the lower load, therefore their sum, $EG + GD = ED$, is equal to one half of the total load, or to the reaction of each support, E or F. From D draw DA parallel with DA of *Fig.* 68, and from E draw EA parallel with EA of *Fig.* 68. The three lines of the triangle AED represent the strains, respectively, in the three lines converging at the point ADE of *Fig.* 68. Draw the other lines of the diagram parallel with the lines of

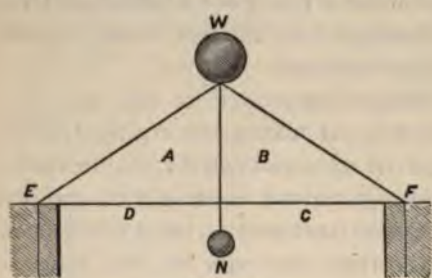


FIG 68.

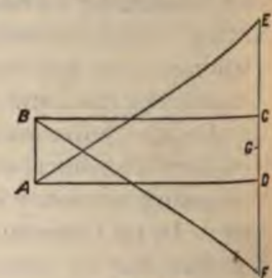


FIG. 69.

Fig. 68, and as directed in *Arts.* 195 and 197. The various lines of *Fig.* 69 will represent the forces in the corresponding lines of *Fig.* 68; bearing in mind (*Art.* 195.) that while a line in the force diagram is designated in the usual manner by the letters at the two ends of it, a line of the frame diagram is designated by the two letters between which it passes. Thus, the horizontal lines AD, the vertical lines AB, and the inclined lines AE have these letters at their ends in *Fig.* 69, while they pass between these letters in *Fig.* 68.

206.—Force Diagram for Truss in *Fig.* 60.—For this truss we have, in *Fig.* 70, a like design, repeated and lettered as required. We here have one load on the tie-beam, and three loads above the truss: one on each rafter and one at the ridge. In the force diagram, *Fig.* 71, make GH, HJ, and JK, by any convenient scale, equal respectively to the weights GH, HJ, and JK of *Fig.* 70. Divide GK into two equal parts at L. Make LE and LF each equal to one half the weight EF (*Fig.* 70). Then GF is equal to one half the

total load, or to the load upon the support G (*Art.* 205). Complete the diagram by drawing its several lines parallel with the lines of *Fig.* 70, as indicated by the letters (see *Art.* 205), commencing with GF, the load on the support G (*Fig.* 70). Draw from F and G the two lines FA and GA parallel with these lines in *Fig.* 70. Their point of intersection defines the point A. From this the several points B, C, and D are developed, and the figure completed. Then the lines in *Fig.* 71 will represent the forces in the corresponding lines of *Fig.* 70, as indicated by the lettering. (See *Art.* 195.)

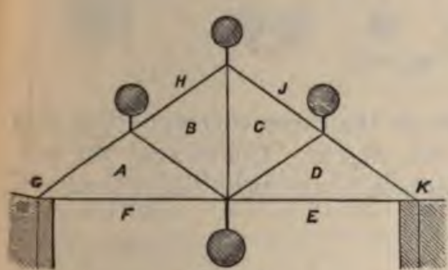


FIG. 70.

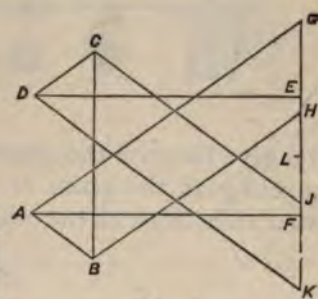


FIG. 71.

207.—Force Diagram for Truss in *Fig* 61.—For this truss we have, in *Fig.* 72, a similar design, properly prepared by weights and lettering; and in *Fig.* 73 the force diagram appropriate to it.

In the construction of this diagram, proceed as directed in the previous example, by first constructing NS, the vertical line of weights; in which line NO, OP, PQ, QR, and RS are made respectively equal to the several weights above the truss in *Fig.* 72. Then divide NS into two equal parts at T. Make TK and TL each equal to the half of the weight KL. Make JK and LM equal to the weights JK and LM of *Fig.* 72. Now, since MN is equal to one half of the weights above the truss plus one half of the weights below the truss, or half of the whole weight, it is therefore the weight upon the support N (*Fig.* 72), and represents the reaction of that support. A horizontal line drawn from M will meet the inclined line drawn from N, parallel with the rafter AN (*Fig.* 72), in the

point *A*, and the three sides of the triangle *A M N*, *Fig.* 73, will give the strains in the three corresponding lines meeting at the point *A M N*, *Fig.* 72. The sides of the triangle *H J S*, *Fig.*

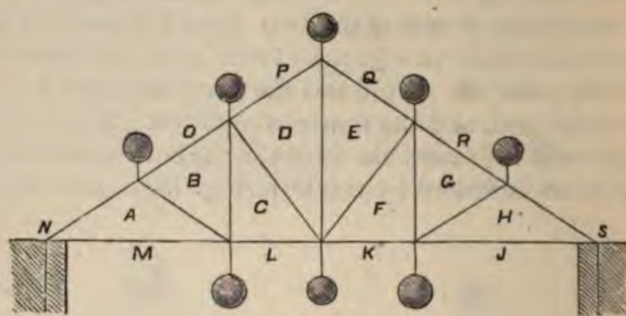


FIG. 72.

73, give likewise the strains in the three corresponding lines meeting at the point *H J S*, *Fig.* 72. Continuing the construction, draw all the other lines of the force diagram parallel

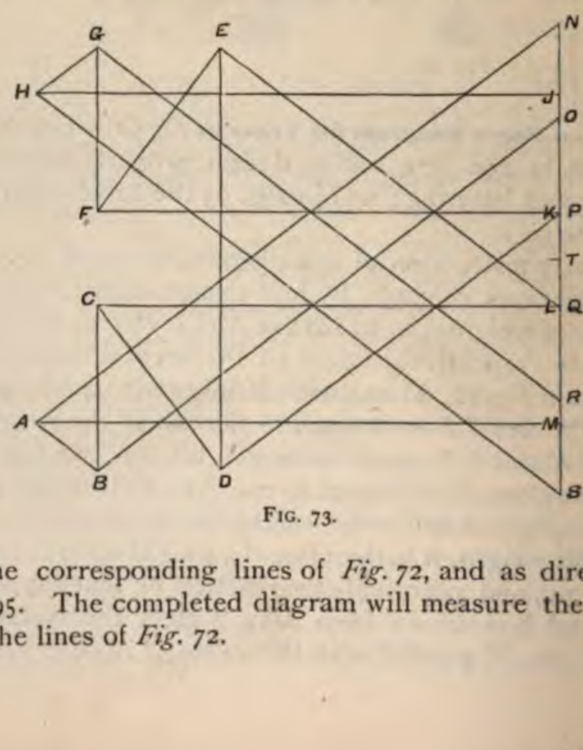


FIG. 73.

with the corresponding lines of *Fig.* 72, and as directed in *Art.* 195. The completed diagram will measure the strains in all the lines of *Fig.* 72.

208.—Force Diagram for Truss in *Fig.* 63.—The roof truss indicated at *Fig.* 63 is repeated in *Fig.* 74, with the ad

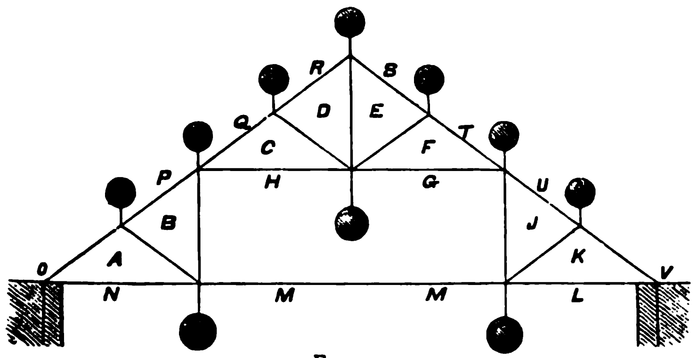


FIG. 74.

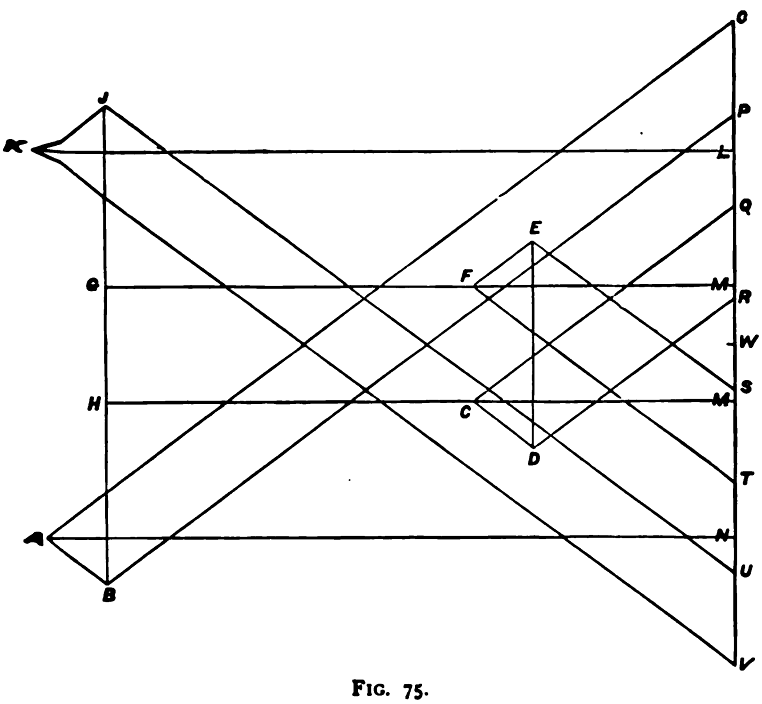


FIG. 75.

dition of the lettering required for the construction of the force diagram, *Fig.* 75.

In this case there are seven weights, or loads, above the truss, and three below. Divide the vertical line *O V* at *W* into two equal parts, and place the lower loads in two equal parts on each side of *W*. Owing to the middle one of these loads not being on the tie-beam with the other two, but on the upper tie-beam, the line *G H*, its representative in the force diagram, has to be removed to the vertical *B J*, and the letter *M* is duplicated. The line *N O* equals half the whole weight of the truss, or $3\frac{1}{2}$ of the upper loads, plus one of the lower loads, plus half of the load at the upper tie-beam. It is, therefore, the true reaction of the support *N O*, and *A N* is the horizontal strain in the beam there. It will be observed also that while *H M* and *G M* (*Fig.* 75), which are equal lines, show the strain in the lower tie-beam at the middle of the truss, the lines *C H* and *F G*, also equal but considerably shorter lines, show the strains in the upper tie-beam. Ordinarily, in a truss of this design, the strain in the upper beam would be equal to that in the lower one, which becomes true when the rafters and braces above the upper beam are omitted. In the present case, the thrusts of the upper rafters produce tension in the upper beam equal to *C M* or *F M* of *Fig.* 75, and thus, by counteracting the compression in the beam, reduce it to *C H* or *F G* of the force diagram, as shown.

209.—Force Diagram for Truss in *Fig.* 64.—The force diagram for the roof-truss at *Fig.* 64 is given in *Fig.* 77, while *Fig.* 78 is the truss reproduced, with the lettering requisite for the construction of *Fig.* 77.

The vertical *E F* (*Fig.* 77) represents the load at the ridge. Divide this equally at *W*, and place half the lower weight each side of *W*, so that *C D* equals the lower weight. Then *E D* is equal to half the whole load, and equal to the reaction of the support *E* (*Fig.* 76). The lines in the triangle *A D E* give the strains in the corresponding lines converging at the point *A D E* of *Fig.* 76. The other lines, according to the lettering, give the strains in the corresponding lines of the truss. (See *Art.* 195.)

210.—Force Diagram for Truss in *Fig.* 65.—This truss is reproduced in *Fig.* 78, with the letters proper for use in the force diagram, *Fig.* 79.

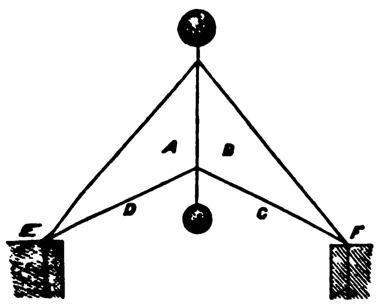


FIG. 76.

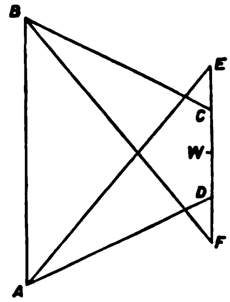


FIG. 77.

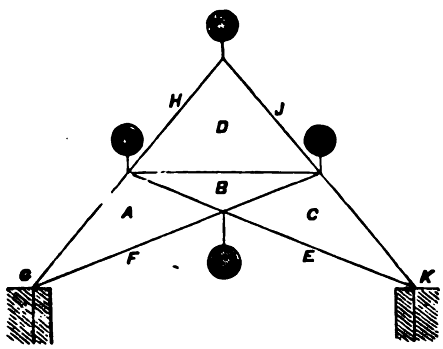


FIG. 78.

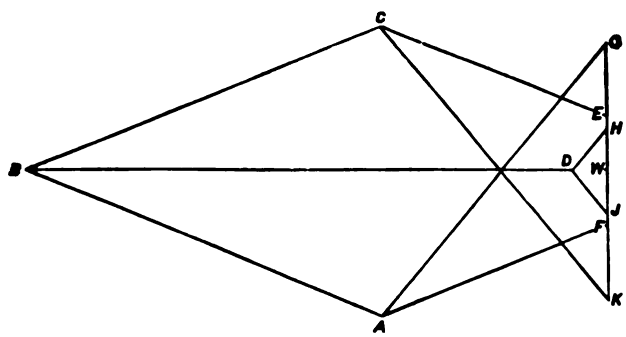


FIG. 79.

Here the vertical $G K$, containing the three upper loads $G H$, $H J$, and $J K$, is divided equally at W, and the lower

load $E F$ is placed half on each side of W, and extends from E to F. Then $F G$ represents one half of the whole load of the truss, and therefore the reaction of the support G (*Fig.* 78). Drawing the several lines of *Fig.* 79 parallel with the corresponding lines of *Fig.* 78, the force diagram is complete, and the strains in the several lines of 78 are measured by the corresponding lines of 79. (See *Art.* 195.)

A comparison of the force diagram of the truss in *Fig.* 76 with that of the truss in *Fig.* 78 shows much greater strains in the latter, and we thus see that *Fig.* 76 or 64 is the more economical form.

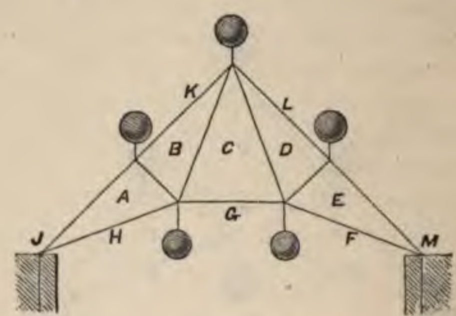


FIG. 80.

211.—Force Diagram for Truss in *Fig.* 66.—This truss is reproduced and prepared by proper lettering in *Fig.* 80, and its force diagram is given in *Fig.* 81.

Here the vertical $J M$ contains the three upper loads $J K$, $K L$, and $L M$. Divide $J M$ into two equal parts at G, and make $F G$ and $G H$ respectively equal to the two loads $F G$ and $G H$ of *Fig.* 80. Then $H J$ represents one half of the whole weight of the truss, and therefore the reaction of the support J. From H and J draw lines parallel with $A H$ and $A J$ of *Fig.* 80, and the sides of the triangle $A H J$ will give the strains in the three lines concentrating in the point $A H J$ (*Fig.* 80). The other lines of *Fig.*

81 are all drawn parallel with their corresponding lines in *Fig.* 80, as indicated by the lettering. (See *Art.* 195.)

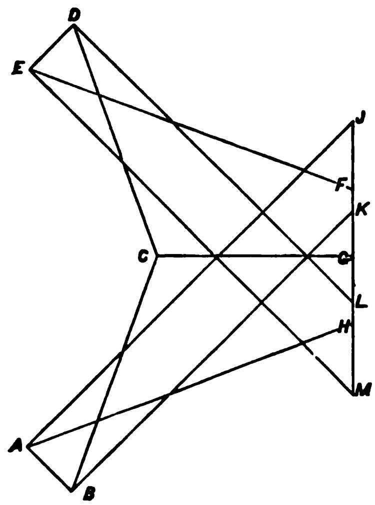


FIG. 81.

212.—Roof-Truss: Effect of Elevating the Tie-Beam.— From Arts. 670, 671, *Transverse Strains*, it appears that the

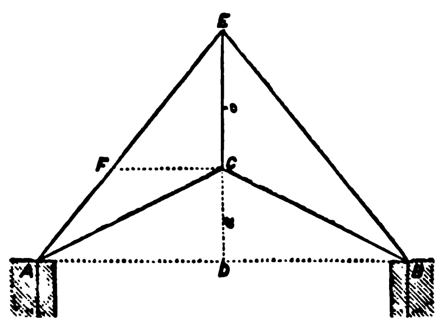


FIG. 82

effect of substituting inclined ties for the horizontal tie at feet of rafters is—

$$V = P\frac{a}{b} \qquad (91.)$$

in which P represents half the weight of the whole truss and the load upon it; $a+b$ = height of the truss at middle above a horizontal line drawn at the feet of the rafters; a equals the height from this line to the point where the two inclined ties meet; b, the height thence to the top of the truss; and V, the additional vertical strain at the middle of the truss due to elevating the tie from a horizontal line.

Examples are given to show that when the elevation of the tie equals $\frac{1}{4}$ of the whole height, the vertical strain thereby induced is equal to a weight which equals $\frac{1}{3}$ of half the whole load; and that when the elevation equals half the whole height, the vertical strain is equal to half the whole load. This is the strain in the vertical rod at middle. The strains in the rafters and inclined ties are proportionately increased.

213.—Planning a Roof.—In designing a roof for a building, the first point requiring attention is the location of the trusses. These should be so placed as to secure solid bearings upon the walls; care being taken not to place either of the trusses over an opening, such as those for windows or doors, in the wall below. Ordinarily, trusses are placed so as to be centrally over the piers between the windows; the number of windows consequently ruling in determining the number of trusses and their distances from centres. This distance should be from ten to twenty feet; fifteen feet apart being a suitable medium distance. The farther apart the trusses are placed, the more they will have to carry: not only in having a larger surface to support, but also in that the roof-timbers will be heavier; for the size and weight of the roof-beams will increase with the span over which they have to reach.

In the roof-covering itself, the roof-planking may be laid upon jack-rafters, carried by purlins supported by the trusses; or upon roof-beams laid directly upon the back of the principal rafters in the trusses. In either case, proper

struts should be provided, and set at proper intervals to resist the bending of the rafter. In case purlins are used, one of these struts should be placed at the location of each purlin.

The number of these points of support rules largely in determining the design for the truss, thus:

For a short span, where the rafter will not require support at an intermediate point, *Fig.* 59 or 64 will be proper.

For a span in which the rafter requires supporting at one intermediate point, take *Fig.* 60, 65, or 66.

For a span with two intermediate points of support for the rafter, take *Fig.* 61 or 67.

For a span with three intermediate points, take *Fig.* 63.

Generally, it is found convenient to locate these points of support at nine to twelve feet apart. They should be sufficiently close to make it certain that the rafter will not be subject to the possibility of bending.

214.—Load upon Roof-Truss.—In constructing the force diagram for any truss, it is requisite to determine the points of the truss which are to serve as points of support (see *Figs.* 70, 72, etc.), and to ascertain the amount of strain, or loading, which will occur at every such point.

The points of support along the rafters will be required to sustain the roofing timbers, the planking, the slating, the snow, and the force of the wind. The points along the tie-beam will have to sustain the weight of the ceiling and the flooring of a loft within the roof, if there be one, together with the loading upon this floor. The weight of the truss itself must be added to the weight of roof and ceiling.

215.—Load on Roof per Superficial Foot.—In any important work, each of the items in *Art.* 214 should be carefully estimated, in making up the load to be carried. For ordinary roofs, the weights may be taken per foot superficial, as follows:

Slate,	about	7·0	pounds.
Roof-plank,	"	2·7	"
Roof-beams or jack-rafters,	"	2·3	"
In all,		12	pounds.

This is for the superficial foot of the inclined roof. For the foot horizontal, the augmentation of load due to the angle of the roof will be in proportion to its steepness. In ordinary cases, the twelve pounds of the inclined surface will not be far from fifteen pounds upon the horizontal foot.

For the roof-load we may take as follows:

Roofing,	about	15	pounds.
Roof-truss,	"	5	"
Snow,	"	20	"
Wind,	"	10	"
Total on roof,		50	pounds

per square foot horizontal.

This estimate is for a roof of moderate inclination, say one in which the height does not exceed ¼ of the span. Upon a steeper roof the snow would not gather so heavily, but the wind, on the contrary, would exert a greater force. Again, the wind acting on one side of a roof may drift the snow from that side, and perhaps add it to that already lodged upon the opposite side. These two, the wind and the snow, are compensating forces. The action of the snow is vertical: that of the wind is horizontal, or nearly so. The power of the wind in this latitude is not more than thirty pounds upon a superficial foot of a vertical surface; except, perhaps, on elevated places, as mountain-tops for example, where it should be taken as high as fifty pounds per foot of vertical surface.

216.—Load upon Tie-Beam.—The load upon the tie-beam must of course be estimated according to the requirements of each case. If the timber is to be exposed to view, the load to be carried will be that only of the tie-beam and the timber struts resting upon it. If there is to be a ceiling attached to the tie-beam, the weight to be added will be in accordance with the material composing the ceiling. If of wood, it need not weigh more than two or three pounds per foot. If of lath and plaster, it will weigh about nine pounds; and if of iron, from ten to fifteen pounds, according to the

thickness of the metal. Again, if there is to be a loft in the roof, the requisite flooring may be taken at five pounds, and the load upon the floor at from twenty-five to seventy pounds, according to the purpose for which it is to be used.

217.—Roof-Weights in Detail.—The load to be sustained by a roof-truss has been referred to in the previous three articles in general terms. It will now be treated more in detail. But first a few words regarding the slope of the roof. In a severe climate, roofs ought to be constructed steeper than in a milder one, in order that snow may have a tendency to slide off before it becomes of sufficient weight to endanger the safety of the roof. In selecting the material with which the roof is to be covered, regard should be had to the requirements of the inclination: slate and shingles cannot be used safely on roofs of small rise. The smallest inclination of the various kinds of covering is here given, together with the weight per superficial foot of each.

MATERIAL.	Least Inclination.	Weight upon a square foot.
Tin	Rise 1 inch to a foot.	$\frac{5}{8}$ to $1\frac{1}{4}$ lbs.
Copper..........................	" 1 " " "	1 to $1\frac{1}{2}$ "
Lead..............................	" 2 inches " "	4 to 7 "
Zinc	" 3 " " "	$1\frac{1}{4}$ to 2 "
Short pine shingles.................	" 5 " " "	$1\frac{1}{2}$ to 2 "
Long cypress shingles...	" 6 " " "	2 to 3 "
Slate..............................	" 6 " " "	5 to 9 "

The weight of the covering as here estimated includes the weight of whatever is used to fix it in place, such as nails, etc. The weight of that which the covering is laid upon, such as plank, boards, or lath, is not included. The weight of plank is about 3 pounds per foot superficial; of boards, 2 pounds; and lath, about half a pound.

Generally, for a slate roof, the weight of the covering, including plank and jack-rafters, amounts to about 12 pounds, as stated in *Art.* 215; but in every case, the weight of each article of the covering should be estimated, and the full load ascertained by summing up these weights.

218.—Load per Foot Horizontal.—The weight of the covering as referred to in the last article is the weight per foot on the *inclined* surface; but it is desirable to know how much per foot, measured *horizontally*, this is equal to. The horizontal measure of one foot of the inclined surface is equal to the cosine of the angle of inclination. Then, to obtain the inclined measure corresponding to one foot horizontal, we have—

$$\cos. : 1 :: p : C = \frac{p}{\cos.};$$

where p represents the pressure on a foot of the inclined surface, and C the weight of so much of the inclined covering as corresponds to one foot horizontal. The cosine of an angle is equal to the base of the right-angled triangle divided by the hypothenuse (see Trigonometrical Terms, *Art.* 474), which in this case is half the span divided by the length of the rafter, or $\frac{s}{2l}$, where s is the span, and l the length of the rafter. Hence, the load per foot horizontal equals—

$$C = \frac{p}{\cos.} = \frac{p}{\frac{s}{2l}} = \frac{2lp}{s}; \qquad (92.)$$

or, twice the pressure per foot of *inclined* surface multiplied by the length of the rafter and divided by the span, both in feet, will give the weight per foot measured horizontally.

219.—Weight of Truss.—The weight of the framed truss will be in proportion to the load and to the span. This, for the weight upon a foot horizontal, will about equal—

$$T = 0{\cdot}077\, Cs;$$

which equals the weight in pounds per foot horizontal to be allowed for a wooden truss with iron suspension-rods and a horizontal tie-beam, near enough for the requirements of our present purpose; where s equals the length or span of the

truss, and C the weight per foot horizontal of the roof covering, as in equation (92.). Substituting for C its value, as in (92.), we have—

$$T = 0{\cdot}0077\, s \frac{2\,l\,p}{s};$$

or—

$$T = 0{\cdot}0154\; l\,p; \tag{93.}$$

which equals the weight in pounds per foot horizontal to be allowed for the truss.

220.—Weight of Snow on Roofs.—The weight of snow will be in proportion to the depth it acquires, which will be in proportion to the rigor of the climate of the place where the building is to be erected. Upon roofs of ordinary inclination, snow, if deposited in the absence of wind, will not slide off; at least until after it has acquired some depth, and then the tendency to slide will be in proportion to the angle of inclination. The weight of snow may be taken, therefore, at its weight per cubic foot (8 pounds) multiplied by the depth it is usual for it to acquire. This, in the latitude of New York, may be taken at about $2\frac{1}{2}$ feet. Its weight would, therefore, be 20 pounds per foot superficial, measured horizontally.

221.—Effect of Wind on Roofs.—The direction of wind is horizontal, or nearly so, when unobstructed. Precipitous mountains or tall buildings deflect the wind considerably from its usual horizontal direction. Its power usually does not exceed 30 pounds per superficial foot except on elevated places, where it sometimes reaches 50 pounds or more. This is the pressure upon a vertical surface; roofs, however, generally present to the wind an inclined surface. The effect of a horizontal force on an inclined surface is in proportion to the sine of the angle of inclination; the direction of this effect being at right angles to the inclined surface. The force thus acting may be resolved into forces acting in two directions—the one horizontal, the other vertical; the former tending, in the case of a roof, to thrust aside the walls

on which the roof rests, and the latter acting directly on the materials of which the roof is constructed—this latter force being in proportion to the sine of the angle of inclination multiplied by the cosine. This will be made clear by the following explanation. Referring to *Fig.* 83, let DKE be the angle of inclination of the roof, DE being equal to one foot. Bisect DK at A; draw AL parallel with EK; make AL equal to the horizontal pressure of the wind upon one foot superficial of a vertical plane. Draw AC perpendicular to DK, and LF parallel with AC from F draw FC parallel with EK; draw AB parallel with DE. The sides of the triangle LAF represent the three several forces in equilibrium: LA is the force of the wind; LF is the pressure upon the roof; and AF is the force with which the wind moves on up the roof towards D. Now, to find the relation of the force of the wind to the strain produced by it in the direction AC, we have—

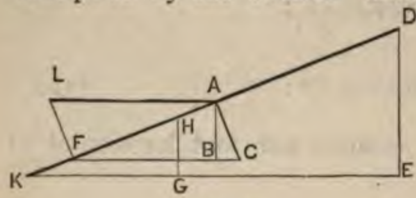


Fig. 83.

$$\text{rad.} : \text{sin.} :: FC : AC;$$
$$FC = LA; \text{ therefore—}$$
$$\text{rad.} : \text{sin.} :: LA : AC = LA \text{ sin.};$$
$$AC = F \text{ sin.};$$

or, the strain perpendicular to the surface of the roof equals the force of the wind multiplied by the sine of the angle of inclination. When AC represents this strain, then, of the two forces referred to above, BC represents the horizontal force, and AB the vertical force. To obtain this last force, we have—

$$\text{rad.} : \text{cos.} :: AC : AB.$$

Putting for AC its value as above, we have—

$$\text{rad.} : \text{cos.} :: F \text{ sin.} : AB = F \text{ sin. cos.};$$
$$V = F \text{ sin. cos.};$$

or, the vertical effect is equal to the product of the force of the wind upon a superficial foot into the sine and the cosine of the angle of inclination. This result is that which is due to the pressure of the wind upon so much of the inclined surface as is covered by one square foot of a vertical surface. The wind, acting horizontally through one foot superficial of vertical section, acts on an area of inclined surface equal to the reciprocal of the sine of inclination, and the horizontal measurement of this inclined surface is equal to the cosine of the angle of inclination divided by the sine. This may be illustrated from *Fig.* 83, thus—

$$\text{sin.} : \text{rad.} :: D\,E : D\,K.$$

$D\,E$ equals 1 foot; therefore—

$$\text{sin.} : \text{rad.} :: 1 : D\,K = \frac{1}{\text{sin.}};$$

or, the surface acted upon by one square foot of sectional area equals the reciprocal of the sine of the angle of inclination. Again, the horizontal measure of this inclined surface may be obtained thus—

$$\text{sin.} : \text{cos.} :: D\,E : K\,E = \frac{\text{cos.}}{\text{sin.}};$$

or, KE, the horizontal measurement, equals the cosine of the angle of inclination divided by the sine.

In the figure, make $K\,G$ equal to one foot; then we have—

$$K\,E : K\,G :: V : W;$$

in which V, as above, represents the vertical pressure due to the wind acting upon the surface $K\,D$, and W the vertical pressure due to the wind acting upon the surface $K\,H$, or so much as covers $K\,G$, one foot horizontal.

Now we have, as above, $K\,E$ equal to $\frac{\text{cos.}}{\text{sin.}}$, $K\,G = 1$, and

$V = F \sin. \cos.$ Substituting these values, we have, instead of the above proportion—

$$\frac{\cos.}{\sin.} : 1 :: F \sin. \cos. : W;$$

from which—

$$W = \frac{F \sin. \cos.}{\frac{\cos.}{\sin.}} = F \sin.^2 \qquad (94.)$$

or, the vertical effect of the wind upon so much of the roof as covers each square foot horizontal, is equal to the product of the force of the wind per square foot into the square of the sine of the angle of inclination.

Example.—When the force of the wind upon a square foot of vertical surface is 30 pounds, what will be the vertical effect per square foot horizontal upon a roof the inclination of which is 26° 33′, or 6 inches to the foot?

Here we have $F = 30$, and the sine of 26° 33′ is 0·44698; therefore—

$$W = 30 \times 0{\cdot}44698^2 = 5{\cdot}9937.$$

This is conveniently solved by logarithms; thus—

$$\begin{array}{rcl} \log.\ 30 & = & 1{\cdot}4771213 \\ 0{\cdot}44698 & = & \bar{9}{\cdot}6502868 \\ 0{\cdot}44698 & = & \bar{9}{\cdot}6502868 \\ \hline 5{\cdot}9937 & = & 0{\cdot}7776949 \end{array}$$

or, the vertical effect is (5·9937, or) 6 pounds.

The form of equation (94.) may be changed; for, in a right-angled triangle, the sine of the angle at the base is equal to the perpendicular divided by the hypothenuse; which, in the case of a roof, is the height divided by the length of the rafter; or—

$$\text{Sine} = \frac{\text{height}}{\text{rafter}} = \frac{h}{2}.$$

Therefore, equation (94.) may be changed to—

$$W = F\frac{h^2}{l^2}; \qquad (95.)$$

or, the vertical effect upon each square foot of a roof is equal to the product of the force of the wind per foot into the square of the height of the roof at the ridge, divided by the square of the length of the rafter (the height and length both in feet.)

Example.—When the force of the wind is 30 pounds, the height of the roof 10 feet, and the length of the rafter 22·36 feet, what will be the vertical effect of the wind? Here we have $F = 30$, $h = 10$, and $l = 22{\cdot}36$; and—

$$W = 30 \times \frac{10^2}{22{\cdot}36^2} = 6.$$

222.—Total Load per Foot Horizontal.—The various items comprising the total load upon a roof are the covering, the truss, the wind, snow, the plastering or other kind of ceiling, and the load which may be deposited upon a floor formed in the roof; or, the total load—

$$M = C + T + W + S + P + L.$$

The value per foot horizontal for these has been found as follows: $C = \frac{2lp}{s}$; $T = 0{\cdot}0154\,lp$; $W = F\frac{h^2}{l^2}$. For S the value must be taken according to circumstances, as in *Art.* 220. So, also, the value of P and L are to be assigned as required for each particular case, as in *Art.* 216. The total load, therefore, with these substitutions, will be—

$$M = \frac{2lp}{s} + 0{\cdot}0154\,lp + F\frac{h^2}{l^2} + S + P + L;$$

which reduces to—

$$M = lp\left(\frac{2}{s} + 0{\cdot}0154\right) + F\frac{h^2}{l^2} + S + P + L; \qquad (96.)$$

in which l is the length of the rafter; p is the weight of the covering per foot superficial, including the roof boards or slats, the jack-rafters, etc.; s is the span of the roof; h is the vertical height above a horizontal line passing through the feet of the rafters; F is the force of the wind per square foot against a vertical surface; S is the weight of snow per square foot horizontal; P is the weight per superficial foot of the ceiling at the tie-beam; and L, the load per superficial foot in the roof, including weight of flooring and floor-timbers. The dimensions, s, l, and h, are each in feet; the weight of p, F, S, P, and L are each in pounds. The value of p is for a square foot of the *inclined* surface.

223.—Strains in Roof-Timbers Computed.—The graphic method of obtaining the strains, as shown in *Arts.* 205 to 211, is, for its conciseness and simplicity, to be preferred to any other method; yet, on some accounts, the method of obtaining the strains by the parallelogram of forces and by arithmetical computations will be found useful, and will now be referred to.

By the parallelogram of forces, the weight of the roof is in proportion to the oblique thrust or pressure in the axis of the rafter as twice the height of the roof is to the length of the rafter; or—

$$R : Y :: 2\,h : l;$$

or, transposing—

$$2\,h : l :: R : Y = \frac{R\,l}{2\,h}; \qquad (97.)$$

where Y equals the pressure in the axis of the rafter, and R the weight of one truss and its load. Again, the weight of the roof is in proportion to the horizontal thrust in the tie-beam as twice the height of the roof is to half the span; or—

$$R : H :: 2\,h : \frac{s}{2};$$

or, transposing—

$$2\,h : \frac{s}{2} :: R : H = \frac{R\,s}{4\,h}; \qquad (98.)$$

where H equals the horizontal thrust in the tie-beam. To obtain R, the weight of the roof, multiply M, the load per foot, as in equation (96.), by s, the span, and by c, the distance from centres at which the trusses are placed; or—

$$R = M c s.$$

With this value of R substituted for it, we have—

$$Y = \frac{M c s l}{2 h}; \qquad (99.)$$

and—

$$H = \frac{M c s^2}{4 h}; \qquad (100.)$$

in which Y equals the strain in the axis of the rafter, and H the strain in the tie-beam. These are the greatest strains in the rafter and tie-beam. At certain parts of these pieces the strains are less, as will be shown in the next article.

224.—Strains in Roof-Timbers Shown Geometrically.— The pressure in each timber may be obtained as shown in *Fig.* 84, where $A B$ represents the axis of the tie-beam, $A C$ the axis of the rafter, $D E$ and $F B$ the axes of the braces, and $D G$, $F E$, and $C B$ the axes of the suspension-rods. In this design for a truss, the distance $A B$ is divided into three equal parts, and the rods located at the two points of division, G and E. By this arrangement the rafter $A C$ is supported at equidistant points, D and F. The point D supports the rafter for a distance extending half-way to A and half-way to F, and the point F sustains half-way to D and half-way to C. Also, the point C sustains half-way to F, and, on the other rafter, half-way to the corresponding point to F. And because these points of support are located at equal distances apart, therefore the load on each is the same in amount. On $D G$ make $D a$ equal by any decimally divided scale to the number of hundreds of pounds in the load on D, and draw the parallelogram $a b D c$. Then, by the same scale, $D b$ represents (*Art.* 71) the pressure in the axis of the rafter by the load at

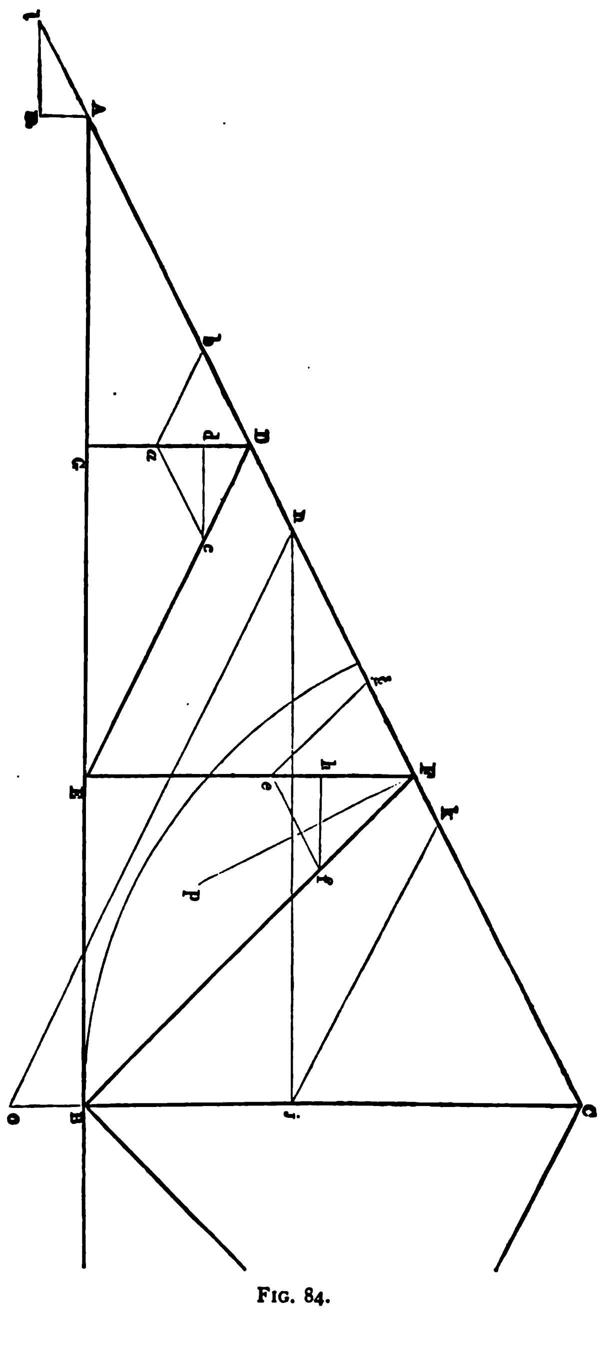


FIG. 84.

D; also, $D\,c$ the pressure in the brace DE. Draw $c\,d$ horizontal; then $D\,d$ is the vertical pressure exerted by the brace DE at E. The point F sustains, besides the common load represented by $D\,a$, also the vertical pressure exerted by the brace DE; therefore, make Fe equal to the sum of $D\,a$ and Dd, and draw the parallelogram $F\,g\,e\,f$. Then Fg, measured by the scale, is the pressure in the axis of the rafter caused by the load at F, and $F\,f$ is the load in the axis of the brace FB. Draw $f\,h$ horizontal; then Fh is the vertical pressure exerted by the brace FB at B. The point C, besides the common load represented by D a, sustains the vertical pressure Fh caused by the brace FB, and a like amount from the corresponding brace on the opposite side. Therefore, make Cj equal to the sum of $D\,a$ and twice Fh, and draw $j\,k$ parallel to the opposite rafter. Then $C\,k$ is the pressure in the axis of the rafter at C. This is not the only pressure in the rafter, although it is the total pressure at its head C. At the point F, besides the pressure $C\,k$, there is Fg. At the point D, besides these two pressures, there is the pressure $D\,b$. At the foot, at A, there is still an additional pressure; for while the point D sustains the load half-way to F and half-way to A, the point A sustains the load half-way to D. This load is, in this case, just half the load at D. Therefore draw $A\,m$ vertical, and equal, by the scale, to half of $D\,a$. Extend $C\,A$ to l; draw $m\,l$ horizontal. Then $A\,l$ is the pressure in the rafter at A caused by the weight of the roof from A half-way to D. Now the total of the pressures in the rafter is equal to the sum of $A\,l + D\,b + Fg$ added to $C\,k$. Therefore make $k\,n$ equal to the sum of $A\,l + Db + F\,g$, and draw $n\,o$ parallel with the opposite rafter, and $n\,j$ horizontal. Then $C\,o$, measured by the same scale, will be found equal to the total weight of the roof on both sides of CB. Since $D\,a$ represents s, the portion of the weight borne by the point D, therefore $C\,o$, representing the whole weight of the roof, should equal six times $D\,a$, as it does, because D supports just one sixth of the whole load. Since $C\,n$ is the total oblique thrust in the axis of the rafter at its foot, therefore $n\,j$ is the horizontal thrust in the tie-beam at A.

225.—Application of the Geometrical System of Strains.—The strains in a roof-truss can be ascertained geometrically, as shown in *Art.* 224. To make a practical application of the results, in any particular case, it is requisite first to ascertain the load at the head of each brace, as represented by the line $D\,a$, *Fig.* 84. The load corresponding to any part of the roof is equal to the product of the superficial area of that particular part (measured horizontally) multiplied by the weight per square foot of the roof. Or, when M equals the weight per square foot, c the distance from centres at which the trusses are placed, and n the horizontal distance between the heads of the braces, then the total load at the head of a brace is represented by—

$$N = M\,c\,n. \qquad (101.)$$

The value of M is given in general terms in equation (96.). To show its actual value, let it be required to find the weight per square foot upon a roof 52 feet span and 13 feet high at middle; or (*Fig.* 84), where $A\ B$ equals half the space, or 26 feet, and $C\,B$ 13 feet, then $A\ C$, the length of the rafter, will be 26·069, nearly. And where the weight of covering per square foot, on the inclination, is 12 pounds, the force of the wind against a vertical plane is 30 pounds; the weight of snow per foot horizontal is 20 pounds; the weight of the plastering forming the ceiling at the tie-beam is 9 pounds; and the load in the roof is nothing;—with these quantities substituted, equation (96.) becomes—

$$M = 29{\cdot}069 \times 12\left(\frac{2}{52} + 0{\cdot}0154\right) + 30\frac{13^2}{29{\cdot}069^2} + 20 + 9 + 0;$$

$$M = (29{\cdot}069 \times 12 \times 0{\cdot}05386) + (30 \times 0{\cdot}2) + 20 + 9;$$

$$M = 18{\cdot}788 + 6 + 29 = 53{\cdot}788;$$

or, say, 53·8 pounds. Then if c, the distance from centres between trusses, is 10 feet, and n, the distance between braces, is one third of $A\ B$, *Fig.* 84, or $\frac{26}{3} = 8\frac{2}{3}$, the total load at the head of a brace will be, as per equation (101.)—

$$N = 53{\cdot}8 \times 10 \times 8\tfrac{2}{3} = 4663;$$

or, say, 4650 pounds. Now, by any decimally divided scale, make $D\,a$, *Fig.* 84, equal to $46\frac{1}{2}$ parts of the scale; this being the number of hundreds of pounds contained in the weight at D, as above. Then, by the same scale, the several lines in the figure drawn as before shown will be found to represent respectively the weights here set opposite to them, as follows:

$D\,d = d\,a = h\,e = 23\frac{1}{4}$,	and represents	2325	pounds;
$D\,a = d\,c = h\,f = F\,h = 46\frac{1}{2}$	"	4650	"
$D\,c = D\,b = A\,l = F\,g = 52$	"	5200	"
$F\,e = D\,a + D\,d = 69\frac{3}{4}$	"	6975	"
$F\,f = 65\frac{3}{4}$	"	6575	"
$C\,j = 3\,D\,a = 139\frac{1}{2}$	"	13950	"
$C\,K = 3\,D\,b = 156$	"	15600	"
$C\,n = C\,k + F\,g + D\,b + A\,l = 312$	"	31200	"
$C\,n = C\,k + 3\,D\,b = 6\,D\,b = 2\,C\,k$ $= 312$	"	31200	"
$N\,j = C\,o = 6\,D\,a = 6 \times 46\frac{1}{2} = 279$	"	27900	"

It should be observed here that the equality of the lines $n\,j$ and $C\,o$ is a coincidence dependent upon the relation which in this particular case the line $C\,B$ happens to bear to the line $A\,B$; $A\,B$ being equal to twice $C\,B$. And so of some other lines in the figure. If the inclination of the roof were made greater or less, the equality of the lines referred to would disappear. It should also be observed that the strains above found are not quite exact; they are, however, correct to within a fraction of a hundred pounds, which is a sufficiently near approximation for the purpose intended. From the results obtained above, we ascertain that the strain in the rafter, from F to C, is represented by $C\,K$, and is equal to 15,600 pounds; while the strain at the foot of the rafter, from A to D, is represented by $C\,n$, and equals 31,200 pounds, or double that which is at the head of the rafter. We ascertain, also, that the maximum strain in the tie-beam, represented by $n\,j$, is 27,900 pounds; that that in the brace $D\,E$, represented by $D\,c$, is 5200 pounds; and that that in the brace $F\,B$, represented by $F\,f$, is 6575 pounds. The strain

in the vertical rod DG is theoretically nothing. There is, however, a small strain in it, for it has to carry a part of the tie-beam and so much of the ceiling as depends for support upon that part. But the manner of locating the weights, adopted in this article, does not recognize any load located at the point G. This is an objection to this system, but it is not material.

For a recognition of weights at the tie-beam, see *Arts.* 205 to 211. The load at G may be found by obtaining the product of the surface carried into the weight per foot of the ceiling; or, say, $10\,cn = 10 \times 10 \times 8\frac{2}{3} = 867$ pounds. The load to be carried by the rod FE is shown at $Dd = he$, which above is found to be 2325 pounds. To this is to be added 867 pounds for the ceiling at E, as before found for the ceiling at G; or, together, 3192 pounds. The central rod CB has to carry the two loads brought to B by the two braces footed there; and also the weight of the ceiling supported by B. The vertical strain from the brace FB is represented at Fh, and equals 4650 pounds; therefore, the total load on CB is $4650 + 4650 + 867 = 10{,}167$ pounds.

226.—Roof-Timbers: the Tie-Beam.—The roof-timbers comprised in the truss shown in *Fig.* 84 are the rafters, tie-beam, two braces, and three rods. Of these, taking first the *tie-beam*, we have a piece subject to tension and sometimes to cross-strain (see Art. 682, *Transverse Strains*). In this case the tensile strain only need be considered. For this a rule is given in *Art.* 117. In this rule, if the factor of safety be taken at 20, the result will be sufficiently large to allow for necessary cuttings at the joints. Therefore, if the beam be of Georgia pine, equation (16.), *Art.* 117, becomes—

$$A = \frac{27900 \times 20}{16000} = 34\tfrac{7}{8};$$

or, say, 35 inches. This is ample to resist the tensile strain; but, to resist the transverse strains to which such a long piece of timber is subjected in the hands of the workman, it would be proper to make it, say, 6×9.

227.—The Rafter.—A rafter, like a post, is subject to a compressive force, and is liable to fail in three ways, namely: by flexure, by being crushed, or by crushing the material against which it presses. To render it entirely safe, therefore, it is requisite to ascertain the requirements for resisting failure in each of these three ways.

Of these it will be convenient to consider, first, that of the liability to being crushed. The rule for this is found in *Art.* 107. Let the rafter be of Georgia pine, then the value of C, Table I., will be 9500. The strain in the rafter (*Art.* 225) is 31,200 pounds. Now, taking the value of a, the factor of safety, at 10, we have, by Rule VI. (*Art.* 107.)—

$$A = \frac{31200 \times 10}{9500} = 32 \cdot 737;$$

or, 33 inches area of cross-section. This is the size of the rafter at its smallest section; for example, at any one of the joints where it is customary to reduce the area by cutting for the struts and rods.

Again: Let the liability of the rafter to flexure be now considered. For this we have a rule in *Art.* 114. The length of the rafter between unsupported points is nearly $9\frac{2}{3}$ feet, or $9\frac{2}{3} \times 12 = 116$ inches. Let the thickness of the rafter be taken at 6 inches. Then, by Rule XI. (*Art.* 114), we have—

$$b = \frac{Wa(1 + \frac{3}{2}cr^2)}{Ct} = \frac{31200 \times 10\,(1 + \overline{\frac{3}{2} \times \cdot 00109 \times r^2})}{9500 \times 6};$$

$$r = \frac{l}{t} = \frac{116}{6} = 19\tfrac{1}{3}; \qquad \overline{19\tfrac{1}{3}}^2 = 373 \cdot 8.$$

$$\begin{aligned} \text{Then, } \tfrac{3}{2} \times \cdot 00109 \times 373 \cdot 8 &= 0 \cdot 611127 \\ \text{adding unity} &= \underline{1 \cdot } \\ &\ 1 \cdot 611127 \end{aligned}$$

Substituting this, we have—

$$b = \frac{31200 \times 10 \times 1 \cdot 611127}{9500 \times 6} = \frac{502671 \cdot 624}{57000} = 8 \cdot 819;$$

or, to resist flexure the breadth is required to be 8·82, or, say, 9 inches; or, the rafter is to be 6×9 inches at the foot. The strain in the rafter at the upper end is only half that at the foot; the area of cross-section, therefore, at the head need not be more than half that which is required at the foot; but it is usual to make it there about $\frac{2}{3}$ of the size at the foot. In this case it would be, therefore, 6×6 inches at the upper end.

Lastly, the requirement to resist crushing the surfaces against which the rafter presses is to be considered.

The fibres of timber yield much more readily when pressed together by a force acting at *right angles* to the direction of their length than when it acts *in a line* with their length.

The value of timber subjected to pressure in these two ways is shown in *Arts.* 94, 98. In Table I., the value per square inch of the first stated resistance is expressed by P, and the ultimate resistance of the other by $\frac{C}{a}$. The value of timber per square inch to safely resist crushing may be expressed by $\frac{C}{a}$, in which a is the factor of safety. Timber pressed in an oblique direction will resist a force exceeding that expressed by P, and less than that expressed by $\frac{C}{a}$. When the angle of inclination at which the force acts is just 45°, then the force will be an average between P and $\frac{C}{a}$. And for any angle of inclination, the force will vary inversely as the angle; approaching P as the angle is enlarged, but approaching $\frac{C}{a}$ as the angle is diminished. It will be equal to $\frac{C}{a}$ when the angle becomes zero, and equal P when the angle becomes 90°. The resistance of timber per square inch to an oblique force is therefore expressed by—

$$M = P + \frac{A^\circ}{90}\left(\frac{C}{a} - P\right); \qquad (102.)$$

where $A°$ equals the complement of the angle of inclination. In a roof, $A°$ is the acute angle formed by the rafter with a vertical line. If no convenient instrument be at hand to measure the angle, describe an arc upon the plan of the truss—thus: with $C\,B$ (*Fig.* 84) for radius, describe the arc $B\,g$, and get the length of this arc in feet by stepping it off with a pair of dividers. Then—

$$\frac{A°}{90} = 0{\cdot}63\tfrac{2}{3}\,\frac{k}{h};$$

where k equals the length of the arc, and h equals $B\,C$, the height of the roof. Therefore—

$$M = P + 0{\cdot}63\tfrac{2}{3}\,\frac{k}{h}\left(\frac{C}{a} - P\right) \qquad (103.)$$

equals the value of timber per square inch in a tie-beam, C and P being obtained from Table I., *Art.* 94. When C for the kind of wood in the tie-beam exceeds C set opposite the kind of wood in the rafter, then the latter is to be used in the rules instead of the former.

The value of M, equation (103.), is the resistance per square inch of the surface pressed at the foot of the rafter. The resistance of the entire surface will therefore be $M\,A$, where A equals the area of the joint. Then, when the resistance equals the strain, we will have—

$$M\,A = S = A\left[P + 0{\cdot}63\tfrac{2}{3}\,\frac{k}{h}\left(\frac{C}{a} - P\right)\right];$$

from which we have—

$$A = \frac{S}{P + 0{\cdot}63\tfrac{2}{3}\,\frac{k}{h}\left(\frac{C}{a} - P\right)}; \qquad (104.)$$

in which S is the strain to be resisted.

Now, the end of the rafter must be of sufficient size to afford a joint the area of which will not be less than that expressed by A in equation (104.).

For example, the strain to which the rafter, *Fig.* 84, is subject at its foot is ascertained to be (*Art.* 225) 31,200 pounds. For Georgia pine, the material of the tie-beam, $P = 900$ (*Art.* 94, Table *I.*), and $C = 9500$.

The length of the arc Bg is about 14·4 feet; the height BC is 13 feet. Let a, the factor of safety, be taken at 10, then we have (104.)—

$$A = \frac{31200}{900 + (0{\cdot}63\tfrac{2}{3} \times \frac{14{\cdot}4}{13})(\frac{9500}{10} - 900)};$$

$$A = \frac{31200}{900 + (0{\cdot}705 \times 50)} = 33{\cdot}36;$$

or, the superficial area of the bearing at the joint required to prevent crushing the tie-beam is $33\frac{1}{3}$ inches.

The results of the computations show that the rafter is required to be 6 inches thick, 9 inches wide at the foot, and 6 inches wide at the top. It is also ascertained that, in cutting for the bearing for the struts and boring for the suspension-rods, it is required that there shall be at least 33 inches area of cross-section left intact; and, farther, that the area of the surface of the joint against the tie-beam should not be less than $33\frac{1}{3}$ inches.

228.—The Braces.—Each brace is subject to compression, and is liable to fail if too small, in the same manner as the rafter. Its size is to be ascertained, therefore, in the manner described for the rafter; which need not be here repeated, except, perhaps, as to the liability to fail by flexure; for in this case we have the breadth given, and need to find the thickness. The breadth of the brace is fixed by the thickness of the rafter, for it is usual to have the two pieces flush with each other. Rule XI. (*Art.* 114) is to be used, but with this difference, namely: instead of the thickness, use the breadth as one of the factors in the divisor. Thus—

$$t = \frac{Wa(1 + \frac{3}{2}cr^2)}{Cb}. \qquad (105.)$$

In working this rule, it is required, in order to get the value of r, the ratio between the height and thickness, to assume the thickness before it is ascertained; and after computation, if the result shows that the assumed value was not a near approximation, a second trial will have to be made. Usually the first trial will be sufficient.

For example, the brace DE is about $9\frac{5}{8}$ feet or 116 inches long. As the strain in it is only 5200 pounds, the thickness will probably be not over 3 inches. Assuming it at this, we have $r = \frac{l}{t} = \frac{116}{3} = 38\frac{2}{3}$; the square of which is about 1495. Therefore, we have—

$$\begin{array}{r} \frac{3}{2} \times 0 \cdot 00109 \times 1495 = 2 \cdot 4445 \\ \text{add unity} = \underline{1.} \\ 3 \cdot 4445 \end{array}$$

The equation reduces, therefore, to this—

$$t = \frac{5200 \times 10 \times 3 \cdot 4445}{9500 \times 6} = 3 \cdot 1424;$$

or, the required thickness of the brace is $3\frac{1}{7}$ inches, or the brace should be, say, $3\frac{1}{4} \times 6$ inches. In this case the result is so near the assumed value, a second trial is not needed.

For the second brace, we have the length equal to about $12\frac{1}{4}$ feet or 147 inches; and the strain equal to 6575 pounds (*Art.* 225). The ratio, therefore, may be obtained by assuming the thickness, say, at 4. With this, we have—

$$r = \frac{l}{t} = \frac{147}{4} = 36 \cdot 75; \text{ the square of which is } 1350\tfrac{9}{16}.$$

With this value of r^2—

$$\begin{array}{r} \frac{3}{2} \times \cdot 00109 \times 1350\frac{9}{16} = 2 \cdot 2081 \\ \text{add unity} = \underline{1.} \\ 3 \cdot 2081 \end{array}$$

Then—

$$t = \frac{6575 \times 10 \times 3 \cdot 2081}{9500 \times 6} = 3 \cdot 7006.$$

Comparing this result with the assumed value of $t = 4$, we find the difference so great as to require a second trial. As the value of r was taken too low, the result obtained is correspondingly low. The true value is somewhere between 3·7 and 4. Assume it now, say, at 3·9. With this value, we have—

$$r = \frac{l}{t} = \frac{147}{3 \cdot 9} = 37 \cdot 692; \text{ the square of which is } 1420 \cdot 7.$$

With this value of r^2—

$$\tfrac{3}{4} \times \cdot 00109 \times 1420 \cdot 7 = 2 \cdot 32282$$
$$\text{add unity} = \underline{1 \cdot \quad\quad}$$
$$3 \cdot 32282$$

Then—

$$t = \frac{6575 \times 10 \times 3 \cdot 32282}{9500 \times 6} = 3 \cdot 833.$$

This result is a trifle less than the assumed value, 3·9. The true value is between these, and probably is about 3·86. This is quite near enough for use. This brace, therefore, is required to be $3 \cdot 86 \times 6$ inches, or, say, 4×6 inches.

229.—The Suspension-Rods.—These are usually made of wrought iron. This metal, when of excellent quality, may be safely trusted with 12,000 pounds per inch sectional area. But it is usual, for good work, to compute the area at only 9000 pounds per inch, and, as ordinarily made, these rods ought not to be loaded with more than 7000 pounds. The strain divided by this value per inch of the metal will give the sectional area of cross-section. For example, the strain in the rod DG, *Fig.* 84, is 867 pounds (*Art.* 225); therefore—

$$A = \frac{867}{7000} = 0 \cdot 124;$$

or, the sectional area required is only an eighth of an inch. By reference to the table of areas of circles in the Appendix, the diameter of a rod containing the required area, as above, will be found to be a little less than half an inch. A rod half an inch in diameter will therefore be of ample strength. For appearance's sake, however, no rod in a truss should be less than $\frac{3}{4}$ of an inch in diameter.

The rod FE has to resist a strain of 3192 pounds. For this, then, we have—

$$A = \frac{3192}{7000} = 0 \cdot 456.$$

A reference to the table of areas shows that a rod contain-

ing this area would be a little more than $\frac{3}{4}$ of an inch in diameter; it would be of ample strength, say, at $\frac{7}{8}$ of an inch in diameter.

The rod $C B$, at the centre, has to carry a strain of 10,167 pounds. For this, then, we have—

$$A = \frac{10167}{7000} = 1 \cdot 452.$$

A reference to the table of areas shows that this rod should be $1\frac{1}{2}$ inches in diameter.

230.—Roof-Beams, Jack-Rafters, and Purlins.—These timbers are subject to loads nearly uniformly distributed, and their dimensions may be obtained by Rule XXX., equation (35.), *Art.* 140. In this equation, $U = c f l$ (*Art.* 152). Substituting this value for U, and $r l$ for δ, equation (35.) becomes—

$$b d^3 = \frac{f c l^3}{1 \cdot 6 F r};$$

and putting for r the rate of deflection, .04, we have—

$$b d^3 = \frac{f c l^3}{0 \cdot 064 F}, \qquad (106.)$$

a formula convenient for roof-timbers.

Example.—In a roof where the roofing is to be supported on white-pine roof-beams 10 feet long, placed $2\frac{1}{2}$ feet from centres, and where the load per foot superficial is to be 40 pounds, including wind and snow: what should be the dimensions of the roof-beams? By equation (106.)—

$$b d^3 = \frac{40 \times 2\frac{1}{2} \times 10^3}{0 \cdot 064 \times 2900} = 538 \cdot 8.$$

Now if b, the breadth, be fixed, say, at 3, then—

$$d^3 = \frac{538 \cdot 8}{3} = 179 6;$$

$$d = 5 \cdot 64 \text{ nearly}.$$

The roof-beams, therefore, require to be $3 \times 5\frac{3}{8}$, or, say, 3×6. All pieces of timber subject to cross-strains will sustain safely much greater strains when extended in one piece over two, three, or more distances between bearings; therefore, roof-beams, jack-rafters, and purlins should, if possible, be made in as long lengths as practicable; the roof-beams and purlins laid on, not framed into, the principal rafters, and extended over at least two spaces, the joints alternating on the trusses; and likewise the jack-rafters laid on the purlins in long lengths.

231.—Five Examples of Roofs: are shown at *Figs.* 85, 86, 87, 88, and 89. In *Fig.* 85, *a* is an iron suspension-rod, *b*, *b* are braces. In *Fig.* 86, *a*, *a*, and *b* are iron rods, and *d*, *d*, *c*, *c* are braces. In *Fig.* 87, *a*, *b* are iron rods, *d*, *d* braces, and *c* the straining beam. In *Fig.* 88, *a*, *a*, *b*, *b* are iron rods, *e*, *e*, *d*, *d* are braces, and *c* is a straining beam. In *Fig.* 89, pur-

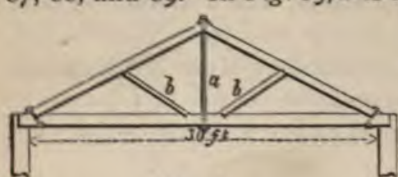


FIG. 85.

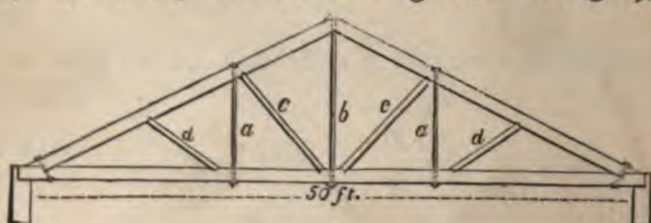


FIG. 86.

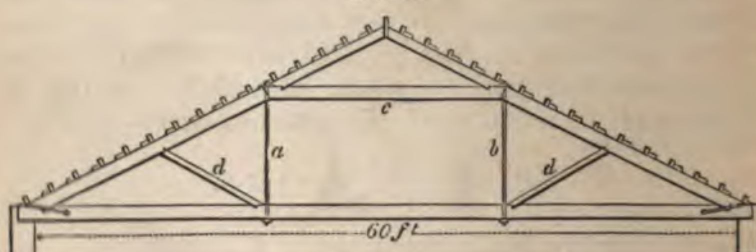


FIG. 87.

lins are located at *P P*, etc.; the inclined beam that lies upon them is the jack-rafter; the post at the ridge is the king-

post, the others are queen-posts. In this design the tie-beam is increased in height along the middle by a strengthening piece (*Art.* 163), for the purpose of sustaining additional weight placed in the room formed in the truss (*Art.* 216).

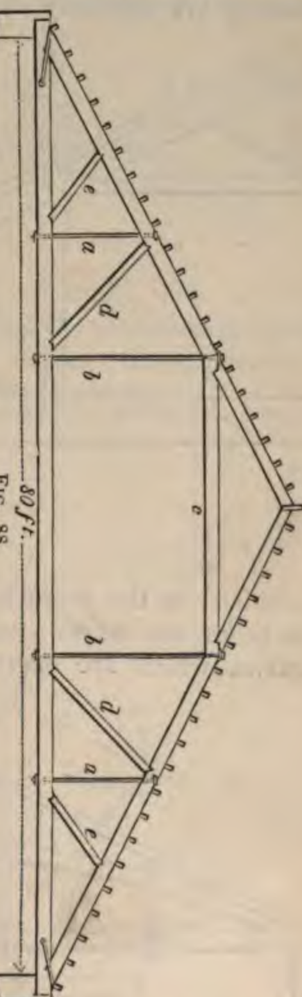


FIG. 88.

Fig. 90 shows a method of constructing a truss having a *built-rib* in the place of principal rafters. The proper form for the curve is that of the parabola (*Art.* 560). This curve, when as flat as is described in the figure, approximates so closely to that of the circle that the latter may be used in its stead. The height, $a\,b$, is just half of $a\,c$, the curve to pass through the middle of the rib. The rib is composed of two series of abutting pieces, bolted together. These pieces should be as long as the dimensions of the timber will admit, in order that there may be but few joints. The suspending pieces are in halves, notched and bolted to the tie-beam and rib, and a purlin is framed upon the upper end of each. A truss of this construction needs, for ordinary roofs, no diagonal braces between the suspending pieces, but if extra strength is required the braces may be added. The best place for the suspending pieces is at the joints of the rib. A rib of this kind will be sufficiently strong if the area of its section contain about one fourth more timber than is required for that of a rafter for a roof of the same size. The proportion of the depth to the thickness should be about as 10 to 7.

232.—Roof-Truss with Elevated Tie-Beam.—Designs such as are shown in *Fig.* 91 have the tie elevated for the accommodation of an arch in the ceiling. This and all similar designs are seriously objectionable, and should always be

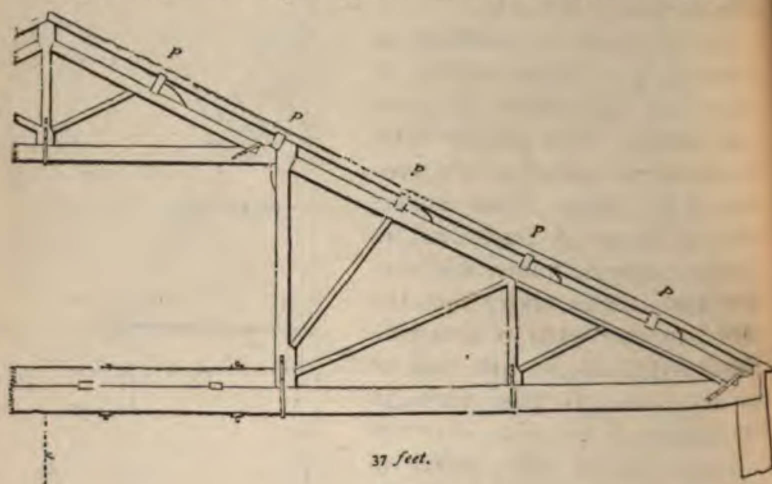


FIG. 89.

avoided; as the small height gained by the omission of the tie-beam can never compensate for the powerful lateral strains which are exerted by the oblique position of the

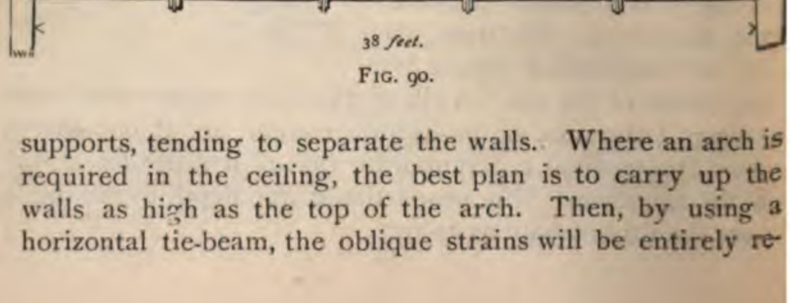


FIG. 90.

supports, tending to separate the walls. Where an arch is required in the ceiling, the best plan is to carry up the walls as high as the top of the arch. Then, by using a horizontal tie-beam, the oblique strains will be entirely re-

moved. It is well known that many a public building has been all but ruined by the settling of the roof, consequent upon a defective plan in the formation of the truss in this respect. It is very necessary, therefore, that the horizontal

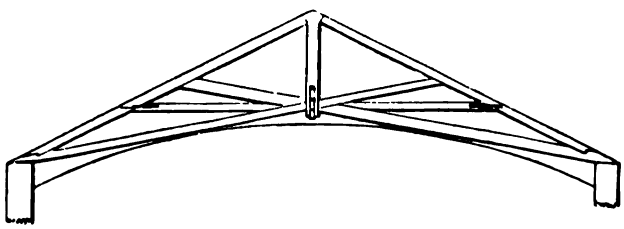


FIG. 91.

tie-beam be used, except where the walls are made so strong and firm by buttresses, or other support, as to prevent a possibility of their separating. (See *Art.* 212.)

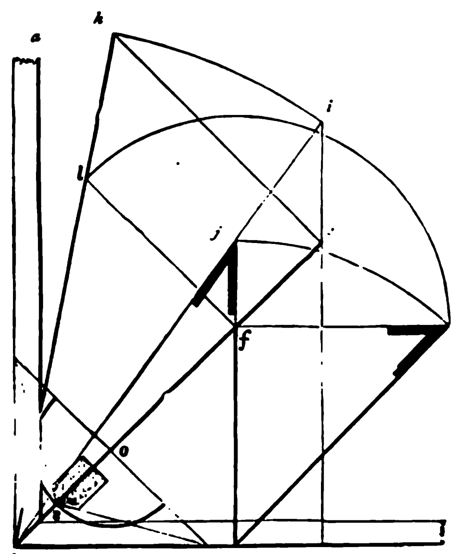


FIG. 92.

233.—Hip-Roofs: Lines and Bevils.—The lines *a b* and *b c*, in *Fig.* 92, represent the walls at the angle of a building; *b e* is the seat of the hip-rafter, and *g f* of a jack or cripple rafter. Draw *e h* at right angles to *b e*, and make it equal

to the rise of the roof; join *b* and *h*, and *h b* will be the length of the hip-rafter. Through *e* draw *d i* at right angles to *b c*; upon *b*, with the radius *b h*, describe the arc *h i*, cutting *d i* in *i*; join *b* and *i*, and extend *g f* to meet *b i* in *j*: then *g j* will be the length of the jack-rafter. The length of each jack-rafter is found in the same manner—by extending its seat to cut the line *b i*. From *f* draw *f k* at right angles to *f g*, also *f l* at right angles to *b e*; make *f k* equal to *f l* by the arc *l k*, or make *g k* equal to *g j* by the arc *j k*; then the angle at *j* will be the *top-bevil* of the jack-rafters, and the one at *k* will be the *down-bevil*.*

234.—The Backing of the Hip-Rafter.—At any convenient place in *b e* (*Fig.* 92), as *o*, draw *m n* at right angles to *b e*; from *o*, tangical to *b h*, describe a semicircle, cutting *b e* in *s*; join *m* and *s* and *n* and *s*; then these lines will form at *s* the proper angle for bevilling the top of the hip-rafter.

DOMES.†

235.—Domes.—The usual form for domes is that of the sphere; the base circular. When the interior dome does not

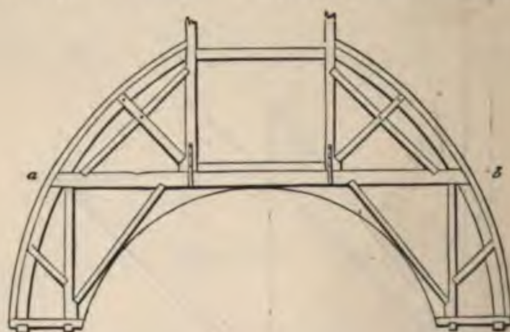


FIG. 93.

rise too high, a horizontal tie may be thrown across, by which any degree of strength required may be obtained

* The lengths and bevils of rafters for roof-*valleys* can also be found by the above process.

† See also *Art.* 68.

Fig. 93 shows a section, and *Fig.* 94 the plan, of a dome of this kind, *a b* being the tie-beam in both. Two trusses of this kind (*Fig.* 93), parallel to each other, are to be placed one on each side of the opening in the top of the dome. Upon these the whole framework is to depend for support,

FIG. 94.

and their strength must be calculated accordingly. (See *Arts.* 70 to 80 and 214 to 222.) If the dome is large and of importance, two other trusses may be introduced at right angles to the foregoing, the tie-beams being preserved in

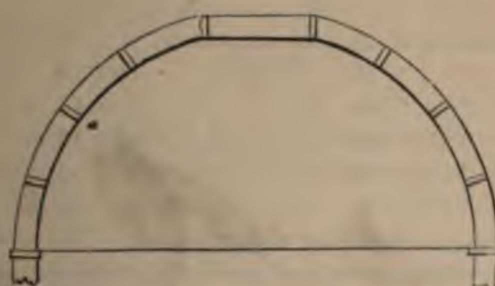


FIG. 95.

one continuous length by framing them high enough to pass over the others.

236.—Ribbed Dome.—When the interior must be kept free, then the framing may be composed of a succession of ribs standing upon a continuous circular curb of timber, as

seen at *Figs.* 95 and 96—the latter being a plan and the former a section. This curb must be well secured, as it serves in the place of a tie-beam to resist the lateral thrust of the ribs. In small domes these ribs may be easily cut from wide plank; but where an extensive structure is required, they must be built in two thicknesses so as to *break joints*, in the same manner as is described for a roof at *Art.* 231. They should be placed at about two feet apart at the base, and strutted as at *a* in *Fig.* 95.

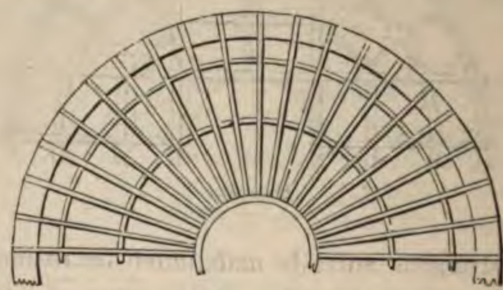


FIG. 96.

The scantling of each thickness of the rib may be as follows:

For domes of	24	feet	diameter,	1	× 8	inches.
" "	" 36	"	"	$1\frac{1}{2}$	× 10	"
" "	" 60	"	"	2	× 13	"
" "	" 90	"	"	$2\frac{1}{2}$	× 13	"
" "	" 108	"	"	3	× 13	"

237.—Dome: Curve of Equilibrium.—The surfaces of a dome may be finished to any curve that may be desired, but the framing should be constructed of such form that the *curve of equilibrium* shall be sure to pass through the middle of the depth of the framing. The nature of this curve is such that, if an arch or dome be constructed in accordance with it, no one part of the structure will be less capable than another of resisting the strains and pressures to which the whole fabric may be exposed. The curve of equilibrium for an arched vault or a roof, where the load is equally diffused

over the whole surface, is that of a parabola (*Art.* 460); for a dome having no *lantern*, tower, or cupola above it, a *cubic parabola* (*Fig.* 97): and for one having a tower, etc., above it, a curve approaching that of an hyperbola must be adopted, as the greatest strength is required at its upper parts. If the curve of a dome be circular (as in the vertical section, *Fig.* 95), the pressure will have a tendency to burst the dome outwards at about one third of its height. Therefore, when this form is used in the construction of an extensive dome, an iron band should be placed around the framework at that height; and whatever may be the form of the curve, a band or tie of some kind is necessary around or across the base.

FIG. 97.

If the framing be of a form less convex than the curve of equilibrium, the weight will have a tendency to crush the ribs inwards, but this pressure may be effectually overcome by strutting between the ribs; and hence it is important that the struts be so placed as to form continuous horizontal circles.

238.—Cubic Parabola Computed.—Let *a b* (*Fig.* 97) be the base, and *b c* the height. Bisect *a b* at *d*, and divide *a d* into 100 equal parts; of these give *d e* 26, *e f* $18\frac{1}{4}$, *f g* $14\frac{1}{2}$, *g h* $12\frac{1}{4}$, *h i* $10\frac{3}{4}$, *i j* $9\frac{1}{2}$, and the balance, $8\frac{3}{4}$, to *j a*; divide *b c* into 8 equal parts, and from the points of division draw lines parallel to *a b*, to meet perpendiculars from the several points

of division in $a\ b$, at the points o, o, o, etc. Then a curve traced through these points will be the one required.

239.—Small Domes over Stairways: are frequently made elliptical in both plan and section; and as no two of the ribs

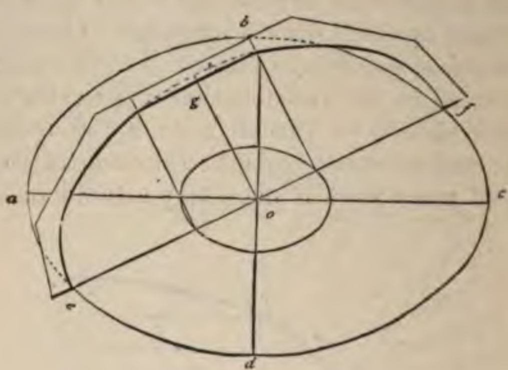


FIG. 98.

in one quarter of the dome are alike in form, a method for obtaining the curves may be useful.

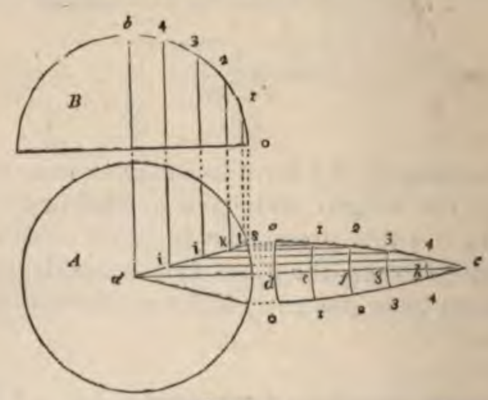


FIG. 99.

To find the curves for the ribs of an elliptical dome, let $a\ b\ c\ d$ (*Fig.* 98) be the plan of a dome, and $e\ f$ the seat of one of the ribs. Then take $e\ f$ for the transverse *axis* and twice the rise, $o\ g$, of the dome for the conjugate, and de-

scribe (according to *Arts.* 548, 549, etc.) the semi-ellipse *e g f*, which will be the curve required for the rib *e g f*. The other ribs are found in the same manner.

240.—Covering for a Spherical Dome.—To find the shape, let *A* (*Fig.* 99) be the plan, and *B* the section, of a given dome. From *a* draw *a c* at right angles to *a b*; find the stretch-out (*Art.* 524) of *o b*, and make *d c* equal to it; divide the arc *o b* and the line *d c* each into a like number of equal parts, as 5 (a large number will insure greater accuracy than a small one); upon *c*, through the several points of division in *c d*, describe the arcs *o d o*, 1 *e* 1, 2 *f* 2, etc.; make *d o* equal to half the width of one of the boards, and draw *o s* parallel

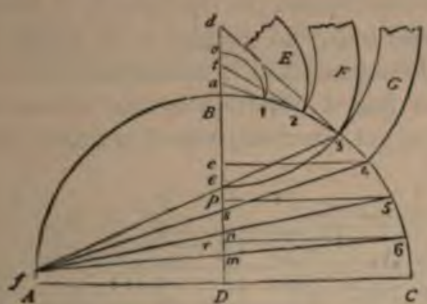


FIG. 100.

to *a c*; join *s* and *a*, and from the points of division in the arc *o b* drop perpendiculars, meeting *a s* in *i j k l*; from these points draw *i* 4, *j* 3, etc., parallel to *a c*; make *d o*, *e* 1, etc., on the lower side of *a c*, equal to *d o*, *e* 1, etc., on the upper side; trace a curve through the points *o*, 1, 2, 3, 4, *c*, on each side of *d c*; then *o c o* will be the proper shape for the board. By dividing the circumference of the base *A* into equal parts, and making the bottom, *o d o*, of the board of a size equal to one of those parts, every board may be made of the same size. In the same manner as the above, the shape of the covering for sections of another form may be found, such as an ogee, cove, etc.

To find the curve of the boards when laid in horizontal courses, let *A B C* (*Fig.* 100) be the section of a given dome,

and $D B$ its axis. Divide $B C$ into as many parts as there are to be courses of boards, in the points 1, 2, 3, etc.; through 1 and 2 draw a line to meet the axis extended at a; then a will be the centre for describing the edges of the board F. Through 3 and 2 draw $3 b$; then b will be the centre for de-

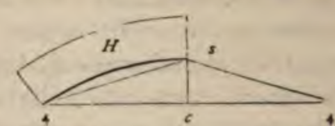


FIG. 101.

scribing F. Through 4 and 3 draw $4 d$; then d will be the centre for G. B is the centre for the arc $1 o$. If this method is taken to find the centres for the boards at the base of the dome, they would occur so distant as to make it impracticable; the following method is preferable for this purpose: G being the last board obtained by the above method, extend the curve of its inner edge until it meets the axis, $D B$,

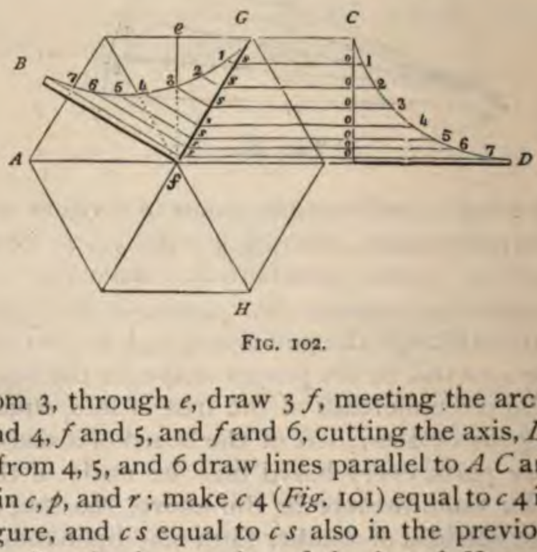


FIG. 102.

in e; from 3, through e, draw $3 f$, meeting the arc $A B$ in f; join f and 4, f and 5, and f and 6, cutting the axis, $D B$, in s, n, and m; from 4, 5, and 6 draw lines parallel to $A C$ and cutting the axis in c, p, and r; make $c 4$ (*Fig.* 101) equal to $c 4$ in the previous figure, and $c s$ equal to $c s$ also in the previous figure; then describe the inner edge of the board H, according to *Art.* 516; the outer edge can be obtained by gauging from the inner edge. In like manner proceed to obtain the next

board—taking p 5 for half the chord, and p n for the height of the segment. Should the segment be too large to be described easily, reduce it by finding intermediate points in the curve, as at *Art.* 515.

241.—Polygonal Dome: Form of Angle-Rib.—To obtain the shape of this rib, let $A\ G\ H$ (*Fig.* 102) be the plan of a given dome, and $C\ D$ a vertical section taken at the line $e\ f$. From 1, 2, 3, etc., in the arc $C\ D$ draw ordinates, parallel to $A\ D$, to meet $f G$; from the points of intersection on $f G$ draw ordinates at right angles to $f G$; make s 1 equal to o 1, s 2 equal to o 2, etc.; then $G f B$, obtained in this way, will be the angle-rib required. The best position for the sheathing-boards for a dome of this kind is horizontal, but if they are required to be bent from the base to the vertex, their shape may be found in a similar manner to that shown at *Fig.* 99.

BRIDGES.

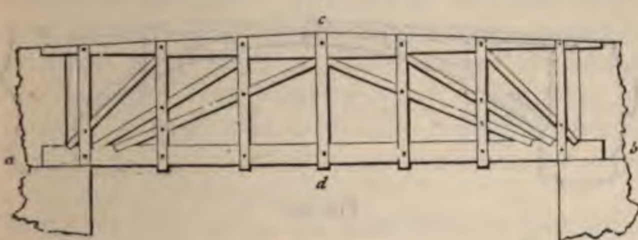


FIG. 103

242.—Bridges.—Of plans for the construction of bridges, perhaps the following are the most useful. *Fig.* 103 shows a method of constructing wooden bridges where the banks of the river are high enough to permit the use of the tie-beam, $a\ b$. The upright pieces, $c\ d$, are notched and bolted on in pairs, for the support of the tie-beam. A bridge of this construction exerts no lateral pressure upon the abutments. This method may be employed even where the banks of the river are low, by letting the timbers for the roadway rest immediately upon the tie-beam. In this case the framework above will serve the purpose of a railing.

Fig. 104 exhibits a wooden bridge without a tie-beam. Where staunch buttresses can be obtained this method may be recommended; but if there is any doubt of their stability, it should not be attempted, as it is evident that such a system of framing is capable of a tremendous lateral thrust.

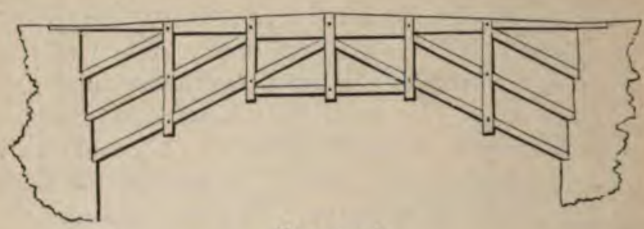


FIG. 104.

243.—Bridges: Built-Rib.—*Fig.* 105 represents a bridge with a *built-rib* (see *Art.* 231) as a chief support. The curve of equilibrium will not differ much from that of a parabola; this, therefore, may be used—especially if the rib is made

FIG. 105.

gradually a little stronger as it approaches the buttresses. As it is desirable that a bridge be kept low, the following table is given to show the least rise that may be given to the rib.

Span in Feet.	Least Rise in Feet	Span in Feet.	Least Rise in Feet	Span in Feet.	Least Rise in Feet
30	0·5	120	7	280	24
40	0·8	140	8	300	28
50	1·4	160	10	320	32
60	2	180	11	350	39
70	2½	200	12	380	47
80	3	220	14	400	53
90	4	240	17		
100	5	260	20		

The rise should never be made less than this, but in all cases greater if practicable; as a small rise requires a greater quantity of timber to make the bridge equally strong. The greatest uniform weight with which a bridge is likely to be loaded is, probably, that of a dense crowd of people. This may be estimated at 70 pounds per square foot, and the framing and gravelled roadway at 230 pounds more; which amounts to 300 pounds on a square foot. The following rule, based upon this estimate, may be useful in determining the area of the ribs.

Rule LXVII.—Multiply the width of the bridge by the square of half the span, both in feet, and divide this product by the rise in feet multiplied by the number of ribs; the quotient multiplied by the decimal 0·0011 will give the area of each rib in feet. When the roadway is only planked, use the decimal 0·0007 instead of 0·0011.

Example.—What should be the area of the ribs for a bridge of 200 feet span, to rise 15 feet and be 30 feet wide, with three curved ribs? The half of the span is 100, and its square is 10000; this multiplied by 30 gives 300000, and 15 multiplied by 3 gives 45; then 300000 divided by 45 gives 6666⅔, which multiplied by 0·0011 gives 7·333 feet or 1056 inches for the area of each rib. Such a rib may be 24 inches thick by 44 inches deep, and composed of 6 pieces, 2 in width and 3 in depth.

The above rule gives the area of a rib that would be requisite to support the greatest possible *uniform* load. But in large bridges, a *variable* load, such as a heavy wagon, is capable of exerting much greater strains; in such cases, therefore, the rib should be made larger.*

In constructing these ribs, if the span be not over 50 feet, each rib may be made in two or three thicknesses of timber (three thicknesses is preferable), of convenient lengths bolted together; but in larger spans, where the rib will be such as to render it difficult to procure timber of sufficient breadth, they may be constructed by bending the pieces to the proper curve and bolting them together. In this case, where tim-

* See Tredgold's *Carpentry* by Hurst, Arts. 174 to 177.

ber of sufficient length to span the opening cannot be obtained, and scarfing is necessary. such joints must be made as will resist both tension and compression (see *Fig.* 114). To ascertain the greatest depth for the pieces which compose the rib, so that the process of bending may not injure their elasticity, multiply the radius of curvature in feet by the decimal 0·05, and the product will be the depth in inches.

Example.—Suppose the curve of the rib to be described with a radius of 100 feet, then what should be the depth? The radius in feet, 100, multiplied by 0·05 gives a product of 5 inches. White pine or oak timber 5 inches thick would freely bend to the above curve; and if the required depth of such a rib be 20 inches, it would have to be composed of at least 4 pieces. Pitch pine is not quite so elastic as white pine or oak—its thickness may be found by using the decimal 0·046 instead of 0·05.

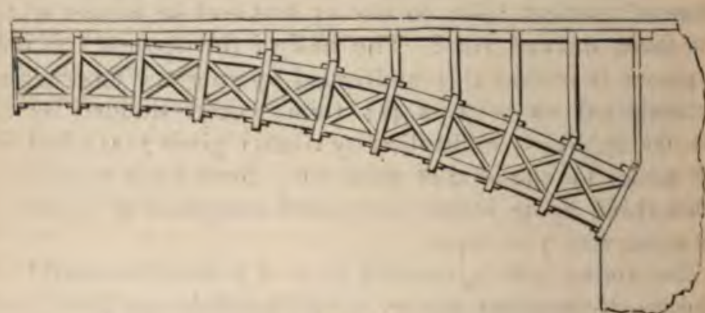


FIG. 106.

244.—Bridges: Framed Rib.—In spans of over 250 feet, a *framed* rib, as in *Fig.* 106, would be preferable to the foregoing. Of this, the upper and the lower edges are formed as just described, by bending the timber to the proper curve. The pieces that tend to the centre of the curve, called *radials*, are notched and bolted on in pairs, and the cross-braces are halved together in the middle, and abut end to end between the radials. The distance between the ribs of a bridge should not exceed about 8 feet. The roadway should be supported by vertical standards bolted to the ribs

at about every 10 to 15 feet. At the place where they rest on the ribs, a double, horizontal tie should be notched and bolted on the back of the ribs, and also another on the underside; and diagonal braces should be framed between the standards, over the space between the ribs, to prevent lateral motion. The timbers for the roadway may be as light as their situation will admit, as all useless timber is only an unnecessary load upon the arch.

245.—Bridges: Roadway.—If a roadway be 18 feet wide, two carriages can pass without inconvenience. Its width, therefore, should be either 9, 18, 27, or 36 feet, according to the amount of travel. The width of the foot-path should be two feet for every person. When a stream of water has a rapid current, as few piers as practicable should be allowed to obstruct its course; otherwise the bridge will be liable to be swept away by freshets. When the span is not over 300 feet, and the banks of the river are of sufficient height to admit of it, only one arch should be employed. The rise of the arch is limited by the form of the roadway, and by the height of the banks of the river (see *Art.* 243). The rise of the roadway should not exceed one in 24 feet, but as the framing settles about one in 72, the roadway should be framed to rise one in 18, that it may be one in 24 after settling. The commencement of the arch at the abutments—the *spring*, as it is termed—should not be below high-water mark; and the bridge should be placed at right angles with the course of the current.

246.—Bridges: Abutments.—The best material for the abutments and piers of a bridge is stone; and no other should be used. The following rule is to determine the extent of the abutments, they being rectangular, and built with stone weighing 120 pounds to a cubic foot.

Rule LXVIII.—Multiply the square of the height of the abutment by 160, and divide this product by the weight of a square foot of the arch, and by the rise of the arch; add unity to the quotient, and extract the square root. Diminish the square root by unity, and multiply the root so dimin-

ished by half the span of the arch, and by the weight of a square foot of the arch. Divide the last product by 120 times the height of the abutment, and the quotient will be the thickness of the abutment.

Example.—Let the height of the abutment from the base to the springing of the arch be 20 feet, half the span 100 feet, the weight of a square foot of the arch, including the greatest possible load upon it, 300 pounds, and the rise of the arch 18 feet: what should be its thickness? The square of the height of the abutment, 400, multiplied by 160 gives 64000, and 300 by 18 gives 5400; 64000 divided by 5400 gives a quotient of 11·852; one added to this makes 12·852, the square root of which is 3·6; this, less one is 2·6; this multiplied by 100 gives 260, and this again by 300 gives 78000; this divided by 120 times the height of the abutment, 2400, gives 32 feet 6 inches, the thickness required.

The dimensions of a pier will be found by the same rule; for, although the thrust of an arch may be balanced by an adjoining arch when the bridge is finished, and while it remains uninjured, yet, during the erection, and in the event of one arch being destroyed, the pier should be capable of sustaining the entire thrust of the other.

Piers are sometimes constructed of timber their principal strength depending on piles driven into the earth; but such piers should never be adopted where it is possible to avoid them; for, being alternately wet and dry, they decay much sooner than the upper parts of the bridge. Spruce and elm are considered good for piles. Where the height from the bottom of the river to the roadway is great, it is a good plan to cut them off at a little below low-water mark, cap them with a horizontal tie, and upon this erect the posts for the support of the roadway. This method cuts off the part that is continually wet from that which is only occasionally so, and thus affords an opportunity for replacing the upper part. The pieces which are immersed will last a great length of time, especially when of elm; for it is a well-established fact that timber is less durable when subject to alternate dryness and moisture than when it is either continually wet or continually dry. It has been ascertained that

the piles under London Bridge, after having been driven about 600 years, were not materially decayed. These piles are chiefly of elm, and wholly immersed.

247.—Centres for Stone Bridges.—*Fig.* 107 is a design for a centre for a stone bridge where intermediate supports, as piles driven into the bed of the river, are practicable. Its timbers are so distributed as to sustain the weight of the arch-stones as they are being laid, without destroying the original form of the centre; and also to prevent its destruction or settlement, should any of the piles be swept away. The most usual error in badly-constructed centres is that the timbers are disposed so as to cause the framing to rise at the crown during the laying of the arch-stones up

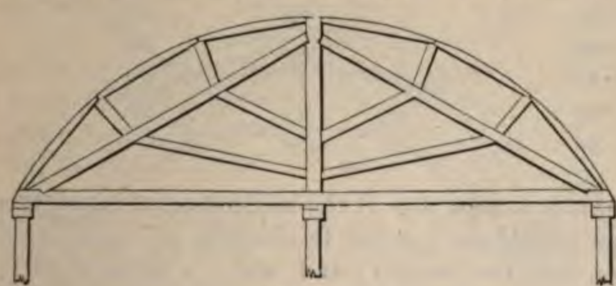


FIG. 107.

the sides. To remedy this evil, some have loaded the crown with heavy stones; but a centre properly constructed will need no such precaution.

Experiments have shown that an arch-stone does not press upon the centring until its bed is inclined to the horizon at an angle of from 30 to 45 degrees, according to the hardness of the stone, and whether it is laid in mortar or not. For general purposes, the point at which the pressure commences may be considered to be at that joint which forms an angle of 32 degrees with the horizon. At this point the pressure is inconsiderable, but gradually increases towards the crown. The following table gives the *portion* of the weight of the arch-stones that presses upon the framing at the various angles of inclination formed by the bed of the

stone with the horizon. The pressure perpendicular to the curve is equal to the weight of the arch-stone multiplied by the decimal—

·0,	when the angle	of inclination	is	32	degrees.
·04	"	"	"	34	"
·08	"	"	"	36	"
·12	"	"	"	38	"
·17	"	"	"	40	"
·21	"	"	"	42	"
·25	"	"	"	44	"
·29	"	"	"	46	"
·33	"	"	"	48	"
·37	"	"	"	50	"
·4	"	"	"	52	"
·44	"	"	"	54	"
·48	"	"	"	56	"
·52	"	"	"	58	"
·54	"	"	"	60	"

From this it is seen that at the inclination of 44 degrees the pressure equals one quarter the weight of the stone; at 57 degrees, half the weight; and when a vertical line, as *a b* (*Fig.* 108), passing through the centre of gravity of the arch-stone, does not fall within its bed, *c d*, the pressure may be considered equal to the whole weight of the stone. This will be the case at about 60 degrees, when the depth of the stone is double its breadth. The direction of these pressures is considered in a line with the radius of the curve. The weight upon a centre being known, the pressure may be estimated and the timber calculated accordingly. But it must be remembered that the whole weight is never placed upon the framing at once—as seems to have been the idea had in view by the designers of some centres. In building the arch, it should be commenced at each buttress at the same time (as is generally the case), and each side should progress equally towards the crown. In designing the fram-

Fig. 108.

ing, the effect produced by each successive layer of stone should be considered. The pressure of the stones upon one side should, by the arrangement of the struts, be counterpoised by that of the stones upon the other side.

Over a river whose stream is rapid, or where it is necessary to preserve an uninterrupted passage for the purposes of navigation, the centre must be constructed without intermediate supports, and without a continued horizontal tie at the base: such a centre is shown at *Fig.* 109. In laying the stones from the base up to *a* and *c*, the pieces *b d* and *b d* act as ties to prevent any rising at *b*. After this, while the stones are being laid from *a* and from *c* to *b*, they act as struts: the piece *f g* is added for additional security. Upon this plan, with some variation to suit circumstances,

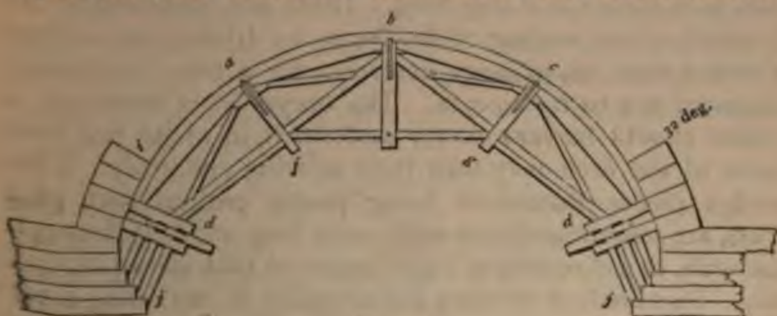


FIG. 109.

Centres may be constructed for any span usual in stone-bridge building.

In bridge centres, the principal timbers should abut, and not be intercepted by a suspension or radial piece between. These should be in halves, notched on each side and bolted. The timbers should intersect as little as possible, for the more joints the greater is the settling; and halving them together is a bad practice, as it destroys nearly one half the strength of the timber. Ties should be introduced across, especially where many timbers meet; and as the centre is to serve but a temporary purpose, the whole should be designed with a view to employ the timber afterwards for other uses. For this reason, all unnecessary cutting should be avoided.

Centres should be sufficiently strong to preserve a staunch and steady form during the whole process of building; for any shaking or trembling will have a tendency to prevent the mortar or cement from *setting*. For this purpose, also, the centre should be lowered a trifle immediately after the key-stone is laid, in order that the stones may take their bearing before the mortar is set; otherwise the joints will open on the underside. The trusses, in centring, are placed at the distance of from 4 to 6 feet apart, according to their strength and the weight of the arch. Between every two trusses diagonal braces should be introduced to prevent lateral motion.

In order that the centre may be easily lowered, the frames, or trusses, should be placed upon wedge-formed sills, as is shown at *d* (*Fig.* 109). These are contrived so as to admit of the settling of the frame by driving the wedge *d* with a maul, or, in large centres, with a piece of timber mounted as a battering-ram. The operation of lowering a centre should be very slowly performed, in order that the parts of the arch may take their bearing uniformly. The wedge pieces, instead of being placed parallel with the truss, are sometimes made sufficiently long and laid through the arch, in a direction at right angles to that shown at *Fig.* 109. This method obviates the necessity of stationing men beneath the arch during the process of lowering; and was originally adopted with success soon after the occurrence of an accident, in lowering a centre, by which nine men were killed.

To give some idea of the manner of estimating the pressures, in order to select timber of the proper scantling, calculate the pressure (*Art.* 247) of the arch-stones from *i* to *b* (Fig. 109), and suppose half this pressure concentrated at *a*, and acting in the direction *a f*. Then, by the parallelogram of forces (*Art.* 71), the strain in the several pieces composing the frame *b d a* may be computed. Again, calculate the pressure of that portion of the arch included between *a* and *c*, and consider half of it collected at *b*, and acting in a vertical direction; then, by the parallelogram of forces, the pressure on the beams *b d* and *b d* may be found. Add the

pressure of that portion of the arch which is included between *i* and *b* to half the weight of the centre, and consider this amount concentrated at *d*, and acting in a vertical direction; then, by constructing the parallelogram of forces, the pressure upon *d j* may be ascertained.

The strains having been obtained, the dimensions of the several pieces in the frames *b a d* and *b c d* may be found by computation, as directed in the case of roof-trusses, from *Arts.* 226 to 229. The tie-beams *b d*, *b d*, if made of sufficient size to resist the compressive strain acting upon them from the load at *b*, will be more than large enough to resist the tensile strain upon them during the laying of the first part of the arch-stones below *a* and *c*.

248.—Arch-Stones: Joints.—In an arch, the arch-stones are so shaped that the joints between them are perpendicular to the curve of the arch, or to its tangent at the point at which the joint intersects the curve. In a circular arch, the

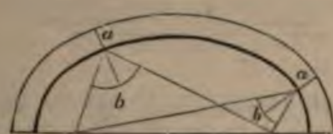


FIG. 110.

joints tend toward the centre of the circle; in an elliptical arch, the joints may be found by the following process:

To find the direction of the joints for an elliptical arch;

FIG. 111.

a joint being wanted at *a* (*Fig.* 110), draw lines from that point to the foci, *f* and *f*; bisect the angle *f a f* with the line *a b*; then *a b* will be the direction of the joint.

To find the direction of the joints for a parabolic arch: a joint being wanted at a (*Fig.* 111), draw $a\,e$ at right angles to the axis $e\,g$; make $c\,g$ equal to $c\,e$, and join a and g: draw $a\,h$ at right angles to $a\,g$; then $a\,h$ will be the direction of the joint. The direction of the joint from b is found in the same manner. The lines $a\,g$ and $b\,f$ are tangents to the curve at those points respectively; and any number of joints in the curve may be obtained by first ascertaining the tangents, and then drawing lines at right angles to them. (See *Art.* 462.)

JOINTS.

249.—Timber Joints.—The joint shown in *Fig.* 112 is simple and strong; but the strength consists wholly in the bolts, and in the friction of the parts produced by screwing the pieces firmly together. Should the timber shrink to

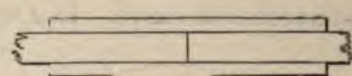


FIG. 112.

even a small degree, the strength would depend altogether on the bolts. It would be made much stronger by indenting the pieces together, as at the upper edge of the tie-beam

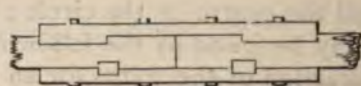


FIG. 113.

in *Fig.* 113, or by placing keys in the joints, as at the lower edge in the same figure. This process, however, weakens the beam in proportion to the depth of the indents.

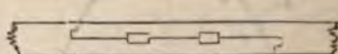


FIG. 114

Fig. 114 shows a method of scarfing, or splicing, a tie-beam without bolts. The keys are to be of well-seasoned, hard wood, and, if possible, very cross-grained. The addi-

on of bolts would make this a very strong splice, or even hite-oak pins would add materially to its strength.

Fig. 115 shows about as strong a splice, perhaps, as can ell be made. It is to be recommended for its simplicity; on account of there being no oblique joints in it, it can readily and accurately executed. A complicated joint is e worst that can be adopted; still, some have proposed nts that seem to have little else besides complication to commend them.

In proportioning the parts of these scarfs, the depths of

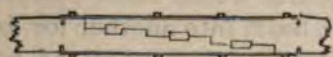


FIG. 115.

the indents taken together should be equal to one third the depth of the beam. In oak, ash or elm, the whole gth of the scarf should be six times the depth, or thicks, of the beam, when there are no bolts; but, if bolts ind of indents are used, then three times the breadth; and en both methods are combined, twice the depth of the m. The length of the scarf in pine and similar soft ods, depending wholly on indents, should be about 12 es the thickness, or depth, of the beam; when depend- wholly on bolts, 6 times the breadth; and when both thods are combined, 4 times the depth.

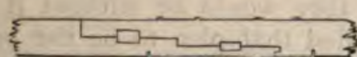


FIG. 116.

Sometimes beams have to be pieced that are required esist cross-strains—such as a girder, or the tie-beam of a f when supporting the ceiling. In such beams, the es of the wood in the upper part are compressed; and refore a simple butt joint at that place (as in *Fig.* 116) ar preferable to any other. In such case, an oblique t is the very worst. The under side of the beam being state of tension, it must be indented or bolted, or both; an iron plate under the heads of the bolts gives a great ition of strength.

Scarfing requires accuracy and care, as all the indents should bear equally; otherwise, one being strained more than another, there would be a tendency to splinter off the parts. Hence the simplest form that will attain the object is by far the best. In all beams that are compressed endwise, abutting joints, formed at right angles to the direction of their length, are at once the simplest and the best. For a temporary purpose, *Fig.* 112 would do very well; it would be improved, however, by having a piece bolted on all four sides. *Fig.* 113, and indeed each of the others, since they have no oblique joints, would resist compression well.

In framing one beam into another for bearing purposes, such as a floor-beam into a trimmer, the best place to make the mortise in the trimmer is in the neutral line (*Arts.* 120, 121), which is in the middle of its depth. Some have thought that, as the fibres of the upper edge are compressed,

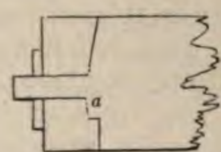


FIG. 117.

a mortise might be made there, and the tenon driven in tight enough to make the parts as capable of resisting the compression as they would be without it; and they have therefore concluded that plan to be the best. This could not be the case, even if the tenon would not shrink; for a joint between two pieces cannot possibly be made to resist compression so well as a solid piece without joints. The proper place, therefore, for the mortise is at the middle of the depth of the beam; but the best place for the tenon, in the floor-beam, is at its bottom edge. For the nearer this is placed to the upper edge, the greater is the liability for it to splinter off; if the joint is formed, therefore, as at *Fig.* 117, it will combine all the advantages that can be obtained. Double tenons are objectionable, because the piece framed into is needlessly weakened, and the tenons are seldom so accurately made as to bear equally. For this reason, unless the tusk

at *a* in the figure fits exactly, so as to bear equally with the tenon, it had better be omitted. And in sawing the shoulders care should be taken not to saw into the tenon in the least, as it would wound the beam in the place least able to bear it.

Thus it will be seen that framing weakens both pieces, more or less. It should, therefore, be avoided as much as possible, and where it is practicable one piece should rest *upon* the other, rather than be framed into it. This remark applies to the bearing of floor-beams on a girder, to the purlins and jack-rafters of a roof, etc.

In a framed truss for a roof, bridge, partition, etc., the joints should be so constructed as to direct the pressures through the axes of the several pieces, and also to avoid every tendency of the parts to slide. To attain this object,

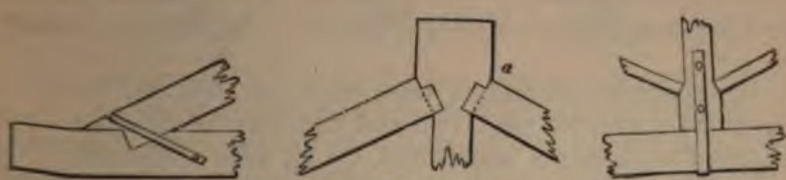


FIG. 118. FIG. 119. FIG. 120.

the abutting surface on the end of a strut should be at right angles to the direction of the pressure; as at the joint shown in *Fig.* 118 for the foot of a rafter (see *Art.* 86), in *Fig.* 119 for the head of a rafter, and in *Fig.* 120 for the foot of a strut or brace. The joint at *Fig.* 118 is not cut completely across the tie-beam, but a narrow lip is left standing in the middle, and a corresponding indent is made in the rafter, to prevent the parts from separating sideways. The abutting surface should be made as large as the attainment of other necessary objects will admit. The iron strap is added to prevent the rafter sliding out, should the end of the tie-beam, by decay or otherwise, splinter off. In making the joint shown at *Fig.* 119, it should be left a little open at *a*, so as to bring the parts to a fair bearing at the settling of the truss, which must necessarily take place from the shrinking of the king-post and other parts. If the joint is made fair at first, when the truss settles it will cause it to open at

the under side of the rafter, thus throwing the whole pressure upon the sharp edge at *a*. This will cause an indentation in the king-post, by which the truss will be made to settle further; and this pressure not being in the axis of the rafter, it will be greatly increased, thereby rendering the rafter liable to split and break.

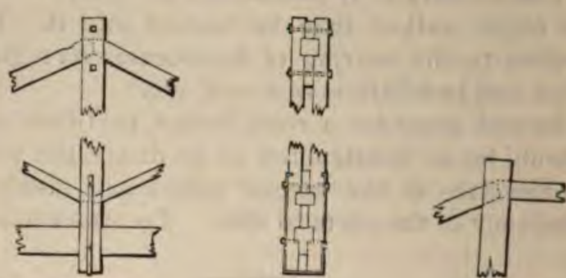


FIG. 121. FIG. 122. FIG. 123.

If the rafters and struts were made to abut end to end, as in *Figs.* 121, 122 and 123, and the king or queen post notched on in halves and bolted, the ill effects of shrinking would be avoided. This method has been practised with success in some of the most celebrated bridges and roofs in Europe; and, were its use adopted in this country, the unseemly sight of a *hogged* ridge would seldom be met with.

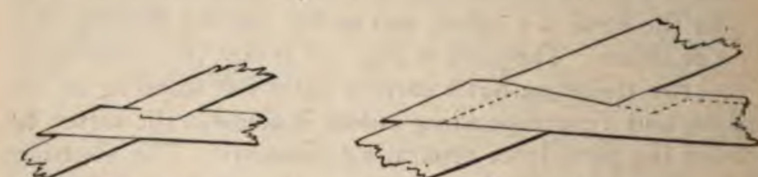


FIG. 124. FIG. 125.

A plate of cast-iron between the abutting surfaces will equalize the pressure.

Fig. 124 is a proper joint for a collar-beam in a small roof: the principle shown here should characterize all tie-joints. The dovetail joint, although extensively practised in the above and similar cases, is the very worst that can be employed. The shrinking of the timber, if only to a small

degree, permits the tie to withdraw—as is shown at *Fig.* 125. The dotted line shows the position of the tie after it has shrunk.

Locust and white-oak pins are great additions to the strength of a joint. In many cases they would supply the place of iron bolts; and, on account of their small cost, they should be used in preference wherever the strength of iron is not requisite. In small framing, good cut nails are of great service at the joints; but they should not be trusted to bear any considerable pressure, as they are apt to be brittle. Iron straps are seldom necessary, as all the joinings in carpentry may be made without them. They can be used to advantage, however, at the foot of suspending-pieces, and for the rafter at the end of the tie-beam. In roofs for ordinary purposes, the iron straps for suspending-pieces may be as follows: When the longest unsupported part of the tie-beam is—

10 feet,	the strap	may be	1	inch	wide by	$\frac{3}{16}$	thick.
15	"	"	$1\frac{1}{2}$	"	"	$\frac{1}{4}$	"
20	"	"	2	"	"	$\frac{1}{4}$	"

In fastening a strap, its hold on the suspending-piece will be much increased by turning its ends into the wood. Iron straps should be protected from rust; for thin plates of iron decay very soon, especially when exposed to dampness. For this purpose, as soon as the strap is made let it be heated to about a blue heat, and, while it is hot, pour over its entire surface raw linseed oil, or rub it with beeswax. Either of these will give it a coating which dampness will not penetrate.

SECTION III.—STAIRS.

250.—Stairs: General Requirements.—The STAIRS is that commodious arrangement of steps in a building by which access is obtained from one story to another. Their position, form, and finish, when determined with discriminating taste, add greatly to the comfort and elegance of a structure. As regards their position, the first object should be to have them near the middle of the building, in order that they may afford an equally easy access to all the rooms and passages. Next in importance is light; to obtain which they would seem to be best situated near an outer wall, in which windows might be constructed for the purpose; yet a skylight, or opening in the roof, would not only provide light, and so secure a central position for the stairs, but may be made, also, to assist materially as an ornament to the building, and, what is of more importance, afford an opportunity for better ventilation.

All stairs, especially those of the most important buildings, should be erected of *stone* or some equally durable and fire-resisting material, that the means of egress from a burning building may not be too rapidly destroyed.

Winding stairs, or those in which the direction is gradually changed by means of *winders*, or steps which taper in width, are interesting by reason of the greater skill required in their construction; but are objectionable, for the reason that children are exposed to accident by their liability to fall when passing over the narrow ends of the steps. Stairs of this kind should be tolerated only where there is not sufficient space for those with *flyers*, or steps of parallel width.

Stairs in one long continuous flight are also objectionable. *Platforms* or landings should be introduced at intervals, so that any one flight may not contain more than about twelve or fifteen steps.

The *width* of stairs should be in accordance with the im-

KHORSABAD.—ASSYRIAN TEMPLE, RESTORED.

tance of the building in which they are placed, varying n 3 to 12 feet. Where two persons are expected to pass i other conveniently the least width admissible is 3 feet. l, in crowded cities, where land is valuable, the space wed for passages is correspondingly small, and in these rs are sometimes made as narrow as 2½ feet.

From 3 to 4 feet is a suitable width for a good dwelling; le 5 feet will be found ample for stairs in buildings occu- l by many people; and from 8 to 12 feet is sufficient for width of stairs in halls of assembly.

To avoid tripping or stumbling, care should be exer- d, in the planning of a stairs, to secure an even grade. this end, the *nosing*, or outer edge, of each step should be ctly in line with all the other nosings. In stairs com- ed of both flyers and winders, precaution in this regard specially needed. In such stairs, the steps—flyers and ders alike—should be of one width on the line along ch a person would naturally walk when having his hand n the rail. This *tread*-line, consequently, would be paral- with the hand-rail, and is usually taken at a distance of n 18 to 20 inches from the centre of it. In the plan of stairs this tread-line should be drawn and divided into al parts, each part being the *tread*, or width of a flyer n the face of one riser to the face of the next.

251.—The Grade of Stairs.—The extra exertion required scending a staircase over that for walking on level and is due to the weight which a person at each step is ired to lift; that is, the weight of his own body. Hence difficulty of ascent will be in proportion to the height of step, or to the *rise*, as it is termed. To facilitate the ration of going up stairs, therefore, the *risers* should be

The grade of a stairs, or its angle of ascent, depends only upon the height of the riser, but also upon the th of the step; and this has a certain relation to the r; for the width of a step should be in proportion to the llness of the angle of ascent.

The distance from the top of one riser to the top of the is the distance travelled at each step taken, and this dis-

tance should vary as the grade of the stairs; for a person who in climbing a ladder, or a nearly vertical stairs, can travel only 12 inches, or less, at a step, will be able with equal or greater facility to travel at least twice this distance on level ground. The distance travelled, therefore, should be in proportion inversely to the angle of ascent; or, the dimensions of riser and step should be reciprocal: a low rise should have a wide step, and a high rise a narrow step.

252.—Pitch-Board: Relation of Rise to Tread.—Among the various devices for determining the relation of the rise to the *tread*, or net width of step, one is to make the sum of the two equal to 18 inches.

For example, for a rise of 6 inches the tread should be 12, for 7 inches the tread should be 11; or—

$6 + 12 = 18$	$8 + 10 = 18$
$6\frac{1}{2} + 11\frac{1}{2} = 18$	$8\frac{1}{2} + 9\frac{1}{2} = 18$
$7 + 11 = 18$	$9 + 9 = 18$
$7\frac{1}{2} + 10\frac{1}{2} = 18$	$9\frac{1}{2} + 8\frac{1}{2} = 18$

This rule is simple, but the results in extreme cases are not satisfactory. If the ascent of a stairs be gradual and easy the length from the top of one rise to that of another, or the hypothenuse of the pitch-board, may be proportionally long but if the stairs be steep, the length must be shorter.

There is a French method, introduced by Blondel in his *Cours d'Architecture*. It is referred to in Gwilt's Encyclopedia, Art. 2813.

This method is based upon the assumed distance of 24 inches as being a convenient step upon level ground, and upon 12 inches as the most convenient height to rise when the ascent is vertical. These are French inches, old system. The 24 inches French equals about $25\frac{7}{12}$ inches English.

With these distances as base and perpendicular, a right-angled triangle is formed, which is used as a scale upon which the proportions of a pitch-board are found. For example, let a line be drawn from any point in the hypothenuse of this triangle to the right angle of the triangle; then this line will equal the length of the pitch-board, along the

rake, for a stairs having a grade equal to the angle formed by this line and the base-line of the scale.

In the absence of the triangular scale, the lengths of the pitch-boards, as found by this rule, may be computed by this expression—

$$W = 25\tfrac{7}{12} - 2\,h; \qquad (107.)$$

in which W equals the tread, or base of the pitch-board, and h the riser, or its perpendicular height.

For example, let $h = 6$; then—

$$W = 25\tfrac{7}{12} - 2 \times 6 = 13\tfrac{7}{12}.$$

This result is greater than would be proper in some cases.

The length of the hypothenuse of the pitch-board should be proportional not only to the angle of ascent (*Art.* 251), but also to the strength and height of the class of people who are to use the stairs. Tall and strong persons will take longer steps than short and feeble people. The hypothenuse of the pitch-board should be made in proportion to the distance taken at a step on level ground by the persons who are to use the stairs.

If people are divided into two classes, one composed of robust workmen and the other of delicate women and infirm men, then there may be two scales formed for the pitch-boards of stairs—one to be used for shops and factories, and the other for dwellings. The distance on level ground travelled per step, by men, varies from about 26 to 32 inches, or on an average 28 inches. The height to which men are accustomed to rise on ladders is from 12 to 16 inches at each step, or on the average 14 inches.

With these dimensions, therefore, of 14 and 28 inches, a scale may be formed for pitch-boards for stairs, in buildings to be used exclusively by robust workmen. And with 12 and 24 inches another scale may be formed for pitch-boards for stairs, in buildings to be used by women and feeble people. These two scales are both shown in *Fig.* 126. They are made thus: Let $C\,A\,B$ be a right angle. Make $A\,B$ equal to 28 inches, and $A\,C$ equal to 14; then join B and

C. At right angles to $C B$, from A, draw $A F$; then with $A F$ for radius describe the arc $F G$. Then a line, as $A K$ or $A L$, drawn from A at any angle with $A B$ and limited by the line $G F B$ will give the length of the hypothenuse of the pitch-board, for shop stairs of a grade equal to the angle which said line makes with $A B$. From K, perpendicular to $A B$, draw $K N$; then $K N$ will be the proper riser for a pitch-board of which $A N$ is the tread. So, likewise, $L M$ will be the appropriate riser for the tread $A M$. The arc $F G$ is introduced to limit the rake-line of pitch-boards occurring between F and C, in order to avoid making them longer than the one at F. The scale for the stairs for dwellings is made in the same manner; $A D = 24$ inches being the base, $A E = 12$ inches the rise, and $J H D$ the line limiting the rake-lines of pitch-boards.

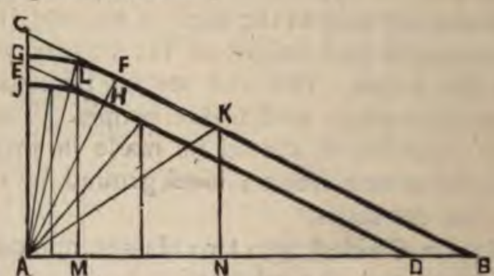


FIG. 126.

To compute the length of risers and treads, we have for the scale for shops, for those occurring between F and B—

$$r = \tfrac{1}{2}(28 - t); \qquad (108.)$$

$$t = 28 - 2r; \qquad (109.)$$

and for those between F and G, we have—

$$r = \sqrt{156 \cdot 8 - t^2}; \qquad (108, A.)$$

$$t = \sqrt{156 \cdot 8 - r^2}; \qquad (109, A.)$$

For the scale for dwellings, we have, for those occurring between H and D—

$$r = \tfrac{1}{2}(24 - t); \qquad (108, B.)$$

$$t = 24 - 2r; \qquad (109, B.)$$

and for those between *H* and *J*, we have—

$$r = \sqrt{115 \cdot 2 - t^2}; \qquad (108, C.)$$

$$t = \sqrt{115 \cdot 2 - r^2}; \qquad (109, C.)$$

where, in each equation, r represents the riser, and t the tread, or net step.

By these formulæ, the following tables have been computed:

STAIRS FOR SHOPS.

Rise.	Tread.	Ratio—Rise to Tread.	Rise.	Tread.	Ratio—Rise to Tread.
2·	24·	1 to 12·	7·40	13·20	1 to 1·78
3·	22·	1 " 7·33	7·60	12·80	1 " 1·68
3·50	21·	1 " 6·	7·80	12·40	1 " 1·59
4·	20·	1 " 5·	8·	12·	1 " 1·50
4·50	19·	1 " 4·22	8·20	11·6	1 " 1·41
5·	18·	1 " 3·60	8·50	11·	1 " 1·29
5·4	17·20	1 " 3·19	8·80	10·40	1 " 1·18
5·7	16·60	1 " 2·91	9·	10·	1 " 1·11
6·	16·	1 " 2·67	9·30	9·40	1 " 1·01
6·25	15·50	1 " 2·48	9·60	8·80	1 " 0·92
6·50	15·	1 " 2·31	10·	8·	1 " 0·80
6·70	14·60	1 " 2·18	10·50	7·	1 " 0·67
6·90	14·20	1 " 2·06	11·	6·	1 " 0·55
7·	14·	1 " 2·	11·50	4·95	1 " 0·43
7·20	13·60	1 " 1·89	12·	3·58	1" 0·30

STAIRS FOR DWELLINGS.

Rise.	Tread.	Ratio—Rise to Tread.	Rise.	Tread.	Ratio—Rise to Tread.
2·	20·	1 to 10·	7·40	9·20	1 to 1·24
3·	18·	1 " 6·	7·50	9·	1 " 1·20
3·50	17·	1 " 4·86	7·60	8·80	1 " 1.16
4·	16·	1 " 4·	7·70	8·60	1 " 1·12
4·50	15·	1 " 3·33	7·80	8·40	1 " 1·08
5·	14·	1 " 2·80	7·90	8·20	1 " 1·04
5·40	13·20	1 " 2·44	8·	8·	1 " 1·
5·70	12·60	1 " 2·21	8·10	7·80	1 " 0·96
6·	12·	1 " 2·	8·30	7·40	1 " 0·89
6·25	11·50	1 " 1·84	8·50	7·	1 " 0·82
6·50	11·	1 " 1·69	8·75	6·50	1 " 0·74
6·75	10·50	1 " 1·56	9·	6·	1 " 0·67
7·	10·	1 " 1·43	9·30	5·40	1 " 0·58
7·10	9·80	1 " 1·38	9·60	4·80	1 " 0·50
7·20	9·60	1 " 1·33	10·	3·90	1 " 0·39
7·30	9·40	1 " 1·29	10·50	2·20	1 " 0·21

These tables will be useful in determining questions in-

volving the proportion between the rise and tread of a pitch-board.

For stairs in which the run is limited, to determine the number of risers which would give an easy ascent: *Divide* the *run* by the *height*, and find in the proper table, above, the *ratio nearest* to the *quotient*, and in a line with this ratio, in the *second* column to the left, will be found the corresponding *riser*. With this divide the *rise* in inches; the *quotient*, or the *nearest whole number* thereto, will be the required *number* of risers in the stairs.

Example.—For the stairs in a dwelling, let the rise be 12′ 8″, or 152 inches. Let the run between the extreme risers be 17′ 2″. To this, for the purpose of obtaining the correct angle of ascent, by having an equal number of risers and treads, add, for one more tread, say 10 inches, its probable width; thus making the total run 18 feet, or 216 inches. Thus we have for the run 216, and for the rise 152. Dividing the former by the latter gives 1·42 nearly. In the table of stairs for dwellings, the ratio nearest to this is 1·43, and in the line to the left, in the second column, is 7, the approximate size of riser appropriate to this case. Dividing the rise, 152 inches, by this 7, we have $21\frac{5}{7}$ as the quotient.

This is nearer to 22 than to 21; therefore, the number of risers required is 22.

When the number of risers is determined, then the rise divided by this number will give the height of each riser; thus, in the above case, the rise is 152 inches. This divided by 22 gives 6·909 inches for the height of the riser.

When the height of the riser is known, then, if the run is unlimited, the width of tread will be found in the proper table above. For example, if the riser is 7 inches or nearly that, then in the table of stairs for dwellings, in the next column to the right, and opposite 7 in the column of risers, is found 10, the approximate width of tread. By the use of equation (109, B.), the width may be had exactly according to the scale. For example, equation (109, B.) with 6·91 for the riser, becomes—

$$t = 24 - 2 \times 6{\cdot}91 = 10{\cdot}18,$$

or about $10\frac{3}{16}$ inches.

When the run is limited and the number of risers is known, then the width of tread is obtained by dividing the run by the number of treads. There are always of treads one less than there are of risers, in each flight.

253.—Dimensions of the Pitch-Board.—The first thing in commencing to build a stairs is to make the *pitch*-board; this is done in the following manner: Obtain very accurately, in feet and inches, the rise, or perpendicular height, of the story in which the stairs are to be placed. This must be taken from the top of the lower floor to the top of the upper floor. Then, to obtain the number of rises and treads and their size, proceed as directed in *Art.* 252. Having obtained these, the pitch-board may be made in the following manner: Upon a piece of well-seasoned board about $\frac{5}{8}$ of an inch thick, having one edge jointed straight and square, lay the corner of a steel square, as shown at *Fig.* 127. Make *a b* equal to the riser, and *b c* equal to the tread; mark along the edges with a knife, and cut by the marks, making the edges of the pitch-board perfectly square. The grain of the wood should run in the direction indicated in the figure, because, in case of shrinkage, the rise and the tread will be equally affected by it. When a pitch-board is first made, the dimensions of the riser and tread should be preserved in figures, in order that, in case of shrinkage or damage otherwise, a second may be made.

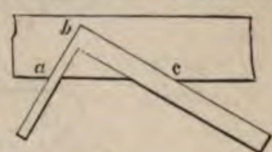


FIG. 127.

254.—The String of a Stairs.—The space required for timber and plastering under the steps is about 5 inches for ordinary stairs, or 6 inches if furred; set a gauge, therefore, at 5 or 6 inches, as the case requires, and run it on the lower edge of the plank, as *a b* (*Fig.* 128). Commencing at one end, lay the longest side of the pitch-board against the gauge-mark, *a b*, as at *c*, and draw by the edges the lines for the first rise and tread; then place it successively as at *d*, *e*,

and *f*, until the required number of risers shall be laid down. To insure accuracy, it is well to ascertain the theoretical raking length of the pitch-board by computation, as in note to *Art.* 536, by getting the square root of the sum of the squares of the rise and run, and using this by which to divide the line *a b* into equal parts.

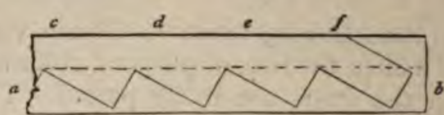


FIG. 128.

255.—Step and Riser Connection.—*Fig.* 129 represents a section of step and riser, joined after the most approved method. In this, *a* represents the end of a block about 2

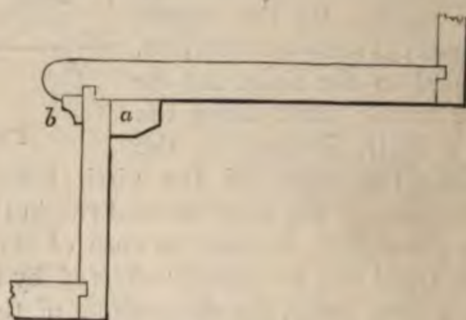


FIG. 129.

inches long, two or three of which, in the length of the step, are glued in the corner. The cove at *b* is planed up square, glued in, and *stuck* or moulded after the glue is set.

PLATFORM STAIRS.

256.—Platform Stairs: the Cylinder.—A platform stairs ascends from one story to another in two or more flights, having platforms or landings between for resting and to change their direction. This kind of stairs, being simple, is

easily constructed, and at the same time is to be preferred to those with *winders*, for the convenience it affords in use (*Art.* 250). The cylinder may be of any diameter desirable, from a few inches to 3 or more feet, but it is generally small, about 6 inches. It may be worked out of one solid piece, but a better way is to glue together 3 pieces, as in *Fig.* 130; in which the pieces *a*, *b*, and *c* compose the cylinder, and *d* and *e* represent parts of the strings. The strings, after being glued to the cylinder, are secured with screws. The joining at *p* and *o* is the most proper for that kind of joint.

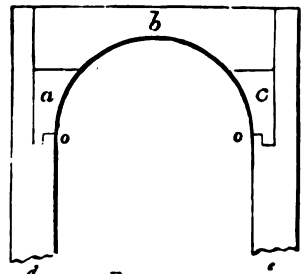


FIG. 130.

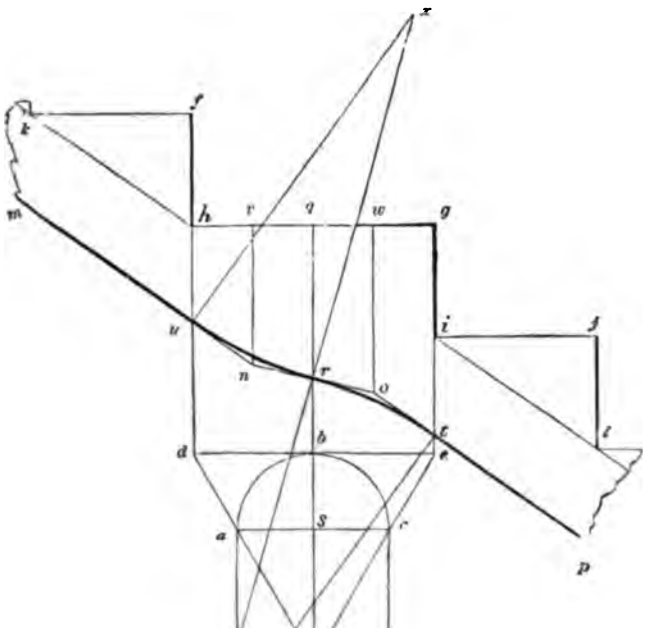


FIG. 131.

257.—Form of Lower Edge of Cylinder.—Find the stretch-out, *de* (*Fig.* 131), of the face of the cylinder, *abc*,

according to *Art.* 524; from *d* and *e* draw *df* and *eg* at right angles to *de*; draw *hg* parallel to *de*, and make *hf* and *gi* each equal to one riser; from *i* and *f* draw *ij* and *fk* parallel to *hg*; place the tread of the pitch-board at these last lines, and draw by the lower edge the lines *kh* and *il*; parallel to these draw *mn* and *op*, at the requisite distance for the dimensions of the string; from *s*, the centre of the plan, draw *sq* parallel to *df*; divide *hq* and *qg* each into two equal parts, as at *v* and *w*; from *v* and *w* draw *vn* and *wo* parallel to *fd*; join *n* and *o*, cutting *qs* in *r*; then the angles *unr* and *rot*, being eased off according to *Art.* 521, will give the proper curve for the bottom edge of the cylinder. A centre may be found upon which to describe these curves, thus: from *u* draw *ux* at right angles to *mn*; from *r* draw *rx* at right angles to *no*; then *x* will be the centre for the curve *ur*. The centre for the curve *rt* may be found in a similar manner. Centres from which to strike these curves are usually quite unnecessary; an experienced workman will readily form the curves guided alone by his practised eye.

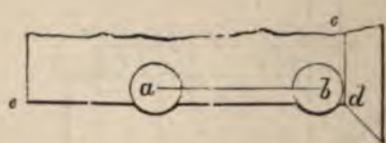


FIG. 132.

258.—Position of the Balusters.—Place the centre of the first baluster, *b* (*Fig.* 132), half its diameter from the face of the riser, *cd*, and one third its diameter from the end of the step, *ed*; and place the centre of the other baluster, *a*, half the tread from the centre of the first. A line through the centre of the rail will occur vertically over the centres of the balusters. The usual length of the balusters is 2 feet 5 inches and 2 feet 9 inches respectively, for the short and long balusters. Their length may be greater than is here indicated, but, for safety, should never be less. The difference in length between the short and long balusters is equal to one half the height of a riser.

259.—Winding Stairs: have the steps narrower at one end than at the other. In some stairs there are steps of parallel width incorporated with the tapering steps; in this case the former are called *flyers*, and the latter *winders*.

260.—Regular Winding Stairs.—In *Fig.* 133, *a b c d* represents the inner surface of the wall enclosing the space allotted to the stairs, *a e* the length of the steps, and *e f g h* the cylinder, or face of the front-string. The line *a e* is given as the face of the first riser, and the point *j* for the limit of

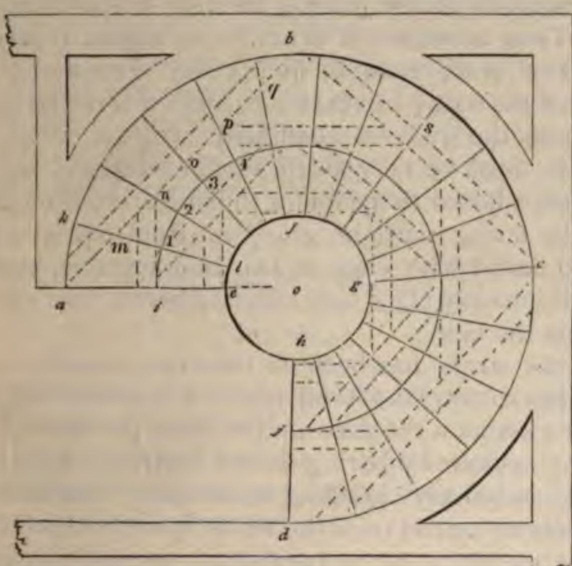


FIG. 133.

the last. Make *e i* equal to 18 inches, and upon *o*, with *o i* for radius, describe the arc *i j*; obtain the number of risers and of treads required to ascend to the floor at *j*, according to *Art.* 252, and divide the arc *i j* into the same number of equal parts as there are to be treads: through the points of division, 1, 2, 3, etc., and from the wall-string to the front-string, draw lines tending to the centre, *o*: then these lines will represent the face of each riser, and determine the form and width of the steps. Allow the necessary projection for the nosing beyond *a e*, which should be equal to the thickness

ness of the step, and then $a\,e\,l\,k$ will be the dimensions for each step. Make a pitch-board for the wall-string having $a\,k$ for the tread, and the rise as previously ascertained: with this lay out on a thicknessed plank the several risers and treads, as at *Fig.* 128, gauging from the upper edge of the string for the line at which to set the pitch-board.

Upon the back of the string, with a 1¼-inch dado plane, make a succession of grooves 1¼ inches apart, and parallel with the lines for the risers on the face. These grooves must be cut along the whole length of the plank, and deep enough to admit of the plank's bending around the curve $a\,b\,c\,d$. Then construct a drum, or cylinder, of any common kind of stuff, made to fit a curve with a radius the thickness of the string less than $o\,a$; upon this the string must be bent, and the grooves filled with strips of wood, called *keys*, which must be very nicely fitted and glued in. After it has dried, a board thin enough to bend around on the outside of the string must be glued on from one end to the other, and nailed with clout-nails. In doing this, be careful not to nail into any place opposite to where a riser or step is to enter on the face.

After the string has been on the drum a sufficient time for the glue to set, take it off, and cut the mortices for the steps and risers on the face at the lines previously made; which may be done by boring with a centre-bit half through the string, and nicely chiselling to the line. The drum need not be made to extend over the whole space occupied by the stairs, but merely so far as requisite to receive one piece of the wall-string at a time; for it is evident that more than one will be required. The front-string may be constructed in the same manner; taking $e\,l$ instead of $a\,k$ for the tread of the pitch-board, dadoing it with a smaller dado plane, and bending it on a drum of the proper size.

261.—Winding Stairs: Shape and Position of Timbers.— The dotted lines in *Fig.* 133 show the position of the timbers as regards the plan; the shape of each is obtained as follows: In *Fig.* 134, the line $1\,a$ is equal to a riser, less the thickness of the floor, and the lines $2\,m$, $3\,n$, $4\,o$, $5\,p$, and $6\,q$ are each

equal to one riser. The line a 2 is equal to $a\ m$ in *Fig.* 133, the line m 3 to $m\ n$ in that figure, etc. In drawing this figure, commence at a, and make the lines a 1 and a 2 of the length above specified, and draw them at right angles to each other; draw 2 m at right angles to a 2, and m 3 at right angles to m 2, and make 2 m and m 3 of the lengths as above specified; and so proceed to the end. Then through the points 1, 2, 3, 4, 5, and 6 trace the line 1 b; upon the points 1, 2, 3, 4, etc., with the size of the timber for radius, describe arcs as shown in the figure, and by these the lower line may be traced parallel to the upper. This will give the proper shape for the timber, $a\ b$, in *Fig.* 133; and that of the others may be found in a similar manner. In ordinary cases, the shape of one face of the timber will be sufficient, for a good workman can easily hew it to its proper level by that; but where great accuracy is desirable, a pattern for the other side may be found in the same man-

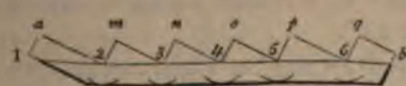


FIG. 134.

ner as for the first. In many cases, the timbers beneath circular stairs are put up after the stairs are erected, and without previously giving them the required form; the workman in shaping them being guided by the form marked out by the lower edge of the risers.

262.—Winding Stairs with Flyers: Grade of Front-String.—In stairs of this kind, if the winders are confined to the quarter circle, the transition from the winders to the flyers is too abrupt for convenience, as well as in appearance. To remove this unsightly bend in the rail and string, it is usual to take in among the winders one or more of the flyers, and thus graduate the width of the winders to that of the flyers. But this is not always done so as to secure the best results. By the method now to be shown, both rail and strings will be gracefully graded. In *Fig.* 135, $a\ b$ represents the line of the facia along the floor of the upper story,

b e c the face of the cylinder, and *c d* the face of the front-string. Make *g b* equal to $\frac{1}{3}$ of the diameter of the baluster, and parallel to *a b*, *b e c*, and *c d* draw the centre-line of the rail, *f g*, *g h i*, and *i j*; make *g k* and *g l* each equal to half the width of the rail, and through *k* and *l*, parallel to the centre-line, draw lines for the convex and the concave sides of the rail; tangical to the convex side of the rail, and parallel to *k m*, draw *n o*; obtain the stretch-out, *q r*, of the semicircle, *k p m*, according to *Art.* 524; extend *a b* to *t*, and *k m* to *s*; make *c s* equal to the length of the steps, and *i u* equal to 18 inches, and parallel to *m p* describe the arcs *s t* and *u* 6; from *t* draw *t w*, tending to the centre of the cylinder; from 6, and on the line 6 *u x*, run off the regular tread, as at 5, 4, 3, 2, 1, and *v*; make *u x* equal to half the arc *u* 6, and make the point of division nearest to *x*, as *v*, the limit of the parallel steps, or flyers; make *r o* equal to *m s*; from *o* draw *o* a^2* at right angles to *n o*, and equal to one riser; from a^2 draw a^2 *s* parallel to *n o*, and equal to one tread; from *s*, through *o*, draw *s* b^2.

Then from *w* draw *w* c^2 at right angles to *n o*, and set up on the line *w* c^2 the same number of risers that the floor, *A*, is above the first winder, *B*, as at 1, 2, 3, 4, 5, and 6; through 5 (on the arc 6 *u*) draw d^2 e^2, tending to the centre of the cylinder; from e^2 draw e^2 f^2 at right angles to *n o*, and through 5 (on the line *w* c^2) draw g^2 f^2 parallel to *n o*; through 6 (on the line *w* c^2) and f^2 draw the line h^2 b^2; make 6 c^2 equal to half a riser, and from c^2 and 6 draw c^2 i^2 and 6 j^2 parallel to *n o*; make h^2 i^2 equal to h^2 f^2; from i^2 draw i^2 k^2 at right angles to i^2 h^2, and from f^2 draw f^2 k^2 at right angles to f^2 h^2; upon k^2, with k^2 f^2 for radius, describe the arc f^2 i^2; make b^2 l^2 equal to b^2 f^2, and ease off the angle at b^2 by the curve f^2 l^2. In the figure, the curve is described from a centre, but as this might be impracticable in a full-size plan, the curve may be obtained accord-

* In the references a^2, b^2, etc., a new form is introduced for the first time. During the time taken to refer to the figure, the memory of the *form* of these may pass from the mind, while that of the *sound* alone remains; they may then be mistaken for *a* 2, *b* 2, etc. This can be avoided in reading by giving them a sound corresponding to their meaning, which is *a second*, *b second*, etc.

ing to *Art.* 521. Then from 1, 2, 3, and 4 (on the line $w c^2$) draw lines parallel to $n o$, meeting the curve in m^2, n^2, o^2, and p^2; from these points draw lines at right angles to $n o$, and meeting it in x^2, r^2, s^2, and t^2; from x^2 and r^2 draw

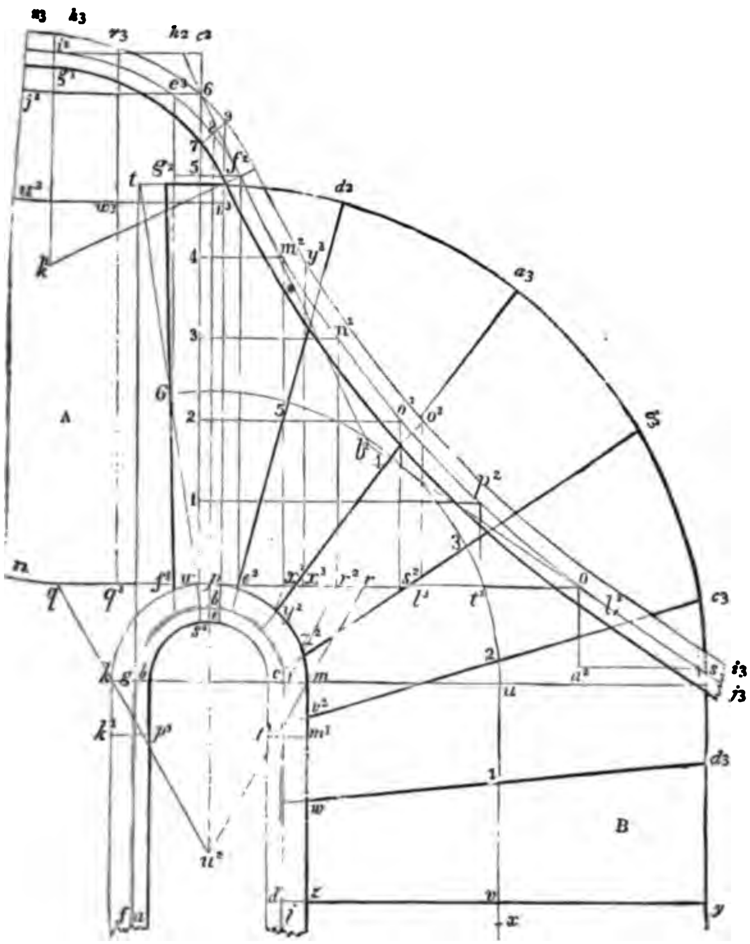


FIG. 135.

lines tending to u^2, and meeting the convex side of the rail in y^2 and z^2; make $m v^2$ equal to $r s^2$, and $m w^2$ equal to $r t^2$; from y^2, z^2, v^2, and w^2, through 4, 3, 2, and 1, draw lines meeting the line of the wall-string in a^3, b^3, c^3, and d^3; from

e^3, where the centre-line of the rail crosses the line of the floor, draw $e^3 f^3$ at right angles to $n\,o$, and from f^3, through 6, draw $f^3 g^2$; then the heavy lines $f^3 g^2$, $e^2 d^2$, $y^2 a^3$, $z^2 b^3$, $v^2 c^3$, $w^2 d^3$, and $z\,y$ will be the lines for the risers, which, being extended to the line of the front-string, $b\ e\ c\ d$, will give the dimensions of the winders and the grading of the front-string, as was required.

HAND-RAILING.

263.—Hand-Railing for Stairs.—A piece of hand-railing intended for the curved part of a stairs, when properly shaped, has a twisted form, deviating widely from plane surfaces. If laid upon a table it may easily be rocked to and fro, and can be made to coincide with the surface of the table in only three points. And yet it is usual to cut such twisted pieces from ordinary parallel-faced plank; and to cut the plank in form according to a *face*-mould, previously formed from given dimensions obtained from the plan of the stairs. The shape of the finished wreath differs so widely from the piece when first cut from the plank as to make it appear to a novice a matter of exceeding difficulty, if not an impossibility, to design a face-mould which shall cover accurately the form of the completed wreath. But he will find, as he progresses in a study of the subject, that it is not only a possibility, but that the science has been reduced to such a system that all necessary moulds may be obtained with great facility. To attain to this proficiency, however, requires close attention and continued persistent study, yet no more than this important science deserves. The young carpenter may entertain a less worthy ambition than that of desiring to be able to form from planks of black-walnut or mahogany those pieces of hand-railing which, when secured together with rail-screws, shall, on applying them over the stairs for which they are intended, be found to fit their places exactly, and to form graceful curves at the cylinders. That railing which requires to be placed upon the stairs before cutting the joints, or which requires the curves or butt-joints to be refitted after leaving the shop, is discredit-

able to the workman who makes it. No true mechanic will be content until he shall be proved able to form the curves and cut the joints in the shop, and so accurately that no alteration shall be needed when the railing is brought to its place on the stairs. The science of hand-railing requires some knowledge of *descriptive geometry*—that branch of geometry which has for its object the solution of problems involving three dimensions by means of intersecting planes. The method of obtaining the lengths and bevils of hip and valley rafters, etc., as in *Art.* 233, is a practical example of descriptive geometry. The lines and angles to be developed in problems of hand-railing are to be obtained by methods dependent upon like principles.

264.—Hand-Railing: Definitions; Planes and Solids.—Preliminary to an exposition of the method for drawing the face-moulds of a hand-rail wreath, certain terms used in descriptive geometry need to be defined. Among the tools used by a carpenter are those well-known implements called planes, such as the jack-plane, fore-plane, smoothing-plane, etc. These enable the workman to straighten and smooth the faces of boards and plank, and to dress them *out of wind*, or so that their surfaces shall be true and unwinding. The term *plane*, as used in descriptive geometry, however, refers not to the implement aforesaid, but to the unwinding surface formed by these implements. A plane in geometry is defined to be such a surface that if any two points in it be joined by a straight line, this line will be in contact with the surface at every point in its length. With like results lines may be drawn in all possible directions upon such a surface. This can be done only upon an unwinding surface: therefore, a plane is an unwinding surface. Planes are understood to be unlimited in their extent, and to pass freely through other planes encountered.

The science of stair-building has to do with *prisms* and *cylinders*, examples of which are shown in *Figs.* 136, 137, and 138. A right prism (*Figs.* 136 and 137) is a solid standing upon a horizontal plane, and with faces each of which is a plane. Two of these faces—top and bottom—are horizontal

and are equal polygons, having their corresponding sides parallel.

The other faces of the prism are parallelograms, each of which is a vertical plane. When the vertical sides of a prism are of equal width, and in number increased indefinitely, the two polygonal faces of the prism do not differ essentially from circles, and thence the prism becomes a *cylinder*. Thus a right cylinder may be defined to be a prism, with circles for the horizontal faces (*Fig.* 138).


FIG. 136.


FIG. 137.

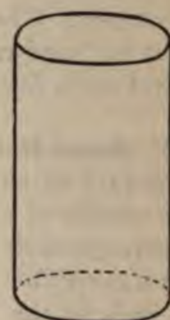


FIG. 138.

265.—Hand-Railing: Preliminary Considerations.—If within the *well-hole*, or stair-opening, of a circular stairs a solid cylinder be constructed of such diameter as shall fill the well-hole completely, touching the hand-railing at all points, and then if the top of this cylinder be cut off on a line with the top of the hand-railing, the upper end of the cylinder would present a winding surface. But if, instead of cutting the cylinder as suggested, it be cut by several planes, each of which shall extend so as to cover only one of the wreaths of the railing, and be so inclined as to touch its top in three points, then the form of each of these planes, at its intersection with the vertical sides of the cylinder, would present the shape of the concave edge of the face-mould for that particular piece of hand-railing covered by the plane. Again, if a hollow cylinder be constructed so as to be in contact with the outer edge of the hand-railing throughout its length, and this cylinder be also cut by the aforesaid

planes, then each of said planes at its intersection with this latter cylinder would present the form of the convex edge of the said face-mould. A plank of proper thickness may now have marked upon it the shape of this face-mould, and the piece covered by the face-mould, when cut from the plank, will evidently contain a wreath like that over which the face-mould was formed, and which, by cutting away the surplus material above and below, may be gradually wrought into the graceful form of the required wreath.

By the considerations here presented some general idea may be had of the method pursued, by which the form of a face-mould for hand-railing is obtained. A little reflection upon what has been advanced will show that the problem to be solved is to pass a plane obliquely through a cylinder at certain given points, and find its shape at its intersection with the vertical surface of the cylinder. Peter Nicholson was the first to show how this might be done, and for the invention was rewarded, by a scientific society of London, with a gold medal. Other writers have suggested some slight improvements on Nicholson's methods. The method to which preference is now given, for its simplicity of working and certainty of results, is that which deals with the *tangents* to the curves, instead of with the curves themselves; so we do not pass a plane through a cylinder, but through a *prism* the vertical sides of which are tangent to the cylinder, and contain the controlling tangents of the face-moulds. The task, therefore, is confined principally to finding the tangents upon the face-mould. This accomplished, the rest is easy, as will be seen.

The method by which is found the form of the top of a prism cut by an oblique plane will now be shown.

266.—A Prism Cut by an Oblique Plane.—A prism is shown in perspective at *Fig.* 139, cut by an oblique plane. The points *a b c d* are the angles of the horizontal base, and *a b g*, *b c f*, *c d e f*, and *a d e g* are the vertical sides; while *e f b g* is the top, the form of which is to be shown.

267.—Form of Top of Prism.—In *Fig.* 139 the form of the top of the prism is shown as it appears in perspective,

not in its *real* shape; this is now to be developed. In *Fig.* 140, let the square $a\,b\,c\,d$ represent by scale the actual form

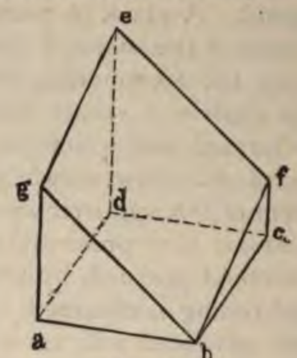


FIG. 139.

and size of the base, $a\,b\,c\,d$, of the prism shown in *Fig.* 139. Make $c\,c_{,}$ and $d\,d_{,}$ respectively equal to the actual heights at

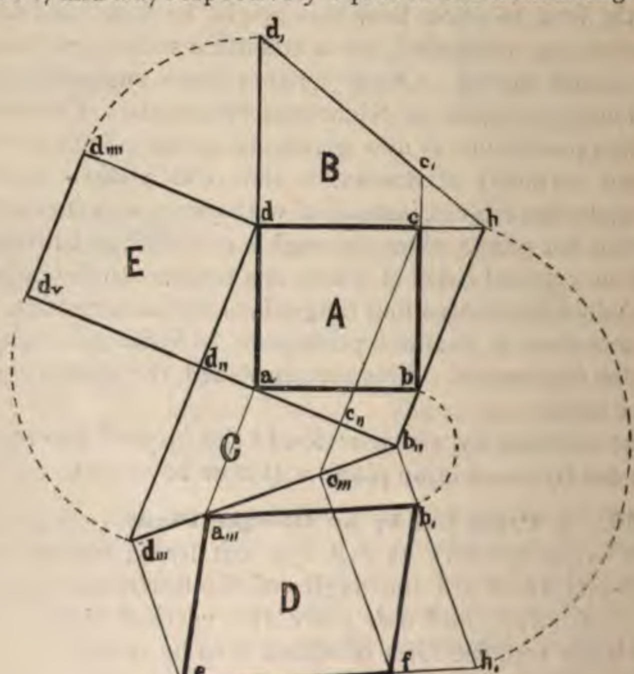


FIG. 140.

$c\,f$ and $d\,e$, *Fig.* 139; the lines $d\,d_{,}$ and $c\,c_{,}$ being set up perpendicular to the line $d\,c$. Extend the lines $d\,c$ and $d_{,}c_{,}$ until

ey meet in h; join b and h. Now this line $b\,h$ is the inter-ction of two planes: one, the base, or horizontal plane upon hich the prism stands; the other, the *cutting* plane, or the ane which, passing obliquely through the prism, cuts it so to produce, by intersecting the vertical sides of the prism, e form $b\,f\,e\,g$, *Fig.* 139.

To show that $b\,h$ is the line of intersection of these two ines, let the paper on which the triangle $d\,h\,d_{,}$ is drawn esignated by the letter B) be lifted by the point $d_{,}$ and volved on the line $d\,h$ until $d_{,}$ stands vertically over d, and over c; then B will be a plane standing on the line $d\,h$, rtical to the base-plane A. The point h being in the line extended, and the line $c\,d$ being in the base-plane A, there-e h is in the base-plane A. Now the line $d_{,}\,c_{,}$ represents line $e\,f$ of *Fig.* 139, and is therefore in the cutting plane; nsequently the point h, being also in the line $d_{,}\,c_{,}$ ex-ded, is also in the cutting plane. By reference to *Fig.* it will be seen that the point b is in both the cutting and e planes; we must therefore conclude that, since the two nts b and h are in both the cutting and base planes, a line ing these two points must be the intersection of these two nes. The determination of the line of intersection of the e and cutting planes is very important, as it is a control-g line; as will be seen in defining the lines upon which form of the face-mould depends. Care should therefore taken that the method of obtaining it be clearly under-od.

It will be observed that the intersecting line $b\,h$, being in horizontal plane A, is therefore a horizontal line. Also, t this horizontal line $b\,h$ being a line in the cutting plane, refore all lines upon the cutting plane which are drawn allel to $b\,h$ must also be horizontal lines. The import-e of this will shortly be seen. Through a, perpendicular $b\,h$, draw the line $b_{,,}\,d_{v}$, and parallel with this line draw $_{,,,}$; on d as centre describe the arc $d_{,}\,d_{,,,,}$; draw $d_{,,,,}\,d_{v}$ allel with $d\,d_{,,}$, and extend the latter to $d_{,,,}$; on $d_{,,}$ as tre describe the arc $d_{v}\,d_{,,,}$; join $b_{,,}$ and $d_{,,,}$. We now e three vertical planes which are to be brought into tion around the base-plane A, as follows: Revolve B

upon $d h$, E upon $d d_{,,}$, and C upon $b_{,,} d_{,,}$, each until it stands perpendicular to the plane A. Then the points $d_{,}$ and $d_{,,,,}$ will coincide and be vertically over d; the points $d_{,,,}$ and $d_{,}$ will coincide and stand vertically over $d_{,,}$; and $c_{,}$ will cover c. These vertical planes will enclose a wedge-shaped figure, lying with one face, $b_{,,} d_{,,} d h$, horizontal and coincident with the base-plane A, and three vertical faces, $b_{,,} d_{,,} d_{,,,}$, $d d_{,,} d_{,} d_{,,,,}$, and $h d d_{,}$. By drawing the figure upon a piece of stout paper, cutting it out at the outer edges, making creases in the lines $h d$, $d d_{,,}$, $d_{,,} b_{,,}$, then folding the three planes B, E, and C at right angles to A, the relation of the lines will be readily seen. Now, to obtain the form of the top or cover to the wedge-shaped figure, perpendicular to $b_{,,} d_{,,,}$ draw $b_{,,} h_{,}$ and $d_{,,,} e$; on $b_{,,}$ as centre describe the arc $h h_{,}$; make $d_{,,,} e$ equal to $d_{,,} d$; join e and $h_{,}$. Now the form of the top of the wedge-shaped figure is shown within the bounds $d_{,,,} b_{,,} h_{,} e$. By revolving this plane D on the line $b_{,,} d_{,,,}$ until it is at a right angle to the plane C, and this while the latter is supposed to be vertical to the plane A, it will be perceived that this movement will place the plane D on top of the wedge-shaped figure, and in such a manner as that the point e will coincide with $d_{,,,,} d_{,}$, and the point $h_{,}$ will fall upon and be coincident with the point h, and the lines of the cover will coincide with the corresponding lines of the top edges of the sides of the figure; for example, the line $b_{,,} d_{,,,}$ is common to the top and the side C; the line $d_{,,,} e$ equals $d_{,,} d$, which equals $d_{v} d_{,,,,}$; therefore, the line $d_{,,,} e$ will coincide with $d_{v} d_{,,,,}$ of the side E; the line $e h_{,}$ will coincide with $d_{,} h$ of the side B; and the line $b_{,,} h_{,}$ will coincide with the line $b_{,,} h$. Thus the figure D bounded by $b_{,,} d_{,,,} e h_{,}$ will exactly fit as a cover to the wedge-shaped figure. Upon this cover we may now develop the form of the top of the prism.

Preliminary thereto, however, it will be observed, as was before remarked, that lines upon the cutting plane which are parallel to the intersecting line $b_{,,} h_{,}$ are horizontal; and each, therefore, must be of the same length as the line in the base-plane A vertically beneath it. For example, the line $d_{,,,} e_{,}$ is a line in the cutting plane D, parallel with the line $b_{,,} h_{,}$ in the same plane, and this line $b_{,,} h_{,}$ will (when the

cutting plane D is revolved into its proper position) be coincident with the intersecting line $b_{,,} h$; therefore, the line $d_{,,,} e$ is a line in the cutting plane D, drawn parallel with the intersecting line $b_{,,} h$. Now this line $d_{,,,} e$, when in position, will be coincident with the line $d_{,,,,} d_{v}$, which lies vertically over the line $d_{,,} d$ of the base-plane A; its length, therefore, is equal to that of the latter. In like manner it may be shown that the length of any line on the plane D parallel to $b_{,,} h_{,,}$ is equal in length to the corresponding line upon the plane A vertically beneath it.

Therefore, to obtain the form of the top of the prism, we proceed as follows: Perpendicular to $b_{,,} d_{v}$ draw $c\, c_{,,,}$ and $a\, a_{,,,}$; perpendicular to $b_{,,} d_{,,,}$ draw $c_{,,,} f$ and equal to $c_{,,} c$; on $b_{,,}$ as centre describe the arc $b\, b_{,}$; join $b_{,} a_{,,,}$, $b_{,} f$, and $a_{,,,} e$. Now we have here in plane D the form of the top of the prism, as shown in the figure bounded by the lines $a_{,,,} b_{,} f e$. This will be readily seen when the plane D is revolved into position. Then the point $a_{,,,}$ will be vertically over a; the point e coincident with $d_{,} d_{,,,,}$ and vertically over d; the point f coincident with $c_{,}$ and vertically over c; while $b_{,}$ will coincide with b of the base-plane A.

The figure $a_{,,,} b_{,} f e$, therefore, represents correctly both in form and size the top of the prism as it is shown in perspective at $b f e g$, *Fig.* 139. The line $e f$, *Fig.* 140, is equal to the line $d_{,} c_{,}$, and so of the other lines bounding the edges of the figure.

The cutting plane $b f e g$, *Fig.* 139, may be taken to represent the surface of the plank from which the wreath of hand-railing is to be cut; the wreath curving around from b to e, as shown in *Fig.* 141, the lines $b g$ and $g e$ being tangent to the curve in the cutting plane; while $a b$ and $a d$ are tangents to the curve on the base plane, or plane of the cylinder. The location of the cutting plane, however, is usually not at the upper surface of the plank, but midway between the upper and under surfaces. The tangents in the plane are found to be more conveniently located here for determining the position of the butt-joints. For a moulded rail two curved lines, each with a pair of tangents, are required upon the cutting plane, one for the outer edge of the rail,

and the other for the inner edge; but for a round rail only one curve with its tangents is required, as that from *b* to *e* in *Fig.* 141, which is taken to represent the curved line running through the centre of the cross-section of the rail. As an easy application of the principles regarding the prism just developed, an example will now be given.

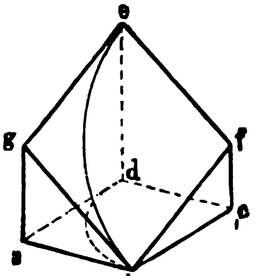


FIG. 141.

268.—Face-Mould for Hand-Railing of Platform Stairs. —Let $j\,k$ and $l\,m$, *Fig.* 142, represent the central or axial lines of the hand-rails of the two flights, one above, the other below the platform; and let the semicircle $j\,d\,l$ be the central line of the rail around the cylinder at the platform, the risers at the platform being located at j and l. Vertically over the platform risers draw $g\,g_{,}$; make $g\,r_{,}$ equal to a riser of the lower flight, and $r_{,}\,g_{,}$ and $s\,s_{,}$ each equal to a riser of the upper flight. Draw $g_{,}\,s$ and $g\,k_{,}$ horizontal and equal each to a tread of each flight respectively. Through $r_{,}$ draw $k_{,}\,a_{,,}$, and through $g_{,}$ draw $s_{,}\,t_{,}$. Vertically over d draw $a_{,}\,t_{,}$. Horizontally draw $a_{,,}\,a_{,,,,}$ and $t_{,}\,t_{,,}$.

It is usual to extend the wreath of the cylinder so as to include a part of the straight rail—such a part as convenience may require. Let the straight part here to be included extend from l to b on the plan. Vertically over b draw $b_{,}\,c_{,,,,}$, and horizontally draw $b_{,}\,w_{,,}$; at any point on $b_{,}\,w_{,,}$ locate $w_{,,}$, and make $w_{,,}\,w_{,}$ equal to $j\,l$, and bisect it in w; erect the perpendiculars $w_{,}\,a_{,,,,}$, $w\,d_{v,,}$, and $w_{,,}\,v$; join $t_{,,}$ and $a_{,,,,}$; from $d_{v,,}$ horizontally draw $d_{v,,}\,d_{v,}$; parallel with $r_{,}\,k_{,}$ draw $d_{v,}\,c_{,,,,}$. We now have the plan and elevations of the prism,

ontaining at its angles the tangents required for the wreath xtending from *b* to *d* on the plan. The elevation *F* is a view f the cylinder looking in the direction *d c*.

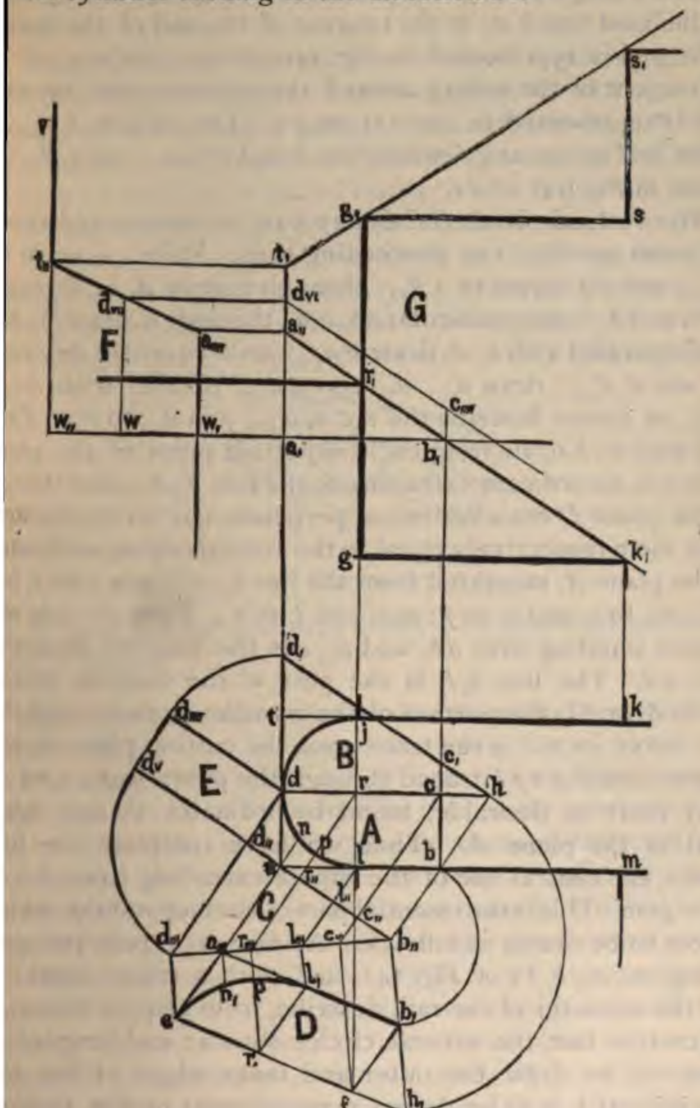


FIG. 142.

Comparing *Fig.* 142 with *Fig.* 141; the line $b_{i}w_{ii}$ is the ice, upon a vertical plane, of the horizontal plane *a b c d*

of *Fig.* 141, or is the ground-line from which the heights of the prism are to be taken.

The triangle $a_{,}\,b_{,}\,a_{,,}$ is represented in *Fig.* 141 at $a\,b\,g$, and the inclined line $b_{,}\,a_{,,}$ is the tangent of the rail of the lower flight, and is represented in *Fig.* 141 at $b\,g$; while $a_{,,,,}\,t_{,,}$ is the tangent of the railing around the cylinder, and the half of it is represented in *Fig.* 141 at $g\,e$. The height $b_{,}\,c_{,,,,}$ is shown in *Fig.* 141 at $c\,f$, while the height $w\,d_{v,,}$, or $a_{,}\,d_{v,}$, is shown in *Fig.* 141 at $d\,e$.

The vertical planes $B\,E\,C$ may now be constructed about the prism as in *Fig.* 140, proceeding thus: Make $c\,c_{,}$ equal to $b_{,}\,c_{,,,,}$, and $d\,d_{,}$ equal to $a_{,}\,d_{v,}$; through $c_{,}$ draw $d_{,}\,h$; through b draw $h\,b_{,,}$; perpendicular to $h\,b_{,,}$ through a draw $b_{,,}\,d_{v}$; from d parallel with $b_{,,}\,d_{v}$ draw $d\,d_{,,,,}$; on d as centre describe the arc $d_{,}\,d_{,,,,}$; draw $d_{,,,,}\,d_{v}$, also $d\,d_{,,,}$, parallel with $h\,b_{,,}$; on $d_{,,}$ as centre describe the arc $d_{v}\,d_{,,,}$; join $d_{,,,}$ to $b_{,,}$. Parallel with $b_{,,}\,h$ draw from each important point of the plan, as shown, an ordinate extending to the line $b_{,,}\,d_{,,,}$, and thence across plane D draw ordinates perpendicular to $b_{,,}\,d_{,,,}$, and make them respectively equal to the corresponding ordinates of the plane A, measured from the line $b_{,,}\,d_{v}$; join e to f, [illegible] to $b_{,}$, $a_{,,,}$ to e, and $b_{,}$ to f; also join $l_{,}$ to $r_{,}$. Then $a_{,,,}\,b_{,}$ is the tangent standing over $a\,b$, and $a_{,,,}\,e$ is the tangent standing over $a\,d$. The line $b_{,}\,l_{,}$ is the part of the tangent which stands over $b\,l_{,}$, the portion of the wreath which is straight. The curve $e\,n_{,}\,p_{,}\,l_{,}$ is the trace upon the cutting plane of the quarter circle $d\,n\,p\,l$, traced through the points $n_{,}\,p_{,}$, and as many more as desirable, found by ordinates as any other point in the plane A. Thus we have complete the line $b_{,}\,l_{,}\,n_{,}\,e$, the central line of the wreath extending from b to d in the plan. This is the essential part of the face-mould, which is now to be drawn as follows: At *Fig.* 143 repeat the parallelogram $a_{,,,}\,b_{,}\,f\,e$ of *Fig.* 142, and, with a radius equal to half the diameter of the rail, describe, from centres taken on the central line, the several circles shown; and tangent to these circles draw the outer and inner edges of the rail. The joint at $b_{,}$ is to be drawn perpendicular to the tangent $b\,a_{,,,}$, while that at e is to be perpendicular to the tangent $e\,a_{,,,}$. This completes the face-mould for the wreath over

b l n d of the plan. If the pitch-board of the upper flight be the same as that of the lower flight, the face-mould at *Fig.* 143 will, reversed, serve also for the wreath over the other half of the cylinder.

In using this face-mould, place it upon a plank equal in thickness to the diameter of the rail, mark its form upon the plank, and saw square through; then chamfer the wreath to an octagonal form, after which carefully remove the angles so as to produce the required round form. The joints, as well as the curved edges, are to be cut square through the plank.

Many more lines have been used in obtaining this face-mould than were really necessary for so simple a case, but no more than was deemed advisable in order properly to elucidate the general principles involved. A very simple method

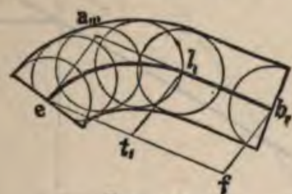


FIG. 143.

for face-moulds of platform stairs with small cylinders will now be shown.

269.—More Simple Method for Hand-Rail to Platform Stairs.—In *Fig.* 144, *j g e* represents a pitch-board of the first flight, and *d* and *i* the pitch-board of the second flight of a platform stairs, the line *e f* being the top of the platform; and *a b c* is the plan of a line passing through the centre of the rail around the cylinder. Through *i* and *d* draw *i k*, and through *j* and *e* draw *j k*; from *k* draw *k l* parallel to *f e*; from *b* draw *b m* parallel to *g d*; from *l* draw *l r* parallel to *k j*; from *n* draw *n t* at right angles to *j k*; on the line *o b* make *o t* equal to *n t*; join *c* and *t*; on the line *j c*, *Fig.* 145, make *e c* equal to *e n* at *Fig.* 144; from *c* draw *c t* at right angles to *j c*, and make *c t* equal to *c t* at *Fig.* 144; through *t* draw *p l* parallel to *j c*, and make *t l* equal to *t l* at *Fig.* 144; join *l* and *c*, and complete the parallelogram *e c l s*; find the points *o*, *o*, *o*, according to *Art.* 551; upon *e*, *o*, *o*, *o*, and *l*,

successively, with a radius equal to half the width of the rail, describe the circles shown in the figure; then a curve traced on both sides of these circles, and just touching them,

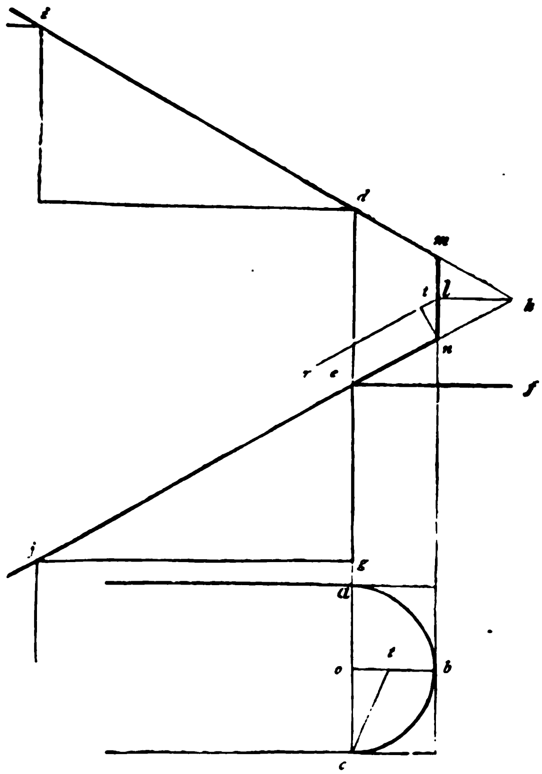


FIG. 144.

will give the proper form for the mould. The joint at *l* is drawn at right angles to *c l*.

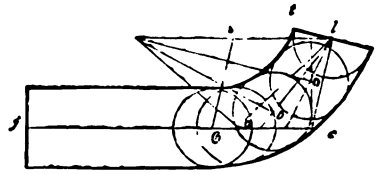


FIG. 145.

This simple method for obtaining the face-moulds for the hand-rail of a platform stairs appeared first in the early editions of this work. It was invented by a Mr. Kells, an

ninent stair-builder of this city. A comparison with *Fig.* ı2 will explain the use of the few lines introduced. For a ıll comprehension of it reference is made to *Fig.* 146, in hich the cylinder, for this purpose, is made rectangular

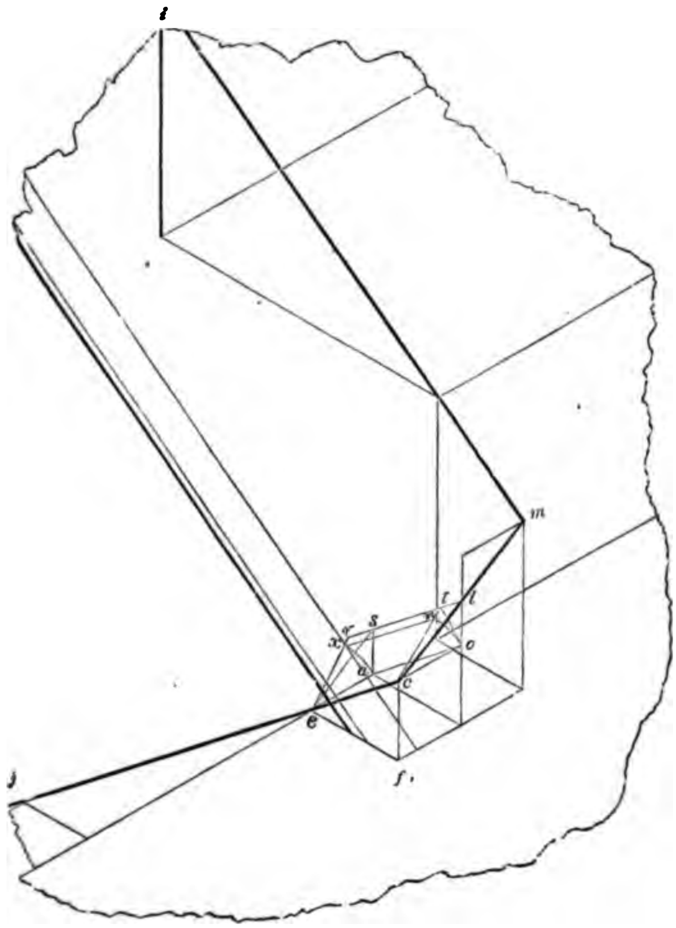


FIG. 146.

stead of circular. The figure gives a perspective view of part of the upper and of the lower flights, and a part of e platform about the cylinder. The heavy lines, *i m*, *m c*, d *c j*, show the direction of the rail, and are supposed to ss through the centre of it. Assuming that the rake of

the second flight is the same as that of the first, as is generally the case, the face-mould for the lower twist will, when reversed, do for the upper flight; that part of the rail, therefore, which passes from *c* to *c*, and from *c* to *l*, is all that will need explanation.

Suppose, then, that the parallelogram *e a o c* represent a plane lying perpendicularly over *e a b f*, being inclined in the direction *e c*, and level in the direction *c o*; suppose this

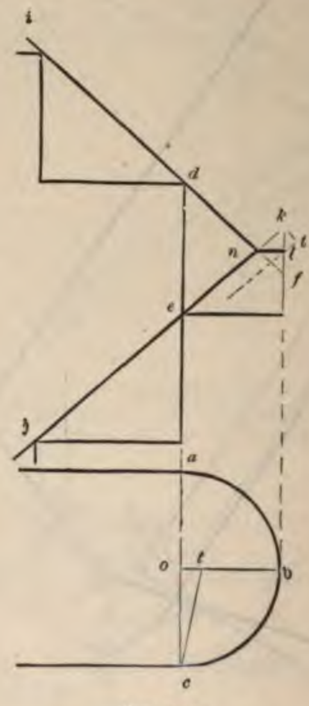


FIG. 147.

plane *e a o c* be revolved on *e c* as an axis, in the manner indicated by the arcs *o n* and *a x*, until it coincides with the plane *e r t c*; the line *a o* will then be represented by the line *x n*; then add the parallelogram *x r t n*, and the triangle *c t l*, deducting the triangle *e r s*; then the edges of the plane *e s l c*, inclined in the direction *e c*, and also in the direction *c l*, will lie perpendicularly over the plane *e a b f*. From this we gather that the line *c o*, being at right angles to *e c*, must, in

order to reach the point *l*, be lengthened the distance *nt*, and the right angle *ect* be made obtuse by the addition to it of the angle *tcl*. By reference to *Fig.* 144, it will be seen that this lengthening is performed by forming the right-angled triangle *cot*, corresponding to the triangle *cot* in *Fig.* 146. The line *ct* is then transferred to *Fig.* 145, and placed at right angles to *ec*; this angle *ect* is then increased by adding the angle *tcl*, corresponding to *tcl*, *Fig.* 146. Thus the point *l* is reached, and the proper position and length of the lines *ec* and *cl* obtained. To obtain the face-mould for a rail over a cylindrical well-hole, the same process is necessary to be followed until the length and position of these lines are found; then, by forming the parallelogram *ecls*, and describing a quarter of an ellipse therein, the proper form will be given.

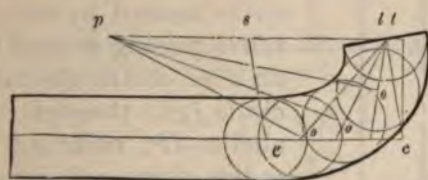


FIG. 148.

270.—Hand-Railing for a Larger Cylinder.—*Fig.* 147 represents a plan and a vertical section of a line passing through the centre of the rail as before. From *b* draw *bk* parallel to *cd*; extend the lines *id* and *je* until they meet *kb* in *k* and *f*; from *n* draw *nl* parallel to *ob*; through *l* draw *lt* parallel to *jk*; from *k* draw *kt* at right angles to *jk*; on the line *ob* make *ot* equal to *kt*. Make *ec* (*Fig.* 148) equal to *ek* at *Fig.* 147; from *c* draw *ct* at right angles to *ec*, and equal to *ct* at *Fig.* 147; from *t* draw *tp* parallel to *ce*, and make *tl* equal to *tl* at *Fig.* 147; complete the parallelogram *ecls*, and find the points *o*, *o*, *o*, as before; then describe the circles and complete the mould as in *Fig.* 145. The difference between this and Case 1 is that the line *ct*, instead of being raised and thrown out, is lowered and drawn in. A method of planning a cylinder so as to avoid the necessity of canting the plank, either up or down, will now be shown.

271.—Face-Mould without Canting the Plank.—Instead of placing the platform-risers at the spring of the cylinder, a more easy and graceful appearance may be given to the rail, and the necessity of canting either of the twists entirely obviated, by fixing the place of the above risers at a certain distance within the cylinder, as shown in *Fig.* 149—the lines indicating the face of the risers cutting the cylinder at *k* and *l*, instead of at *p* and *q*, the spring of the cylinder. To ascertain the position of the risers, let *a b c* be the pitch-board of the lower flight, and *c d e* that of the upper flight, these being placed so that *b c* and *c d* shall form a right line. Extend *a c* to cut *d e* in *f*; draw *f g* parallel to *d b*, and of indefinite length; draw *g o* at right angles to *f g*, and equal in length to the radius of the circle formed by the centre of the rail in passing around the cylinder; on *o* as centre describe the semi-circle *j g i*; through *o* draw *i s* parallel to *d b*; make *o h* equal to the radius of the cylinder, and describe on *o* the face of the cylinder *p h q*; then extend *d b* across the cylinder, cutting it in *l* and *k*—giving the position of the face of the risers, as required. To find the face-mould for the twists is simple and obvious: it being merely a quarter of an ellipse, having *o j* for semi-minor axis, and *s f* for the semi-major axis; or, at *Fig.* 151, let *d c i* be a right angle; make *c i* equal to *o j*, *Fig.* 149, and *d c* equal to *s f*, *Fig.* 149; then draw *d o* parallel to *c i*, and complete the curve as before.

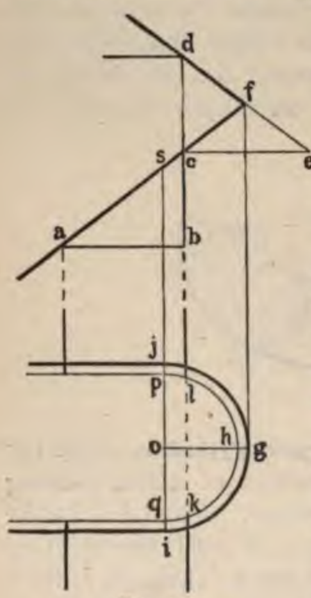


FIG. 149.

272.—Railing for Platform Stairs where the Rake meets the Level.—In *Fig.* 150, *a b c* is the plan of a line passing through the centre of the rail around the cylinder as before, and *j e* is a vertical section of two steps starting from the floor, *h g*. Bisect *e h* in *d*, and through *d* draw *d*

parallel to *hg*; bisect *fn* in *l*, and from *l* draw *lt* parallel to *nj*; from *n* draw *nt* at right angles to *jn*; on the line *ob* make *ot* equal to *nt*. Then, to obtain a mould for the twist going up the flight, proceed as at *Fig.* 145; making *ec* in that figure equal to *en* in *Fig.* 150, and the other lines of a length and position such as is indicated by the letters of reference in each figure. To obtain the mould for the level

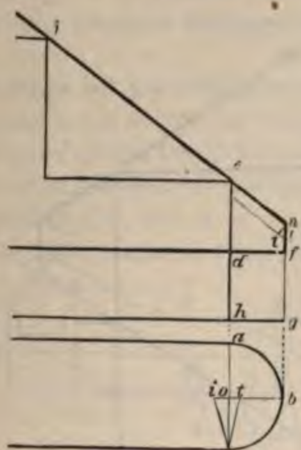


FIG. 150.

rail, extend *bo* (*Fig.* 150) to *i*; make *oi* equal to *fl*, and join *i* and *c*; make *ci* (*Fig.* 151) equal to *ci* at *Fig.* 150; through

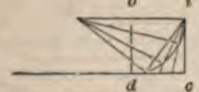


FIG. 151.

c draw *cd* at right angles to *ci*; make *dc* equal to *df* at *Fig.* 150, and complete the parallelogram *odci*; then proceed as in the previous cases to find the mould.

273.—Application of Face-Moulds to Plank.—All the moulds obtained by the preceding examples have been for round rails. For these, the mould may be applied to a plank of the same thickness as the rail is intended to be, and the

plank sawed square through, the joints being cut square from the face of the plank. A twist thus cut and truly rounded will hang in a proper position over the plan, and present a perfect and graceful wreath.

274.—Face-Moulds for Moulded Rails upon Platform Stairs.—In *Fig.* 152, *a b c* is the plan of a line passing through

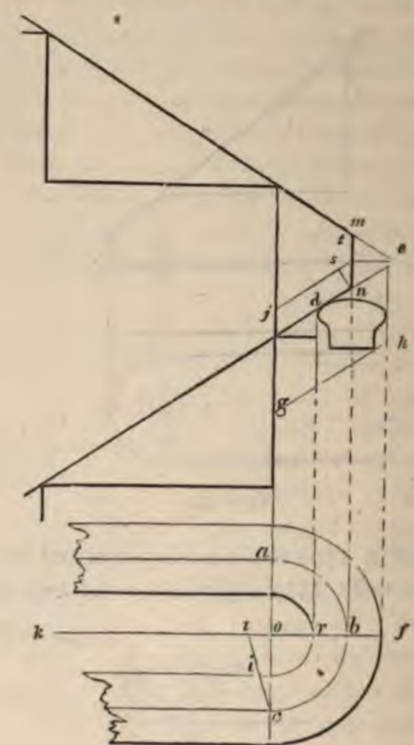


FIG. 152.

the centre of the rail around the cylinder, as before, and the lines above it are a vertical section of steps, risers, and platform, with the lines for the rail obtained as in *Fig.* 144. Set half the width of the rail from *b* to *f* and from *b* to *r*, and from *f* and *r* draw *fe* and *rd* parallel to *ca*. At *Fig.* 153 the centre-lines of the rail *jc* and *cl* are obtained as in the previous examples, making *fc* equal *jn* of *Fig.* 152, *ct*

equal ct of *Fig.* 152, and tl equal sl of *Fig.* 152. Make ci and ck each equal to ci at *Fig.* 152, and draw the lines im and kg parallel to cj; make le and lr equal to ne and nd at *Fig.* 152, and draw dn and eq parallel to lc; also, through j draw og parallel to lc; then, in the parallelograms $mnro$ and $goeq$, find the elliptic curves, dm and eg, according to *Art.* 551, and they will define the curves. The line dp, being drawn through l perpendicular to lc, defines the joint which is to be cut square through the plank.

275.—Application of Face-Moulds to Plank.—In *Fig.* 152 make a drawing, from d to h, of the cross-section of the hand-rail, and tangent to the lower corner draw the line gh. The distance between the lines fe and gh is the thickness of the plank from which the rail is to be cut. Lay the face-mould upon the plank, mark its shape upon the plank, and

FIG. 153.

saw it *square* through. To proceed strictly in accordance with the requirements of the principles upon which the face-mould is formed, the cutting ought to be made vertically through the plank, the latter being in the position which it would occupy when upon the stairs. Formerly it was the custom to cut it thus, with its long raking lines. But, owing to the great labor and inconvenience of this method, efforts were made to secure an easier process. By investigation it was found that it was possible, without change in the face-mould, to cut the plank square through and still obtain the correct figure for the railing, and this method is the one now usually pursued. Not only is the labor of sawing much reduced by this change; but to the workman it is an entire relief, as he now, after marking the form of the wreath upon the plank, sends it to a steam saw-mill, and, at a small cost, has it

cut out with an upright scroll-saw. When thus cut out in the square, the upper surface of the plank is to be faced up true and unwinding, and the outer edge jointed straight and square from the face. Then a figure of the cross-section of the hand-railing is to be carefully drawn on the ends of the squared block as shown in *Figs.* 154 and 155, and which are regulated so as to be correctly in position, as follows. *First*, as to the end *h* of the straight part *h j*: In *Fig.* 154, let *a b c d* be an end view of the squared block, of which *a e f d* is the shape of the end of the straight part. Let the point *g* be the centre of this end of the straight part; through *g* draw upon the end *a e f d* the line *j k*, so that the angle *b j k* shall be equal to the angle *k t c*, *Fig.* 152. This is the angle at which the plank is required to be canted, revolving it on

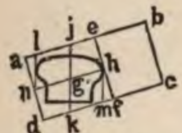


FIG. 154.

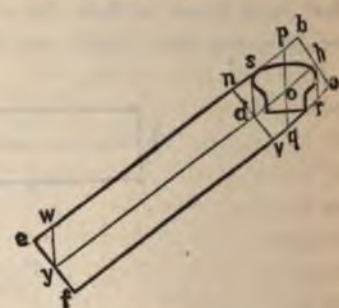


FIG. 155.

the axis of the straight part of the rail. Through *g* draw the line *n h* parallel with *a b*. Upon a thin sheet of metal (zinc is preferable) mark carefully the exact figure of the cross-section of the rail, drawing a vertical line through its centre, cut away the surplus metal, then, with this *template* as a pattern, mark upon the end *a e f d*, *Fig.* 154, the figure of the rail as shown, the vertical line upon the template being made to coincide with the line *j k*. From *n* and *h* draw the vertical lines *h m* and *l n* parallel with *j k*.

Now, as to the other end of the square block: Let *b c f e*, *Fig.* 155, represent the block, of which *b c v n* is the form of the end at the curved part, and *o* its centre. Through *o* draw *p q*, so that the angle *e p q* shall be equal to the angle *j n b*, *Fig.* 152. Also, through *o* draw *d h* parallel with *e b*;

from d and h draw the vertical lines $h\ r$ and $d\ s$ parallel with $p\ q$. Place the template on $b\ c\ v\ n$, the end of the block, so that the vertical line through its centre shall coincide with $p\ q$: mark its form, then from y, at mid-thickness, draw $w\ y$ parallel with $p\ q$.

In applying the mould, let *Fig.* 156 represent the upper face of the squared block, with the face-mould lying upon it. With the distance $a\ l$, *Fig.* 154, and by the edge $a\ x$, mark a gauge-line upon the upper face of the squared block. Set the outer edge of the face-mould to coincide with this gauge-line. Let the end of the face-mould be set at w, $e\ w$ being equal to $e\ w$, *Fig.* 155; then mark the block by the edge of the face-mould.

FIG. 156.

Now turn the block over and apply the face-mould to the underside, as in *Fig.* 157. With the distance $d\ m$, *Fig.* 154, and by the outer edge of the block, mark a gauge-line from m, *Fig.* 157. Set the inner edge of the face-mould to this gauge-line, and slide it endwise till the distance $e\ m$ shall equal $e\ w$, *Fig.* 155, then mark the block by the edges of the face-mould. The over wood may now be removed as indicated by the vertical lines at the sides of the cross-section marked on each end of the block (see also *Fig.* 167): the direction of the cutting at the curves must be vertical; the inner curve will require a round-faced plane. A comparison of the several figures referred to, with the directions given, together with a little reflection, will manifest the reasons for the method here given for applying the face-mould. Especially so when it is remembered that the face-mould was obtained not for the *top* of the rail, but for the rail at the mid-thickness of the block. So, therefore, in the application to the upper surface of the block, the face-mould is slid up the rake far enough to put the mould in position vertically over its true position at mid-thickness; and on the

FIG. 157.

contrary, in applying the face-mould to the underside of the plank, it is slid down until it is vertically beneath its true position at the mid-thickness of the block.

When the vertical faces are completed, the over wood above and below the wreath is to be removed. In doing this, the form at the ends, as given by the template, is a sufficient guide there. Between these the upper and under surfaces are to be warped from one end to the other, so as to form a graceful curve. With a little practice an intelligent mechanic will be able to work these surfaces with facility. The form of cross-section produced by this operation is that of a parallelogram, tangent to the top, bottom, and two sides of the rail; and which at and near the ends of the block is not quite full. The next operation is that of working the moulding at the sides and on top, first by rebates at the sides, then chamfering, and finally moulding the curves. Templates to fit the rail, one at the sides, another on top, are useful as checks against cutting away too much of the wood.

The joints are all to be worked square through the plank in the line drawn perpendicular to the tangent, as shown in *Fig.* 153.

276.—Hand-Railing for Circular Stairs.—Let it be required to furnish the face-moulds for a circular stairs similar to that shown in *Fig.* 133.

Preliminary to making the face-moulds it is requisite to make a plan, or horizontal projection of the stairs, and on this to locate the projections of the tangents and develop their vertical projections. For this purpose let *b c d e f g*, *Fig.* 158, be the horizontal projection of the centre of the rail, and the lines numbered from 1 to 19 be the risers. At any point, *a*, on an extension of the line of the first riser locate the centre of the newel. On *a* as a centre describe the two circles; the larger one equal in diameter to the diameter of the newel-cap, the inner one distant from the outer one equal to half the width of the rail. Let the first joint in the hand-rail be located at *b*, at the fourth riser; through *b* draw *h k* tangent to the circle. Select a point, *h*, on this

tangent which shall be equally distant from b and from the inner circle of the newel-cap, measured on a line tending to a; join h and a, and from a point, q, on the line $b\,o$ describe

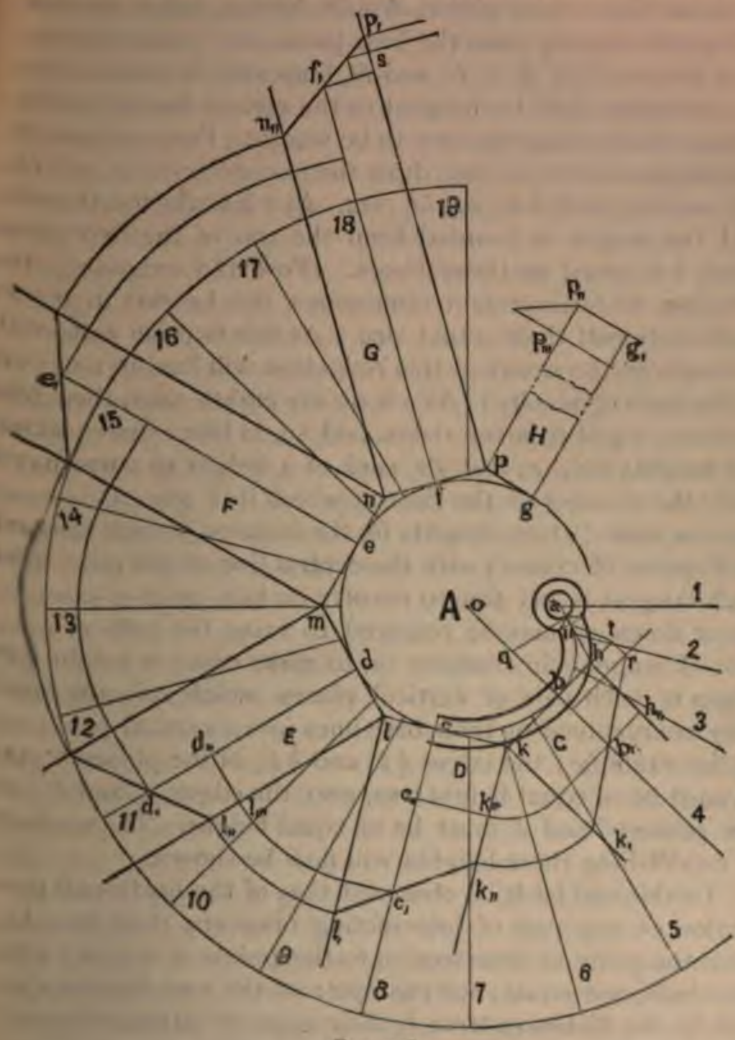


FIG. 158.

the curve from b to the point of the mitre of the newel-cap, the curve being tangent, at this point, to the line $a\,h$. Select positions for the other joints in the hand-rail as at c, d, e, and f.

Through these draw lines tangent to the circle.* Then the horizontal projection of the tangents will be the lines $h\,k$, $k\,l$, $l\,m$, $m\,n$, and $n\,p$. Now, if a vertical plane stand upon each of these lines, these planes would form a prism not quite complete standing upon the base-plane, A. Upon these vertical planes, C, D, E, F, G, and H, lines may be drawn which at each joint shall be tangent to the central line of the rail. These are the tangents now to be sought. Perpendicular to the tangents at b, c, d, etc., draw the lines $b\,b_{,}$, $c\,c_{,}$, $d\,d_{,}$, $e\,e_{,}$, $f\,f_{,}$, $g\,g_{,}$, and $h\,h_{,,}$, $k\,k_{,}$, $k\,k_{,,}$, $l\,l_{,}$, $l\,l_{,,}$, etc. As b is at the fourth riser, and the height is counted from the top of the first riser, make $b\,b_{,}$ equal to three risers. (To avoid extending the drawing to inconvenient dimensions, the heights in it are made only half their actual size. As this is done uniformly throughout the drawing, this reduction will lead to no error in the desired results.) As c is on the eighth riser, therefore make $c\,c_{,}$ equal to seven risers, and so, in like manner, make the heights $d\,d_{,}$, $e\,e_{,}$, and $f\,f_{,}$ each of a height to correspond with the number of the riser at which it is placed, deducting one riser. These heights fix the location of each tangent at its point of contact with the central line of the rail. But each tangent is yet free to revolve on this point of contact, up or down, as may be required to bring the ends of each pair of tangents in contact; or, to make equal in height the edges of each pair of vertical planes, which coincide after they are revolved on their base-lines into a vertical position; as, for example: the edges $k\,k_{,}$ and $k\,k_{,,}$ of the planes C and D must be of equal height; so, also, the edges $l\,l_{,}$ and $l\,l_{,,}$ of the planes D and E must be of equal height. The method of establishing these heights will now be shown.

To this end let it be observed, that of the horizontal projection of any pair of intersecting tangents, their lengths, from the point of intersection to the points of contact with the circle, are equal; for example: of the two tangents $h\,k$ and $l\,k$, the distances from k, their point of intersection, to b and c, their points of contact with the circle, are equal; and so also $c\,l$ equals $d\,l$, $d\,m$ equals $e\,m$, etc. It will be observed

* A tangent is a line perpendicular to the radius, drawn from the point of contact.

that this equality is not dependent on b, c, d, etc., the points of contact, being disposed at equal distances; for, in this example, they are placed at unequal distances, some being at three treads apart and others at four; and yet while this unequal distribution of the points b, c, d, etc., has the effect of causing the point of contact, as b, c, or e, to divide each whole tangent into two unequal parts, it does not disturb the equality of the two adjoining parts of any two adjacent tangents. Now, because of this equality of the two adjoining parts of a pair of tangents, the height to be overcome in passing from one point of contact to the next must be divided equally between the two; each tangent takes half the distance. Therefore, for stairs of this kind, the arrangement being symmetrical, we have this rule by which to fix the height of the ends of any two adjoining tangents, namely: To the height at the lower point of contact add half the difference between the heights at the two points of contact; the sum will be the required height of the two adjoining ends of tangents. For example: the heights at b and c, two adjacent points of contact, are respectively three and seven risers; the difference is four risers; half this added to three, the height of the lower rise, gives five risers as the height of $k k_{,}$, $k k_{,,}$, the height at the adjoining ends of the tangents $h k$ and $l k$. Again, the heights at c and d are respectively seven and ten risers; their difference is three; half of which, or one and a half risers, added to seven, the height at the lower point of contact, makes nine and a half risers as the heights $l l_{,}$, $l l_{,,}$, at the ends of the adjoining tangents $k l$ and $m l$. In a similar manner are established the heights of the tangents at m, n, and p.

The rule for finding the heights of tangents as just given is applicable to circular stairs in which the treads are divided equally at the front-string, as in *Fig.* 158. Stairs of irregular plan require to have drawn an elevation of the rail, stretched out into a plane, upon which the tangents can be located. This will be shown farther on.

The locations of the joints c, d, e, in this example, were disposed at unequal distances merely to show the effect on the tangents as before noticed. In practice it is proper to

locate them at equal distances, for then one face-mould in such a stairs will serve for each wreath.

When the tangent at G has been drawn, the level tangent for the landing may be obtained in this manner: As the joint f is located at the eighteenth riser, one riser below the landing, draw a horizontal line at s, one riser above the point $f_{,}$, and at half a riser above this draw the level line at $p_{,}$; then this line is the level tangent, and p its point of intersection with the raking tangent. Draw the vertical line $p_{,} p$, and from p draw the tangent $p\,g$, which is the horizontal projection of the tangent $p_{,} g_{,}$ on plane H (which, to avoid undue enlargement of the drawing, is reduced in height), where $p\,p_{,,}$ equals $p\,p_{,}$.

To obtain the horizontal tangent $t\,u$ at the newel, proceed thus: Fix the point r, in the tangent $r\,k_{,}$, at a height above $b\,t$ equal to the elevation of the centre of the newel above the height of a short baluster—for example, from 5 to 8 inches—and draw a line through r parallel to $b\,t$; this is a horizontal line through the middle of the height of the newel-cap, and upon which and the rake the easement to the newel is formed. Perpendicular to $b\,t$ draw $r\,t$, and join t and u; then $t\,u$ is the horizontal tangent.

277.—Face-Moulds for Circular Stairs.—At *Fig.* 159 the plan of the newel and the adjacent hand-rail are repeated, but upon an enlarged scale; and in which $b\,b_{,}$ is the reduced height of the point b, or is equal to $b\,b_{,}$ less $t\,r$, *Fig.* 158, and the angle $b\,b_{,}\,t$ equals the angle $b\,b_{,}\,r$ of *Fig.* 158. In this plan the *actual* heights must now be taken. Join t and u; then $t\,u$ is the level tangent, as also the line of intersection of the cutting plane C and the horizontal plane A. Perpendicular to $t\,u$, at a point t or anywhere above it, draw $u_{,} b_{,,}$. Parallel with $t\,u$ draw $b\,b_{,,,}$; make $b_{,,} b_{,,,}$ equal to $b\,b_{,}$; join $b_{,,,}$ and $u_{,}$; then the angle $b\,b_{,,,}\,u_{,}$ is the angle which the plank in position makes with a vertical line, or what is usually termed the *plumb-bevil*. Perpendicular to $b_{,,,}\,u_{,}$ draw $u_{,}\,u_{,,}$ and $b_{,,,}\,b_{,,,,}$; make $b_{,,,}\,b_{,,,,}$ equal to $b\,b_{,,}$; make $u_{,}\,t_{,}$ equal to $u_{,}\,t$, and $u_{,}\,u_{,,}$ to $u_{,}\,u$; join $b_{,,,,}$ and $t_{,}$; then $b_{,,,,}\,t_{,}$ is the tangent in the cutting plane, the horizontal projection of

n is bt. The butt-joint at $b_{,,,,}$ is drawn square to the nt $b_{,,,,}\,t_{,}$. Parallel to the intersecting line $t\,u_{,}$ draw ates across the plane A from as many points as desir- and extend them to the rake-line $u_{,}\,b_{,,,}$; through the s of their intersection with this line, and perpendicular draw corresponding ordinates across the plane C. Make $_{,,}$ equal to $d_{,}\,d$, and so in like manner, for all other points,

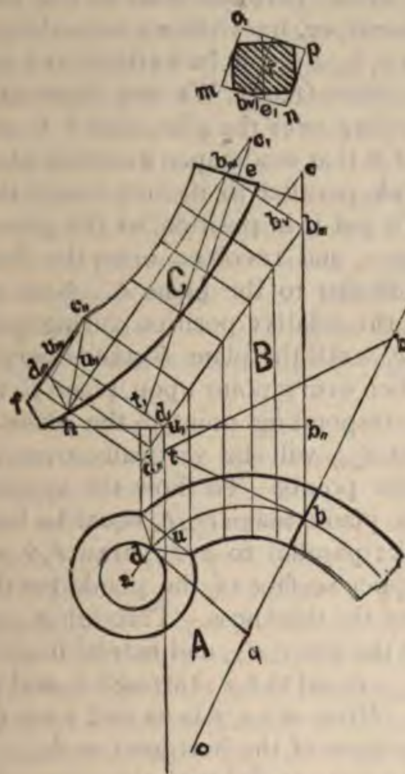


FIG. 159.

n in the plane C for each point in the horizontal plane corresponding point in the plane C: in each case g the distance to the point in the plane A from the line nd applying it in the plane C from the rake-line $u_{,}\,b_{,,,}$. the curves bend a flexible strip to coincide with the al points obtained, and draw the curve by the side of trip. The point of the mitre is at $d_{,,,}$, the mitre-joint is

shown at $h\,d_{,,,}$ and $d_{,,,}\,c_{,,}$. The line $f\,c_{,,}$ is drawn through $c_{,,}$, the most projecting point of the mitre, and parallel to the rake-line $u_{,}\,b_{,,,}$. Additional wood is left attached, extending from h to f; this is an allowance to cover the mitre, which has to be cut vertically; the butt-joint at $b_{,,,,}$ and the face at $f\,c_{,,}$ are both to be cut square through the plank. The face $f\,c_{,,}$, because it is parallel to the rake-line $u_{,}\,b_{,,,}$, is a vertical face, as well as being perpendicular to the surface of the plank. On it, therefore, lines drawn according to the rake, or like the angle $u_{,}\,b_{,,,}\,b_{,,}$, will be vertical and will give the direction of the mitre-faces. We now have at C the face-mould for the railing over the plan from b to d in A. The mould thus found is that made upon a cutting plane C, passed through the plank, parallel to its face, but at the middle of its thickness. To put it in position, let the plane C be lifted by its upper edge $c_{,,}$ and revolved upon the line $u_{,}\,b_{,,,}$ until it stands perpendicular to the plane B. Now revolve both C and B (kept in this relative position during the revolution) upon the line $u_{,}\,b_{,,}$ until the plane B stands perpendicular to the plane A. Then every point upon plane C will be vertically over its corresponding point in the plane A. For example, the point $b_{,,,,}$ will be vertically over b, $t_{,}$ over t, and so of all other points. To show the application of the face-mould to the plank, make $b_{,,,}\,b_{v}$ equal to half the thickness of the plank; parallel to $u_{,}\,b_{,,,}$ draw $b_{v}\,c$, a line which represents the upper surface of the plank, for the line $u_{,}\,b_{,,,}$ is at the middle of the thickness. Through $b_{,,,,}$, and parallel with $b_{,,,}\,u_{,}$, draw the line $c_{,}\,b_{,,,,}$ and extend it across the face-mould; make $b_{,,,,}\,c_{,}$ equal to $b_{v}\,c$; through $c_{,}$, and parallel with $b_{,,,,}\,t_{,}$, draw $c_{,}\,e$. Now, $m\,n\,o_{,}\,p$ is an end view of the plank, showing the face view of the butt-joint at $b_{,,,,}$. Through r, the centre, draw a line parallel with the sides. Then b_{vi} represents the point $b_{,,,,}$; make $b_{vi}\,e_{,}$ equal to $b_{,,,,}\,e$; through r, the centre, draw $e_{,}\,r$ across the face of the joint; then $e_{,}\,r$ is a vertical line (see *Art.* 284), parallel and perpendicular to which the four sides of the squared-up wreath are to be drawn as shown. In applying the face-mould to the plank at first, for the purpose of marking by its edges the form of the face-mould, it will be observed that the face-mould is understood to have the position indicated by the line $u_{,}\,b_{,,,}$, or at

the middle of the thickness of the plank. By this marking the rail-piece is cut square through the plank, and this cutting gives the correct form of the wreath, but *only* at the middle of the thickness of the plank. After it is cut square through the plank, then, to obtain the form at the upper and under surfaces, the face-mould is required to be moved endwise, but parallel with the auxiliary plane *B*, and so far as to bring the face-mould into a position vertically over or under its true position at the middle of the thickness of the plank. For example, the point $b_{,,,,}$, if the mould were placed at the middle of the thickness of the plank, would be at the height of the point $b_{,,,}$; but when upon the top of the plank, the point $b_{,,,,}$ would have to be at the height of the point $c_{,}$, therefore the mould must be so moved that the point $b_{,,,,}$ shall pass from b_{v} to c; consequently $b_{v}\,c$ is the distance the mould must be moved, or, as it is technically termed, the sliding distance; hence $b_{,,,,}\,c_{,}$, which is equal to $b_{v}\,c$, is the distance the mould is to be moved: up when on top, and down when underneath. This is more fully explained in *Art.* 284.

278.—Face-Moulds for Circular Stairs.—At *Fig.* 160 so much of the horizontal projection of the hand-railing of stairs in *Fig.* 158 is repeated as extends from the joint b to that at d, but at an enlarged scale. Upon the tangent $c\,k$ set up the heights as given in *Fig.* 158; for example, make $k\,k_{,}$ equal to $k_{,,,}\,k_{,,}$ of *Fig.* 158, and $c\,c_{,}$ equal to $c_{,,}\,c_{,}$ of *Fig.* 158. Join $c_{,}$ and $k_{,}$ and extend the line to meet $c\,k$, extended, in a. Join a and b; then $a\,b$ is the line of intersection of the cutting and horizontal planes; it is therefore a horizontal line, parallel to which the ordinates are to be drawn. Perpendicular to $a\,b$ draw $b_{,}\,c_{,,,,}$. Parallel to $a\,b$ draw $c\,c_{,,}$ and $k\,k_{,,}$; join $b_{,}$ and $c_{,,}$; the angle $c\,c_{,,}\,b_{,}$ is the plumb-bevil; perpendicular to $b_{,}\,c_{,,}$ draw $b_{,}\,b_{,,}$, $k_{,,}\,k_{,,,}$, and $c_{,,}\,c_{,,,}$; make $b_{,}\,b_{,,}$ equal to $b_{,}\,b$, and so of the other two points, $k_{,,,}$ and $c_{,,,}$, make them respectively equal to their horizontal projections upon the plane *A*. Join $c_{,,,}$ and $k_{,,}$; also, $k_{,,}$ and $b_{,,}$; then $b_{,,}\,k_{,,,}$ and $k_{,,,}\,c_{,,,}$ are the tangents. From $c_{,,,}$ draw the line $c_{,,,}\,b_{,,}$ parallel to $b_{,}\,c_{,,}$; this is the slide-line. In this example, this

line passes through the point $b_{,,}$; the slide-line doe always pass through the ends of the two tangents; it required to pass through both, but it is indispensable th be drawn parallel with the rake-line $b_{,}c_{,,}$. The lines for joints at each end are drawn square to the tangent lin Points in the curves, as many as are desirable, are now be found by ordinates as shown in the figure, and as bef

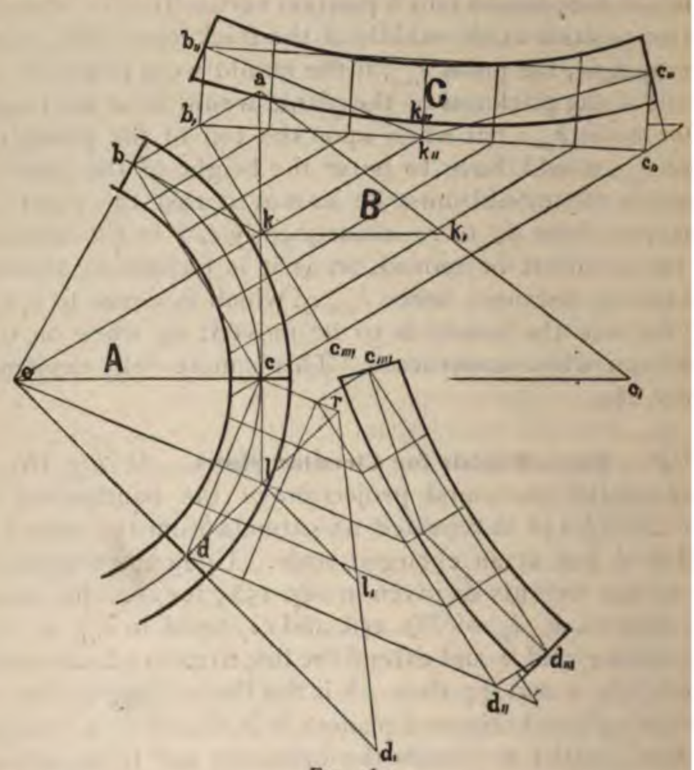


FIG. 160.

explained for the points in the tangents. The curves made by drawing a line against the side of a flexible st bent to coincide with the points.

The face-mould may be put in position by revolving planes C and B, as explained in the last article, for the at the newel.

The face-mould for the rail over the plan from c to d is

be obtained in a similar manner, taking the heights from *Fig.* 158. For example, make $d\,d_,$ equal to $d_{,,}d_,$ of *Fig.* 158, and $l\,l_,$ equal to $l_{,,,}l_{,,}$ of *Fig.* 158 (taking the heights at their *actual* measurement now). Join $d_,$ and $l_,$, and extend the line to meet the line $d\,l$ extended in r; join r and c; then $r\,c$ is the line of intersection, and parallel to which the ordinates are to be drawn. The points in the face-mould may now be obtained as in the previous cases, giving attention first to the tangent and slide-line; drawing the lines for the joints perpendicular to the tangents.

It may be remarked here that the chord-line $b\,c$ is parallel with the measuring line $b_,c_{,,,,}$, and that the line $o\,k$ bisects the chord-line; so, also, the line $o\,l$ bisects the chord-line $c\,d$. This coincidence is not accidental; it will always occur in a regular circular stairs.

Hence in cases of this kind it is not necessary to go through the preliminaries by which to obtain the intersecting line $a\,b$, but draw it at once parallel to the line $o\,k$, bisecting the chord $b\,c$ and passing through the point of intersection of the two tangents. For the distance to slide the mould in its after-application, the lines are given at $c_{,,}$ and $d_{,,}$, and their use is explained in the last article, and more fully in *Art.* 284.

279.—Face-Moulds for Circular Stairs, again.—At *Fig.* 161 so much of the plan of the hand-railing of the stairs of *Fig.* 158 is repeated as is required to show the rail from f to g, but drawn at a larger scale. To prepare for the face-moulds, perpendicular to $p\,f$ draw $p\,p_,$, and make $p\,p_,$ equal to $p_{,,,}p_{,,}$ of *Fig.* 158 (taking this height now at its actual measurement); join $p_,$ and f; then $f\,p_,$ is the tangent of the vertical plane C, and f is a point in the cutting plane at its intersection with the base-plane A. Now since $r\,s$, the tangent over $p\,g$, is horizontal and is in the cutting plane, therefore from f draw $f\,a$ parallel with $r\,s$ or $p\,g$; then $f\,a$ is the line of intersection of the cutting and horizontal planes, and gives direction to the ordinates. Draw $f_,p_{,,,}$ perpendicular to $f\,a$; make $p_{,,,}p_{,,}$ equal to $p\,p_,$; join $p_{,,}$ and $f_,$; then the angle $p\,p_{,,}f_,$ is the plumb-bevil; perpendicular

to $p_{,,}f_{,}$ draw $f_{,}f_{,,}$ and $p_{,,}p_{,,,,}$; make $p_{,,}p_{,,,,}$ equal t
$p_{,,}d$ equal to $p_{,,,}p$; join d and $f_{,,}$; then $df_{,,}$ and d
the tangents. Make $p_{,,}e$ equal to half the thickness
plank; draw $f_{,,}a$ parallel with $f_{,}p_{,,}$; make $f_{,,}a$ equa
draw $ac_{,}$ parallel with the tangent $f_{,,}d$; through
pendicular to $f_{,,}d$, draw the line for the butt-joint; th
is the distance required to determine the vertical line
face of the joint at $f_{,,}$, as shown at *A*. Through $p_{,}$

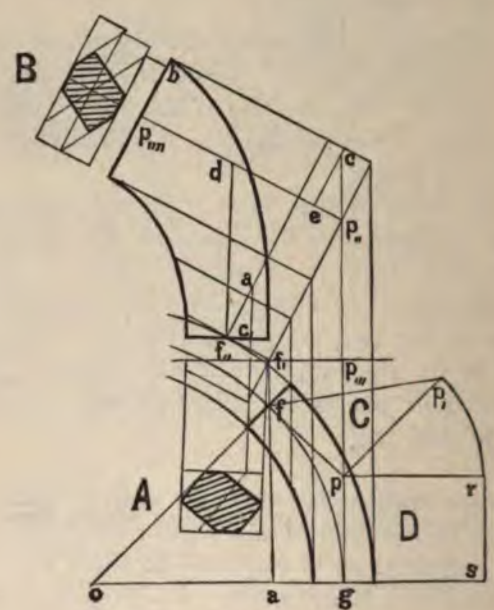


FIG. 161.

pendicular to the tangent $p_{,,,,}d$, draw the line for th
joint; make $p_{,,,,}b$ equal to ec; then $p_{,,,,}b$ is the d
required for determining the vertical line on the face
joint at $p_{,,,,}$, as shown at *B* (see *Art.* 284). The curv
are obtained by drawing a line against the edges of
ble rod bent to as many points as desirable, obtai
measuring the ordinates of the plan at *A* and trans
them to the face-mould by the corresponding ordin
before explained.

280.—**Hand-Railing for Winding Stairs.**—The term *winding* is applied more particularly to a stairs having steps of parallel width compounded with those which taper in width, as in *Fig.* 135, and as is here shown in *Fig.* 162, in which $f\,a\,b\,c$ represents the central line of the rail around the cylinder, and the quadrant $d\,e$, distant from the first quadrant 20 inches, is the tread-line, upon which from d, a point taken at pleasure, the treads are run off. Through e, perpendicu-

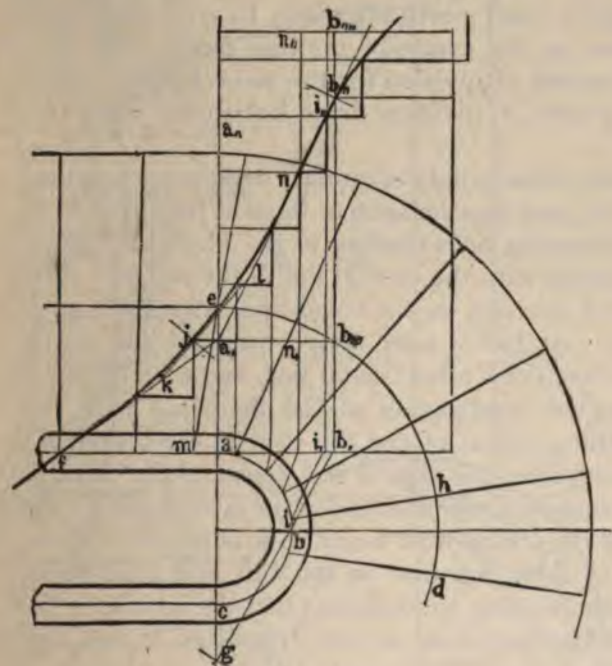


FIG. 162.

lar to $a\,f$, draw $a\,e$ (the occurrence here of one of the points of division on the tread-line perpendicularly opposite a, the spring of the circle, is only an accidental coincidence); make $a\,a_,$ equal to two risers; join $a_,$ and f. With the diameter $a\,c$, on b as a centre, describe the arc at g, crossing $a\,c$ extended; through b draw $g\,b_,$; then $a\,b_,$ is the stretch-out, or development of the quadrant $a\,b$.

Through h draw $h\,i$, tending toward the centre of the

cylinder; make $b_, i_,$ equal to $b\,i$; perpendicular to $f\,b_,$ draw $b_, b_{,,}$ and $i_, i_{,,}$. As there are four risers from e to h, make $a_, a_{,,}$ equal to four risers, and draw $a_{,,} i_{,,}$ parallel with $f\,a$; through $i_{,,}$ draw $a_, b_{,,}$; by intersecting lines, or in any convenient manner, ease off to any extent the angle $f\,a_, i_{,,}$. Through j, a point in this curve (chosen so as to be perpendicularly over m, a point between a and f, nearer to a), draw $k\,l$, a tangent to the curve. Perpendicularly to this tangent, through j, draw the line for a butt-joint; also through $b_{,,}$, and perpendicularly to $a_, b_{,,}$ draw the line for the joint at the centre of the half circle. On the line $a\,a_{,,}$ set up points of division for the riser heights, and through these points of division draw horizontal lines to the line $b_{,,} j\,f$.

From these points of contact drop perpendiculars to the line $f\,a\,b_,$, and transfer such of them as require it to the circle $a\,i$, by drawing lines tending to g. Through these points of intersection with the central line of the rail, and through the points of division on the tread-line, draw the riser-lines $m\,d$, $a\,n$, etc. At half a riser above the floor-line, on top of the upper riser draw a horizontal line, and ease off the angle as shown; the intersection of the floor-line with this curve gives the position of the top riser at the centre of the rail. This completes the plan of the steps and the elevation of the rail—requisite preliminaries for the face-moulds. The graduation of the treads from flyers to winders obviates an abrupt angle at their junction in the rail and front-string. The objection to the graduation, that it interferes with the regularity of stepping at the tread-line, is not realized in practice.

281.—Face-Moulds for Winding Stairs.—At *Fig.* 163 so much of the plan at *Fig.* 162 is repeated as is required for the face-moulds, but for perspicuity at twice the size. The horizontal projection of the tangents for the first wreath are $a\,d$ and $d\,b$ drawn at right angles to each other, tangent to the circle at a and b. Let those tangents be extended beyond d; through m, the lower end of the wreath, draw $m\,d_,$, making an angle with $m\,d$ equal to that in *Fig.* 162, between the

line $a f$ and $a_, f$; or let the angle $d m d_,$ equal $a f a_,$ of *Fig.* 162. Make $d d_{,,}$ equal to $d d_,$. Make $b b_{,,}$ equal to $b_{,,,} b_{,,}$ of *Fig.* 162; join $d_{,,}$ and $b_{,,}$ and extend the line to $e_{,,}$; make $b_{vi}\ b_v$ equal to $b_{,,} b_{,,,,}$ of *Fig.* 162, and draw $b_v e_{,,}$ parallel with

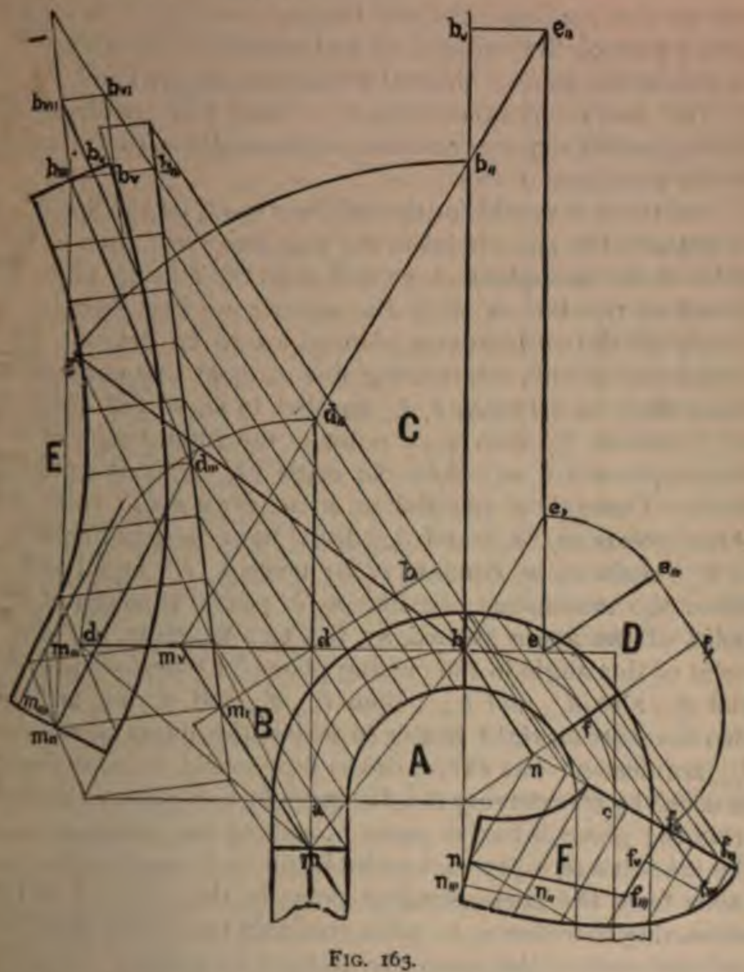


FIG. 163.

$d e$. From $e_{,,}$ draw $e_{,,} e$ parallel with $b_{,,} b$; through e and f draw $e f$ tangent to the circle at f; then $b e$ and $e f$ are the horizontal projections of the tangents for the upper wreath. Then if the plane B be revolved on $a d$, the plane C on $d e$,

and the plane D on ef until they each stand vertical to the plane A, the lines $m\,d_{,}$, $d_{,,}\,e_{,,}$, and $e_{,,,}f_{,}$ will constitute the tangents of the two wreaths in position. This arrangement locates the upper joint of the upper wreath at f, leaving fc, a part of the circle, to be worked as a part of the long level rail on the landing. As the tangent over ef is level, the raking part of the rail will all be included in the wreath bf. so that at the joint f the rail terminates on the level.

The portion fc, therefore, is a level rail requiring no canting, and it requires no other face-mould than that afforded by the plan from f to c.

For the face-mould for the rail over $m\,a\,b$, let the line $e_{,,}\,d_{,,}$ be extended to m_{v}, a point in the base-line $b\,m_{v}$; then m_{v} is a point in the base-plane A, as well as in the cutting plane E; therefore the line $m_{v}\,m$ is the *intersecting* line parallel to which all the ordinates on plane A are to be drawn. Perpendicular to this intersecting line $m_{v}\,m$, at any convenient place draw $m_{,}\,b_{,}$; make $b_{,}\,b_{,,,}$ parallel to $m_{v}\,m$ and equal to $b\,b_{,,}$; connect $b_{,,,}$ with $m_{,}$, a point at the intersection of the lines $m_{v}\,m$ and $b_{,}\,m_{,}$; then the angle $b\,b_{,,,}\,m_{,}$ is the plumb-bevil. Through d, parallel to $m_{v}\,m$, draw $d\,d_{,,,}$; from the three points $m_{,}$, $d_{,,,}$, and $b_{,,,}$ draw lines perpendicular to $m_{,}\,b_{,,,}$; make $m_{,}\,m_{,,}$ equal to $m_{,}\,m$; make $b_{,,,}\,b_{,,,,}$ equal to $b_{,}\,b$. Since the measuring base-line $m_{,}\,b_{,}$ passes through d, the point of the angle formed by the two tangents, $d_{,,,}$ is the point of this angle in the cutting plane E; therefore join $m_{,,}$ and $d_{,,,}$, also $d_{,,,}$ and $b_{,,,,}$; then $b_{,,,,}\,d_{,,,}$ and $d_{,,,}\,m_{,,}$ are the two tangents at right angles to which the joints at $m_{,,}$ and $b_{,,,,}$ are drawn. The curves of the face-mould are now found as usual, by transferring the distances by ordinates, as shown, from the plane A to the plane E, making the distance from the rake-line $m_{,}\,b_{,,,}$ to each point in plane E equal to the distance from the corresponding point in the plane A to the measuring base-line $m_{,}\,b_{,}$. Now, to obtain the sliding distance and the vertical line upon the butt-joints, make $b_{,,,}\,b_{v}$ equal to half the thickness of the plank; parallel with $m_{,}\,b_{,,,}$ draw $b_{v}\,b_{vi}$; also, $b_{,,,,}\,b_{vii}$ and $m_{,,}\,m_{,,,}$; make $b_{,,,,}\,b_{vii}$ and $m_{,,}\,m_{,,,}$ each equal to $b_{v}\,b_{vi}$; through b_{vii} and $m_{,,,}$, and parallel to the respective tangents, draw $b_{vii}\,b_{x}$ and $m_{,,,}\,m_{,,,,}$; then b_{x} and

$m_{,,,,}$ are the points from which, through the centre of the butt-joints, a line is to be drawn which will be vertical when the wreath is in position. (See *Art.* 284.)

For the face-mould for the upper quarter, through b, *Fig.* 163, draw $b\,e_,$ parallel with $d_{,,}\,e_{,,}$; make $e\,e_{,,,}$ equal to $e\,e_,$; draw $e_{,,,}\,f_,$ parallel with $e\,f$. Now, since $e_{,,,}\,f_,$ is a horizontal line and is in the cutting plane F, therefore, parallel with $e_{,,,}\,f_,$ and through $b_,$, draw $b\,n$; then $b\,n$ is the required intersecting line. Extend $e\,f$ to $f_{,,}$; make $f\,f_{,,}$ equal to $f\,f_,$; join $f_{,,}$ and n; then the angle $f\,f_{,,}\,n$ is the plumb-bevil. Perpendicular to $n\,f_{,,}$ draw $f_{,,}\,f_{,,,}$ and $n\,n_,$, and make these lines respectively equal to $e\,f$ and $b\,n$; join $f_{,,}$ and $f_{,,,}$; also $f_{,,,}$ and $n_,$; then $f_{,,}\,f_{,,,}$ and $f_{,,,}\,n_,$ are the required tangents. The butt-joints at $f_,$ and $n_,$ are drawn perpendicular to their respective tangents. To get the slide distance and vertical lines on the butt-joints, make $f_{,,}\,f_{v}$ equal to half the thickness of the plank; parallel with $n\,f_{,,}$, through f_{v} draw $f_{v}\,f_{,,,,}$; also, through $n_,$ draw $n_,\,n_{,,}$; make $n_,\,n_{,,}$ equal to $f_{v}\,f_{,,,,}$; through $n_{,,}$, parallel with $n_,\,f_{,,,}$, draw $n_{,,}\;n_{,,,}$; then $n_{,,,}$ is the point through which a line is to be drawn to the centre of the butt-joint, and this line will be in the vertical plane containing the tangent. So, also, parallel with the tangent $f_{,,}\;f_{,,,}$, and through $f_{,,,,}$, draw $f_{,,,,}\,f_{vi}$; then f_{vi} is the point through which a line is to be drawn to the centre of the butt-joint (see *Art.* 284). The curve is now to be obtained by the ordinates, as before explained.

282.—Face-Moulds for Winding Stairs, again.—In the last article, in getting the face-moulds for a winding stairs, the two wreaths are found to be very dissimilar in length. This dissimilarity may be obviated by a judicious location of the butt-joint connecting the two wreaths, as shown in *Fig.* 164. Instead of locating the joint precisely at the middle of the half circle, as was done in *Fig.* 163, place it farther down, say at n, which is at n in *Fig.* 162, two risers down from the top, or at any other point at will. Then through n in the plan draw $m_,\,s$ tangent to the circle at n; and perpendicular to this tangent draw $n\,n_{,,,}$ and $d\,d_{,,}$; make $n\,n_{,,}$ equal to $n_,\,n$ of *Fig.* 162; from d erect $d\,d_,$ perpendicular to $m\,d$;

make the angle $d\ m\ d_{,}$ equal to that of $b_{,,,}\ j\ l$ of *Fig.* 162. Make $d\ d_{,,}$ equal to $d\ d_{,}$; join $d_{,,}$ and $n_{,,}$ and extend the line to $m_{,}$, a point of intersection with the base-line $n\ n_{,}$; then $n_{,}$ is a point in the base-plane, as also in the cutting plane;

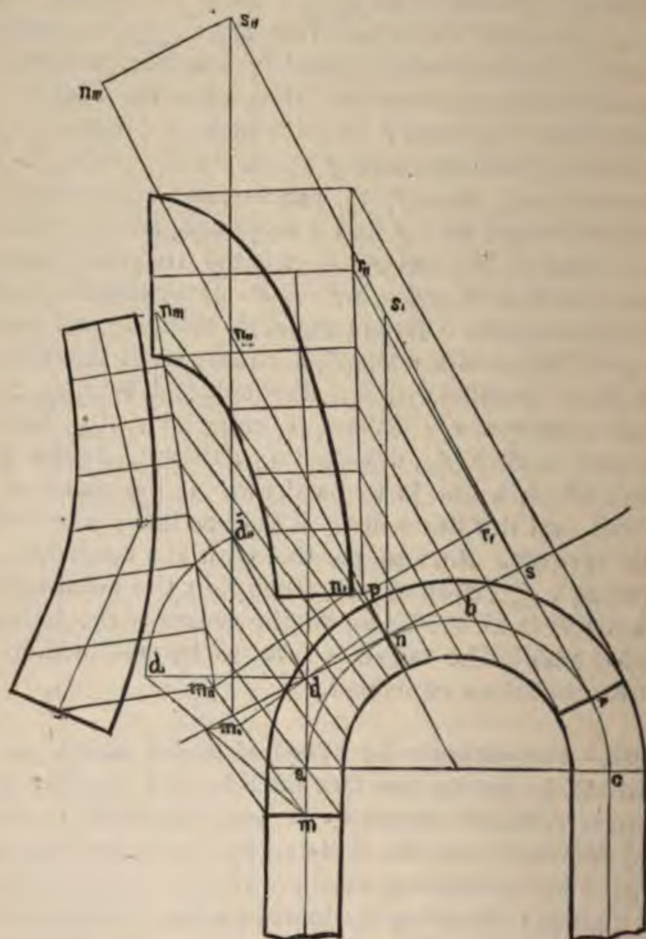


FIG. 164.

therefore $m_{,}\ m$ is the intersecting line parallel to which all the ordinates of the plan are to be drawn, and perpendicular to which $m_{,,}\ n_{,}$, the measuring base-line, is drawn. Make $n_{,}\ n_{,,,,}$ equal to $n\ n_{,,}$; connect $m_{,,}$ and $n_{,,,,}$, and then transfer

by the ordinates to the cutting plane $m\,d$ and n the three points of the plan at the ends of the tangents, as before described, as also such points in the curve as may be required to mark the curve upon the face-mould, all as shown in previous examples. For the face-mould of the upper wreath, make $n_{,,}\,n_{,,,}$ equal to $n\,n_{,,}$ of *Fig.* 162. From $n_{,,,}$ draw $n_{,,,}\,s_{,,}$ parallel with $m_{,}\,s$; extend the line $d_{,,}\,n_{,,}$ to intersect $n_{,,,}\,s_{,,}$ in $s_{,,}$; parallel with $n_{,,,}\,n$ draw $s_{,,}\,s$; from s draw $s\,r$ tangent to the circle at r ($s\,n$ equals $s\,r$); through r, tending to the centre of the cylinder, draw the butt-joint; then $r\,s$ and $s\,n$ are the horizontal projections of the tangents for the upper wreath-piece, the tangent $s\,r$ being level and, consequently, parallel to the intersecting line drawn through n. Perpendicular to $r\,s$ draw $r_{,}\,p$; parallel with $n_{,,}\,s_{,,}$ draw $n\,s_{,}$; make $r_{,}\,r_{,,}$ equal to $s\,s_{,}$; join $r_{,,}$ and p. From this line and the measuring base-line $r_{,}\,p$, the points for the tangents are first to be obtained and then the points in the curve, all as before described. The part of the circle from r to c is on the level, as before shown, and may be worked upon the end of the long level rail, its form being just what is shown in the plan from c to r.

283.—Face-Moulds: Test of Accuracy.—The methods which have been advanced for obtaining face-moulds are based upon principles of such undoubted correctness that there can be no question as to the results, when the methods given are thoroughly followed. And yet, notwithstanding the correctness of the system and its thorough comprehension by the stair-builder, he will fail of success unless he exercises the greatest care in getting his dimensions, his perpendiculars, and his angles. The slightest deviation in a perpendicular terminated by an oblique line will result in a magnified error at the oblique line. To secure the greatest possible degree of accuracy, care must be exercised in the choice of the instruments by which the drawings are to be made: care to know that a straight-edge is what it purports to be; that a square, or right-angle, is truly a right-angle; that the compasses or dividers be well made, the joint perfect, and the ends neatly ground to a point. Then let the drawing-board be carefully planed to a true surface; and,

if possible, let the drawing, full size, be made upon large, stout roll-paper rather than upon the drawing-board itself, as then the points for the face-mould may be pricked through upon the board out of which the face-mould is to be cut, and thus a correct transfer be made. For long straight lines it is better to use a fine chalk-line than the edge of a wooden straight-edge. The line is more trustworthy. Perpendiculars, especially when long, are better obtained by measurement or by calculation (*Art.* 503) than by a square. The pencil used should be of fine quality—rather hard, in order that its point may be kept fine. With these precautions in regard to the instruments used, and with due care in the manipulations, the face-moulds may be correctly drawn, accurate in size and form. As a test of the accuracy of the work, it will be well to observe in regard to the tangents, that the length of a tangent, as found upon the face-mould, should always equal its length as shown upon the vertical plane. For example, in *Fig.* 160, the tangent $k_{,,} c_{,,,}$ on the face-mould should be equal to $k_{,} c_{,}$, the tangent on the vertical plane B; and in cases like this, where the stairs are quite regular, with equal treads at the front-string, the two tangents of a face-mould are equal to each other, or $k_{,,} c_{,,,}$ equals $k_{,,} b_{,,}$; and in this case, the line $b_{,,} c_{,,,}$ should equal the rake-line $b_{,} c_{,,}$.

Again, as another example, in *Fig.* 161, $d f_{,,}$, the tangent upon the face-mould, should be equal to $f p_{,}$, the tangent of the vertical plane C; while $d p_{,,,,}$, the other tangent on the face-mould, should be equal to $r s$, the tangent of the vertical plane D. But the more important test is in the length of the chord-line joining the ends of the two tangents; as, for example, the chord $m_{,,} b_{,,,,}$ of *Fig.* 163, the horizontal projection of which is the chord $m b$ in plane A. Perpendicular to $m b$ draw $b g$; make $b g$ equal to $b b_{,,}$, and join g and m; then $m_{,,} b_{,,,,}$, the chord of the face-mould, should be equal to $m g$. After fully testing the accuracy of the drawing for the face-mould, choose a well-seasoned thin piece of white-wood, or any other wood not liable to split, and plane it to an even thickness throughout; mark upon it the curves, joints, tangents, and slide-line, and cut the edges true to the curve-

lines and joints square through the board; then square over such marks as are required to draw each tangent and the slide-line also upon the reverse side of the board. This completes the face-mould.

284.—Application of the Face-Mould.—In order that a more comprehensive idea of the lines given for applying a face-mould may be had, let *A*, *Fig.* 165, represent one end of a wreath-piece as it appears when first cut from a plank, and when held up in the position it is to occupy at completion over the stairs. Also, let *B* represent the corresponding face-mould, laid upon the wreath-piece *A* in the position which it should have after sliding. And, for the purpose of a clearer illustration, let it be supposed that the two pieces, *A* and *B*, are transparent. Then let $a_, a\ b\ d\ c_, e_,$ represent a solid of wedge form, having a triangular level base, $a\ b\ d$, upon the three lines of which stand these three vertical planes, namely: on the line $a\ b$ the plane $a_, a\ b\ c_,$, upon the line $a\ d$ the plane $a_, a\ d\ e_,$, and on the line $d\ b$ the plane $b\ d\ e_, c_,$; the top of the solid is an inclined plane, $a_, c_, e_,$, and the vertical line $a_, a$ is the edge of the wedge. Now, it will be observed that the point a in the base of the solid is identical with a, the centre of the butt-joint, and the point $a_,$ (at the intersection of two vertical planes and the inclined plane of the solid) is vertically over a, and is identical with $a_,$, a point in the upper surface of the plank. Also, the inclined plane $e_, c_, a_,$, which forms the top of the solid, coincides with the upper surface of the plank *A*, from which the wreath-piece has been squared; and the line $c_, a_,$ (at the angle formed by the inclined plane $e_, c_, a_,$ and the vertical plane $a_, a\ b\ c_,$) coincides with $f\ g$, the slide-line drawn upon the top of the plank; also, the line $e_, a_,$ (at the angle formed by the inclined plane $e_, c_, a_,$ and the vertical plane $a_, a\ d\ e_,$) coincides with $a_, k$, the tangent line upon the underside of the face-mould after it has been slid to its new position, vertically over its true position at the middle of the thickness of the plank. From a the line $a\ c$ is drawn parallel with $a_, c_,$; so, also, the line $a\ e$ is drawn parallel with $a_, e_,$; consequently the line $e\ c$ is parallel with $e_, c_,$; and the plane $e\ c\ a$ is parallel

with the plane $e_, c_, a_,$, and coincides with a plane passing through the middle of the thickness of the plank, and, consequently, is the cutting plane referred to in previous articles, upon which the lines are drawn which give shape to the

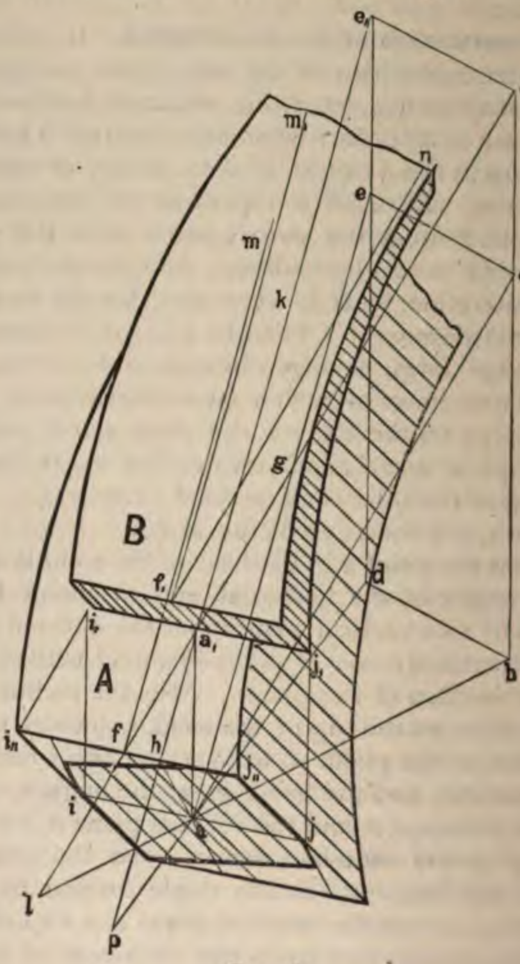


FIG. 165.

face-mould. When the face-mould is first laid upon the plank the line $i_, j_,$ coincides with $i_{,,}$ $j_{,,}$, and when in that position, its form marked upon the plank is the form by which the plank is sawed square through; but this gives the form of

the wreath, not as it is at the surface of the plank, but as it is at the middle of the thickness of the plank, or upon the plane $a\,c\,e$; so that, for example, the line $i_{,,}\,j_{,,}$ represents the line $i\,j$ drawn through a, the centre of the butt-joint; and when the mould B is slid to the position shown in the figure, the line $i_{,}\,j_{,}$ comes into a position vertically over $i\,j$; hence the three lines $i_{,}\,i$, $a_{,}\,a$, and $j_{,}\,j$ are each vertical and in a vertical plane, $i\,i_{,}\,j_{,}\,j$. By these considerations it will be seen that the face-mould B, located as shown in the figure, is in its true position for the second marking, by which the additional cutting is now to be performed vertically. This being established, it will now be shown how to get upon the butt-joint a line in the vertical plane containing the tangent. If the top and bottom lines of the vertical plane $a_{,}\,a\,b\,c_{,}$ be extended, they will meet in the point l, and will extend the plane into a triangle $l\,b\,c_{,}$, cutting the upper edge of the butt-joint in f, the end of the tangent, and the point in which the point $a_{,}$ of the underside of the face-mould was located when the mould was first applied to the plank. The line $f\,a$ on the butt-joint is perpendicular to $i\,j$ or $i_{,,}\,j_{,,}$. Again, if the top and bottom lines of the plane $a_{,}\,a\,d\,e_{,}$ be extended, they will meet in p, and will extend the plane into the triangle $p\,d\,e_{,}$, cutting the edge of the butt-joint in h, a point from which, if a line be drawn upon the butt-joint to a, its centre, this line will be in the vertical plane $p\,d\,e_{,}$, which plane contains the tangent perpendicular to which the butt-joint is drawn; consequently lines upon the butt-joint parallel to $h\,a$ will each be in a vertical plane parallel to the vertical tangent plane, and lines drawn upon the butt-joint perpendicular to these lines will be horizontal lines; hence the line $h\,a$ is the required line by which to square the wreath at the butt-joint. Now, it will be observed that the triangle $a\,f\,a_{,}$ is like that given in the various figures for obtaining face-moulds, to regulate the sliding of the face-mould and the squaring at the butt-joint. For example, in *Fig.* 163, the right-angled triangle $b_{,,,}\,b_{v}\,b_{vi}$ is the one referred to. This triangle is in a vertical plane parallel to one containing the slide-line; its longer side is a vertical line; one of the sides containing the right angle is equal to half the thickness

of the plank, while the other, drawn parallel to the face of the plank, is the distance the face-mould is required to slide. Precisely like this, the triangle $a\ f\ a_{,}$ of *Fig.* 165 is in the vertical plane $l\ b\ c_{,}$, containing $f\ g$, the slide-line; its longer side, $a_{,}\ a$, is a vertical line; $f\ a$, one of the sides containing the right angle, is equal to half the thickness of the plank, while the other side, drawn coincident with the surface of

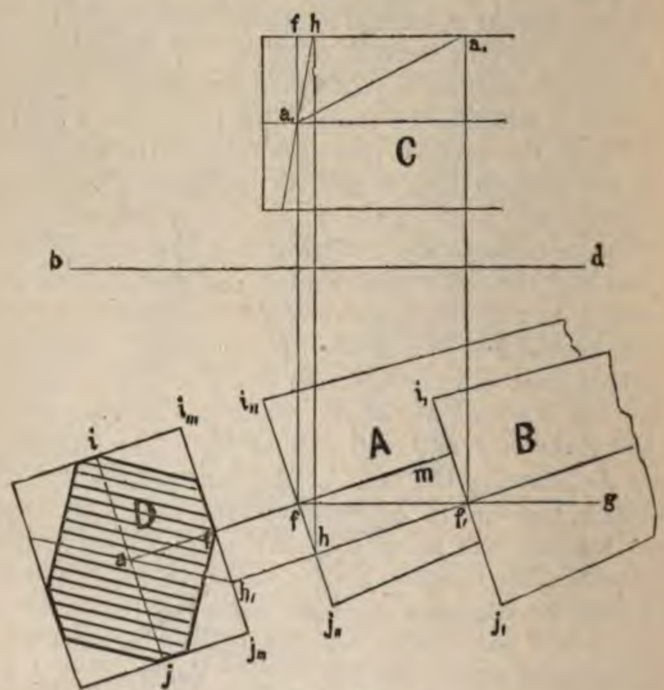


FIG. 166.

the plank, is the distance to slide the face-mould. Therefore the triangle $a_{,}\ f\ a$ of *Fig.* 165 gives the required lines by which to regulate the application of the face-moulds. The relative position of the more important of these lines is geometrically shown in *Fig.* 166, in which *A* and *B* are upon the horizontal plane of the paper, *C* is in a vertical plane standing on the ground-line $b\ d$, and *D* is a plan of the butt-joint, revolved upon the line $i_{,,}\ j_{,,}$ into the horizontal plane, and

hen perpendicularly removed to the distance $f f_{,}$. The lettering corresponds with that in *Fig.* 165. The shaded part of *D* shows the end of the squared wreath. When the blocked piece has been marked by the face-mould in its second application, its edges are to be trimmed vertically as shown in *Fig.* 167, after which the top and bottom surfaces of the wreath are to be formed from the shape marked on the butt-joints.

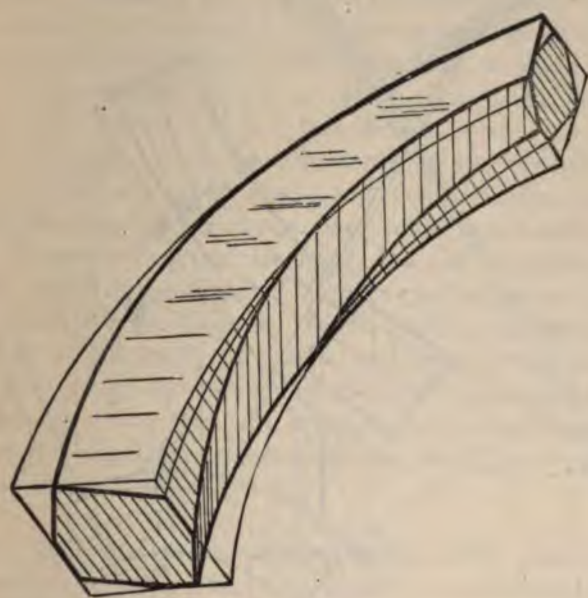


FIG. 167.

285.—Face-Mould Curves are Elliptical.—The curves of he face-mould for the hand-railing of any stairs of circular lan are elliptical, and may be drawn by a trammel, or n any other convenient manner. The trouble, however, ttending the process of obtaining the axes, so as to be able o employ the trammel in describing the curves, is, in many ases, greater than it would be to obtain the curves through oints found by ordinates, in the usual manner. But as

this method for certain reasons may be preferred by
an example is here given in which the curves are dra
a trammel, and in which the method of obtaining the
shown.

Let *Fig.* 168 represent the plan of a hand-rail arour

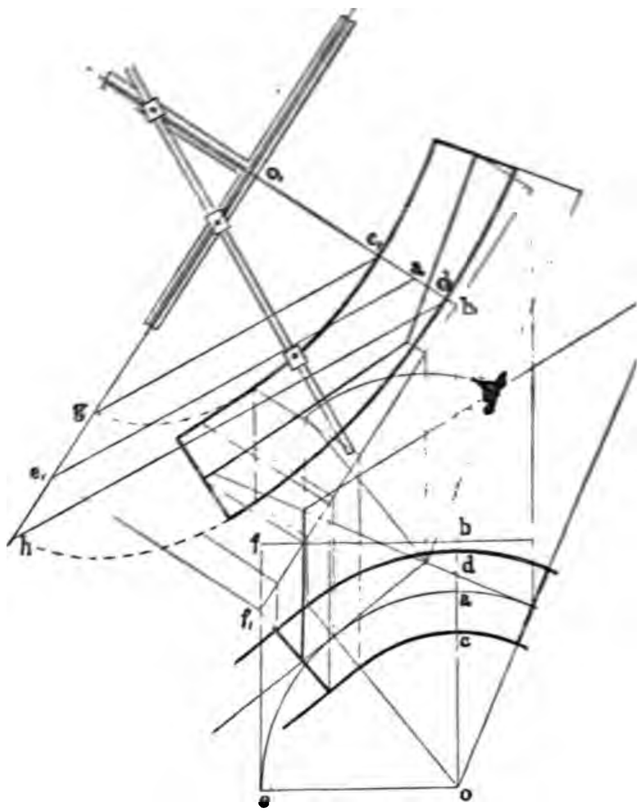


Fig. 168.

of a cylinder and with the heights set up, the inter
line obtained, the measuring base-line drawn, the ra
established, and the tangents on the face-mould locat
in the usual manner as hereinbefore described. Tl
prepare for the trammel, from o, the centre of the cy
draw $o\,b_{,}$ parallel with the intersecting line, and $b_{,}\,o_{,}$ p

dicular to $b_, f_,$, the rake-line; make $b_, o_,$ equal to $b o$, and $o_, a_,$ equal to $o a$; through $o_,$ draw $o_, h$ parallel with $b_, f_,$. From o draw $o e$ perpendicular to $o b_,$; continue the central circular line of the rail around to e; parallel with $o b_,$ draw $e f$, and from $f_,$, the point of intersection of $e f$ with $b_, f_,$, and perpendicular to $b_, f_,$, draw $f_, e_,$; make $f_, e_,$ equal to $f e$; then $o_,$ is the centre of the ellipse, and $o_, a_,$ the semi-conjugate diameter and $o_, e_,$ the semi-transverse diameter of an ellipse drawn through the centre of the face-mould. To get the diameters for the edges of the face-mould, make $a_, c_,$ and $a_, d_,$ each equal to half the width of the rail, as at $c a d$; parallel to a line drawn from $a_,$ to $e_,$, and through $c_,$, draw the line $c_, g$; also, parallel with a line drawn from $a_,$ to $e_,$ draw $d_, h$ (see *Art.* 559); then for the curve at the inner edge of the face-mould, $o_, g$ is the semi-transverse diameter, and $o_, c_,$ the semi-conjugate; while for the curve at the outer edge $o_, h$ is the semi-transverse diameter, and $o_, d_,$ the semi-conjugate. So much of the edges of the face-mould as are straight are parallel with the tangent. Now, placing the trammel at the centre, as shown in the figure, and making the distance on the rod from the pencil to the first pin equal to the semi-conjugate diameter, and the distance to the second pin equal to the semi-transverse diameter, each curve may be drawn as shown. (See *Art.* 549.)

286.—Face-Moulds for Round Rails.—The previous examples given for finding face-moulds are intended for *moulded* rails. For *round* rails the same process is to be followed, with this difference: that instead of finding curves on the face-mould for the sides of the rail, find one for a centre-line and describe circles upon it, as at *Fig.* 145; then trace the sides of the mould by the points so found. The thickness of stuff for the twists of a round rail is the same as for the straight part. The twists are to be sawed square through.

287.—Position of the Butt-Joint.—When a block for the wreath of a hand-rail is sawed square through the plank, the joint, in all cases, is to be laid on the face-mould square to the tangent and cut square through the plank.

Managed in this way, the butt-joint is in a plane
perpendicularly by the tangent. But if the block
sawed square through, but vertically from the edge

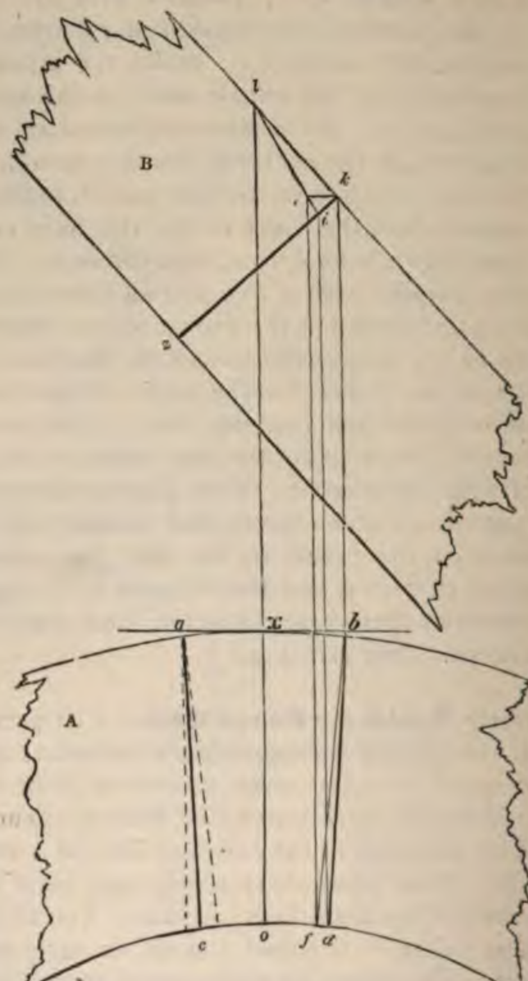


FIG. 169.

face-mould, then, especially, care is required in loca
joint. The method of sawing square through is
with so many advantages that it is now generally f
yet, as it is possible that for certain reasons some ma

me cases, to saw vertically, it is proper that the method nding the position of the joint for that purpose should iven. Therefore, let A, *Fig.* 169, be the plan of the rail, B the elevation, showing its side; in which kz is the ection of the butt-joint. From k draw kb parallel to lo, kc at right angles to kb; from b draw bf, tending to centre of the plan, and from f draw fe parallel to bk; m l, through e, draw li, and from i draw id parallel to ; join d and b, and db will be the proper direction for the int on the plan. The direction of the joint on the other de, ac, can be found by transferring the distances xb and xd to xa and oc. Then the allowance for over wood to over the butt-joint is shown as that which is included be- ween the lines ox and db. The face-mould must be so drawn as to cover the plan to the line bd for the wreath at the left, and to the line ac for that at the right. By some the direction of the joint is made to radiate toward the centre of the cylinder; indeed, even Mr. Nicholson, in his *Carpenter's Guide*, so advised. That this is an error may be shown as follows: In *Fig.* 170, $arji$ is the plan of a part of the rail about the joint, su is the stretch-out of ai, and gp is the helinet, or vertical projection of the plan $arji$. This is found by drawing a horizontal line from the height set upon each perpendicular standing upon the stretch-out line su. The lines upon the plan $arji$ are drawn radiating to the centre of the cylinder, and therefore correspond to the horizontal lines of the helinet drawn upon its upper and under surfaces.

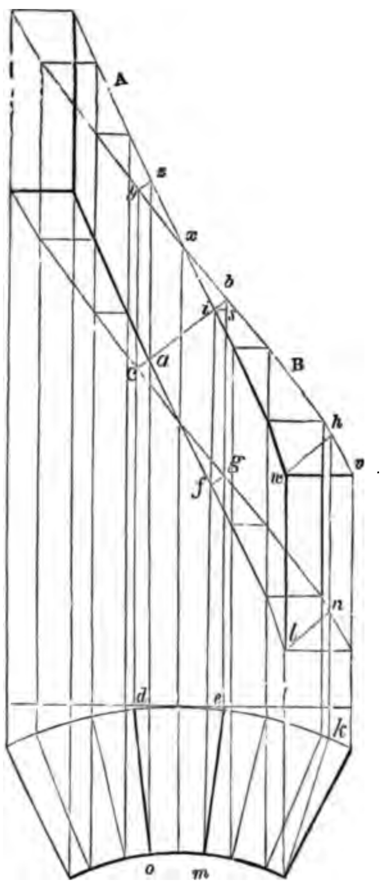
Bisect rt on the ordinate drawn from the centre of the plan, and through the middle draw cb at right angles to gv; from b and c draw cd and be at right angles to su; from d and e draw lines radiating toward the centre of the plan; then do and em will be the direction of the joint on the plan, according to Nicholson, and cb its direction on the falling-mould. It must be admitted that all the lines on the upper or the lower side of the rail which radiate toward the centre of the cylinder, as do, em, or ij, are level; for instance, the level line wv, on the top of the rail in the helinet, is a true representation of the radiating line ji on

the plan. The line $b\,h$, therefore, on the top of the rail i the helinet, is a true representation of $e\,m$ on the plan, an $k\,c$ on the bottom of the rail truly represents $d\,o$. From draw $k\,l$ parallel to $c\,b$, and from h draw $h\,f$ parallel to b

FIG. 170.

join l and b, also c and f; then $c\,k\,l\,b$ will be a true rep sentation of the end of the lower piece, B, and $c\,f\,h\,b$ of end of the upper piece, A; and $f\,k$ or $h\,l$ will show h much the joint is open on the inner, or concave, side of rail.

ıow that the process followed in *Art.* 287 is correct, nd *em* (*Fig.* 171) be the direction of the butt-joint at *Fig.* 169. Now, to project, on the top of the rail linet, a line that does not radiate toward the centre



r FIG. 171. i

linder, as *jk*, draw vertical lines from *j* and *k* to *w* d join *w* and *h*; then it will be evident that *wh* is epresentation in the helinet of *jk* on the plan, it the same plane as *jk*, and also in the same winding s *wv*. The line *ln*, also, is a true representation on

the bottom of the helinet of the line $j k$ in the plan. The line of the joint $e m$, therefore, is projected in the same way, and truly, by $i b$ on the top of the helinet, and the line $d o$ by $c a$ on the bottom. Join a and i, and then it will be seen that the lines $c a$, $a i$, and $i b$ exactly coincide with $c b$, the line of the joint on the convex side of the rail; thus proving the lower end of the upper piece, A, and the upper end of the lower piece, B, to be in one and the same plane, and that the direction of the joint on the plan is the true one. By reference to *Fig.* 169 it will be seen that the line $l i$ corresponds to $x i$ in *Fig.* 171; and that $e k$ in that figure is a representation of $f b$, and $i k$ of $d b$.

288.—Scrolls for Hand-Rails: General Rule for Size and Position of the Regulating Square.—The breadth which the scroll is to occupy, the number of its revolutions, and the relative size of the regulating square to the eye of the scroll being given, multiply the number of revolutions by 4, and to the product add the number of times a side of the square is contained in the diameter of the eye, and the sum will be the number of equal parts into which the breadth is to be divided. Make a side of the regulating square equal to one of these parts. To the breadth of the scroll add one of the parts thus found, and half the sum will be the length of the longest ordinate.

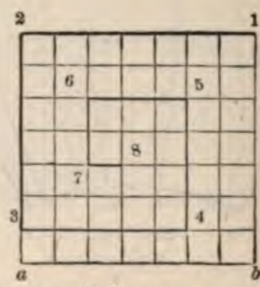


FIG. 172.

289.—Centres in Regulating Square.—Let a 2 1 b (*Fig.* 172) be the size of a regulating square, found according to the previous rule, the required number of revolutions being

$1\frac{3}{4}$. Divide two adjacent sides, as *a* 2 and 2 1, into as many equal parts as there are quarters in the number of revolutions, as seven; from those points of division draw lines across the square at right angles to the lines divided; then 1 being the first centre, 2, 3, 4, 5, 6, and 7 are the centres for the other quarters, and 8 is the centre for the eye; the heavy lines that determine these centres being each one part less in length than its preceding line.

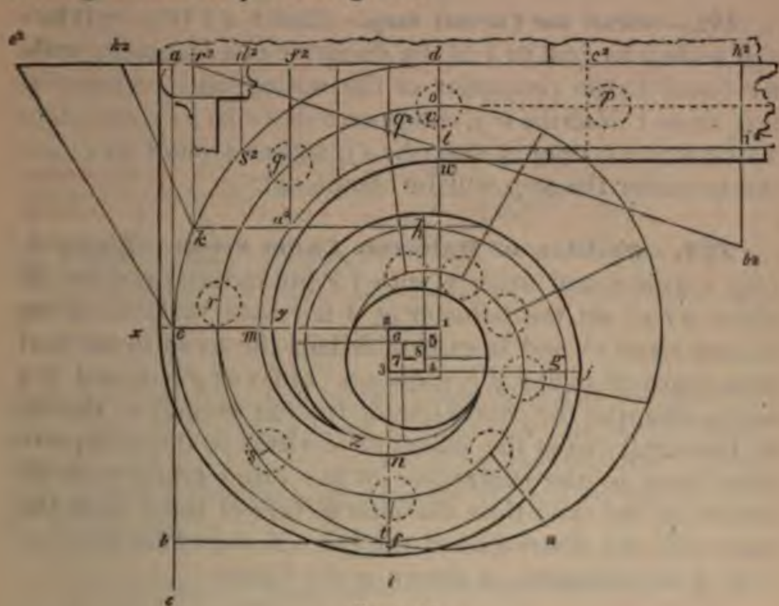


FIG. 173.

290.—Scroll for Hand-Rail Over Curtail Step.—Let *a b* (*Fig.* 173) be the given breadth, $1\frac{3}{4}$ the given number of revolutions, and let the relative size of the regulating square to the eye be $\frac{1}{3}$ of the diameter of the eye. Then, by the rule, $1\frac{3}{4}$ multipled by 4 gives 7, and 3, the number of times a side of the square is contained in the eye, being added, the sum is 10. Divide *a b*, therefore, into 10 equal parts, and set one from *b* to *c*; bisect *a c* in *e*; then *a e* will be the length of the longest ordinate (1 *d* or 1 *e*). From *a* draw *a d*, from *e* draw *e* 1, and from *b* draw *b f*, all at right angles to *a b*; make *e* 1 equal to *a*, and through 1 draw 1 *d* parallel

to $a\ b$; set $b\ c$ from 1 to 2, and upon 1 2 complete the regulating square; divide this square as at *Fig.* 172; then describe the arcs that compose the scroll, as follows: upon 1 describe $d\ e$, upon 2 describe $e\ f$, upon 3 describe $f\ g$, upon 4 describe $g\ h$, etc.; make $d\ l$ equal to the width of the rail, and upon 1 describe $l\ m$, upon 2 describe $m\ n$, etc.; describe the eye upon 8, and the scroll is completed.

291.—Scroll for Curtail Step.—Bisect $d\ l$ (*Fig.* 173) in o, and make $o\ v$ equal to $\frac{1}{8}$ of the diameter of a baluster; make $v\ w$ equal to the projection of the nosing, and $e\ x$ equal to $w\ l$; upon 1 describe $w\ y$, and upon 2 describe $y\ z$; also, upon 2 describe $x\ i$, upon 3 describe $i\ j$, and so around to z; and the scroll for the step will be completed.

292.—Position of Balusters Under Scroll.—Bisect $d\ l$ (*Fig.* 173) in o, and upon 1, with 1 o for radius, describe the circle $o\ r\ u$; set the baluster at p fair with the face of the second riser, c^2, and from p, with half the tread in the dividers, space off as at o, q, r, s, t, u, etc., as far as q^2; upon 2, 3, 4, and 5 describe the centre-line of the rail around to the eye of the scroll; from the points of division in the circle $o\ r\ u$ draw lines to the centre-line of the rail, tending to 8, the centre of the eye; then the intersection of these radiating lines with the centre-line of the rail will determine the position of the balusters, as shown in the figure.

293.—Falling-Mould for Raking Part of Scroll.—Tangical to the rail at h (*Fig.* 173) draw $h\ k$ parallel to $d\ a$; then $k\ a^2$ will be the joint between the twist and the other part of the scroll. Make $d\ e^2$ equal to the stretch-out of $d\ e$, and upon $d\ e^2$ find the position of the point k, as at k^2; at *Fig.* 174, make $e\ d$ equal to $e^2\ d$ in *Fig.* 173, and $d\ c$ equal to $d\ c^2$ in that figure; from c draw $c\ a$ at right angles to $e\ c$, and equal to one rise; make $c\ b$ equal to one tread, and from b, through a, draw $b\ j$; bisect $a\ c$ in l, and through l draw $m\ q$ parallel to $e\ h$; $m\ q$ is the height of the level part of a scroll, which should always be about $3\frac{1}{2}$ feet from the floor; ease off the angle $m\ f\ j$, according to *Art.* 521, and draw

$g\,w\,n$ parallel to $m\,x\,j$, and at a distance equal to the thickness of the rail; at a convenient place for the joint, as i, draw $i\,n$ at right angles to $b\,j$; through n draw $j\,h$ at right angles to $e\,h$; make $d\,k$ equal to $d\,k^2$ in *Fig.* 173, and from k draw $k\,o$ at right angles to $e\,h$; at *Fig.* 173, make $d\,h^2$ equal to $d\,h$ in *Fig.* 174, and draw $h^2\,b^2$ at right angles to $d\,h^2$;

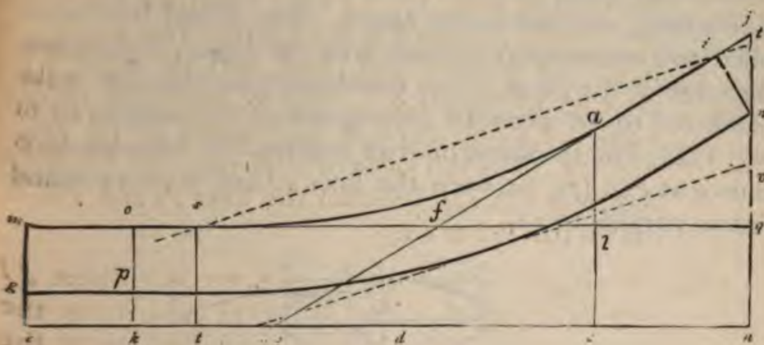


FIG. 174.

then $k\,a^2$ and $h^2\,b^2$ will be the position of the joints on the plan, and, at *Fig.* 174, $o\,p$ and $i\,n$ their position on the falling-mould; and $p\,o\;i\,n$ (*Fig.* 174) will be the required falling-mould which is to be bent upon the vertical surface from h^2 to k (*Fig.* 173).

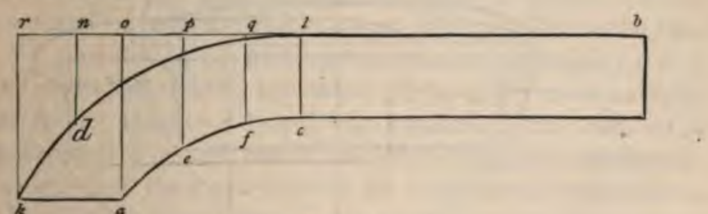


FIG. 175.

294.—Face-Mould for the Scroll.—At *Fig.* 173, from k draw $k\,r^2$ at right angles to $r^2\,d$: at *Fig.* 172, make $h\,r$ equal to $h^2\,r^2$ in *Fig.* 173, and from r draw $r\,s$ at right angles to $r\,h$; from the intersection of $r\,s$ with the level line $m\,q$, through i, draw $s\,t$; at *Fig.* 173, make $h^2\,b^2$ equal to $q\,t$ in *Fig.* 172, and join b^2 and r^2; from a^2, and from as many

other points in the arcs, $a^2 l$ and $k d$, as is thought necessary, draw ordinates to $r^2 d$ at right angles to the latter; make $r b$ (*Fig.* 175) equal in its length and in its divisions to the line $r^2 b^2$ in *Fig.* 173; from r, n, o, p, q, and l draw the lines $r k$, $n d$, $o a$, $p e$, $q f$, and $l c$ at right angles to $r b$, and equal to $r^2 k$, $d^2 s^2$, $f^2 a^2$, etc., in *Fig.* 173; through the points thus found trace the curves $k l$ and $a c$, and complete the face-mould, as shown in the figure. This mould is to be applied to a square-edged plank, with the edge, $l b$ parallel to the edge of the plank. The rake-lines upon the edge of the plank are to be made to correspond to the angle $s t k$ in *Fig.* 174. The thickness of stuff required for this mould is shown at *Fig.* 174, between the lines $s t$ and $u v$—$u v$ being drawn parallel to $s t$.

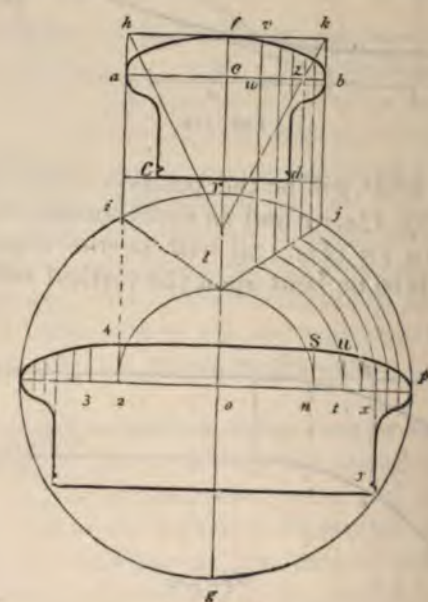


FIG. 176.

295.—Form of Newel-Cap from a Section of the Rail. —Draw $a b$ (*Fig.* 176) through the widest part of the given section, and parallel to $c d$; bisect $a b$ in e, and through a, e, and b draw $h i$, $f g$, and $k j$ at right angles to $a b$; at a con-

venient place on the line $f g$, as o, with a radius equal to half the width of the cap, describe the circle $i j g$; make $r l$ equal to $e b$ or $e a$; join l and j, also l and i; from the curve $f b$ to the line $l j$ draw as many ordinates as is thought necessary parallel to $f g$; from the points at which these ordinates meet the line $l j$, and upon the centre, o, describe arcs in continuation to meet $o p$; from $n t x$, etc., draw $n s$, $t u$, etc., parallel to $f g$; make $n s$, $t u$, etc., equal to $e f$, $w v$, etc.; make $x y$, etc., equal to $z d$, etc.; make $o 2$, $o 3$, etc., equal to $o n$, $o t$, etc.; make $2 4$ equal to $n s$, and in this way find the length of the lines crossing $o m$; through the points thus found describe the section of the newel-cap as shown in the figure.

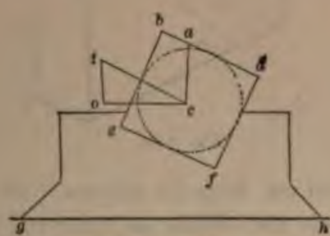


FIG. 177.

296.—Boring for Balusters in a Round Rail before it is Rounded.—Make the angle $o c t$ (*Fig.* 177) equal to the angle $o c t$ at *Fig.* 144; upon c describe a circle with a radius equal to half the thickness of the rail; draw the tangent $b d$ parallel to $t c$, and complete the rectangle $e b d f$, having sides tangical to the circle; from c draw $c a$ at right angles to $o c$; then, $b d$ being the bottom of the rail, set a gauge from b to a, and run it the whole length of the stuff; in boring, place the centre of the bit in the gauge-mark at a, and bore in the direction $a c$. To do this easily, make *chucks* as represented in the figure, the bottom edge, $g h$, being parallel to $o c$, and having a place sawed out, as $e f$, to receive the rail. These being nailed to the bench, the rail will be held steadily in its proper place for boring vertically. The distance apart that the balusters require to be, on the underside of the rail, is one half the length of the *rake-side* of the pitch-board.

SPLAYED WORK.

297.—The Bevels in Splayed Work.—The principles employed in finding the lines in stairs are nearly allied to those required to find the bevels for *splayed* work—such as hoppers, bread-trays, etc. A method by which these may be

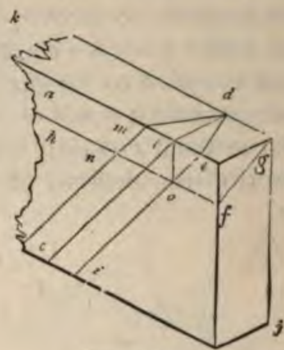


FIG. 178.

obtained will, therefore, here be shown. In *Fig.* 178, *a b c* is the angle at which the work is splayed, and *b d*, on the upper edge of the board, is at right angles to *a b*; make the angle *f g j* equal to *a b c*, and from *f* draw *f h* parallel to *e a*; from *b* draw *b o* at right angles to *a b*; through *o* draw *i e* parallel to *c b*, and join *e* and *d*; then the angle *a e d* will be the proper bevil for the ends from the inside, or *k d e* from the outside. If a mitre-joint is required, set *f g*, the thickness of the stuff on the level, from *e* to *m*, and join *m* and *d*; then *k d m* will be the proper bevil for a mitre-joint.

If the upper edge of the splayed work is to be bevelled, so as to be horizontal when the work is placed in its proper position, then *f g j*, the same as *a b c*, will be the proper bevel for that purpose. Suppose, therefore, that a piece indicated by the lines *k g*, *g f*, and *f h* were taken off; then a line drawn upon the bevelled surface from *d* at right angles to *k d* would show the true position of the joint, because it would be in the direction of the board for the other side; but a line so drawn would pass through the point *o*, thus proving the principle correct. So, if a line were drawn upon the bevelled surface from *d* at an angle of 45 degrees to *k d*, it would pass through the point *n*.

VIEW IN THE ALHAMBRA.

SECTION IV.—DOORS AND WINDOWS.

DOORS.

298.—General Requirements.—Among the architectural arrangements of an edifice, the door is by no means the least in importance; and if properly constructed, it is not only an article of use, but also of ornament, adding materially to the regularity and elegance of the apartments. The dimensions and style of finish of a door should be in accordance with the size and style of the building, or the apartment for which it is designed. As regards the utility of doors, the principal door to a public building should be of sufficient width to admit of a free passage for a crowd of people; while that of a private apartment will be wide enough if it permit one person to pass without being incommoded. Experience has determined that the least width allowable for this is 2 feet 8 inches; although doors leading to inferior and unimportant rooms may, if circumstances require it, be as narrow as 2 feet 6 inches; and doors for closets, where an entrance is seldom required, may be but 2 feet wide. The width of the principal door to a public building may be from 6 to 12 feet, according to the size of the building; and the width of doors for a dwelling may be from 2 feet 8 inches to 3 feet 6 inches. If the importance of an apartment in a dwelling be such as to require a door of greater width than 3 feet 6 inches, the opening should be closed with two doors, or a door in two folds; generally, in such cases, where the opening is from 5 to 8 feet, folding or sliding doors are adopted. As to the height of a door, it should in no case be less than about 6 feet 3 inches; and generally not less than 6 feet 8 inches.

299.—The Proportion between Width and Height: of single doors, for a dwelling, should be as 2 is to 5; and, for entrance-doors to public buildings, as 1 is to 2. If the width is given and the height required of a door for a

dwelling, multiply the width by 5, and divide the product by 2; but if the height is given and the width required, divide by 5 and multiply by 2. Where two or more doors of different widths show in the same room, it is well to proportion the dimensions of the more important by the above rule, and make the narrower doors of the same height as the wider ones; as all the doors in a suit of apartments, except the folding or sliding doors, have the best appearance when of one height. The proportions for folding or sliding doors should be such that the width may be equal to $\frac{4}{5}$ of the height; yet this rule needs some qualification; for if the width of the opening be greater than one half the width of the room, there will not be a sufficient space left

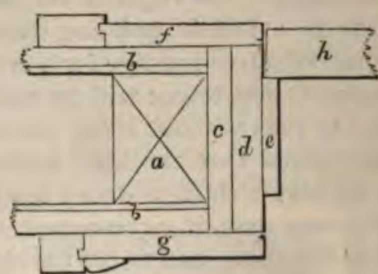


Fig. 179.

for opening the doors; also, the height should be about one tenth greater than that of the adjacent single doors.

300.—Panels.—Where doors have but two panels in width, let the stiles and muntins be each $\frac{1}{7}$ of the width; or, whatever number of panels there may be, let the united widths of the stiles and the muntins, or the whole width of the solid, be equal to $\frac{3}{7}$ of the width of the door. Thus: in a door 35 inches wide, containing two panels in width, the stiles should be 5 inches wide; and in a door 3 feet 6 inches wide, the stiles should be 6 inches. If a door 3 feet 6 inches wide is to have 3 panels in width, the stiles and muntins should be each $4\frac{1}{2}$ inches wide, each panel being 8 inches. The bottom rail and the lock-rail ought to be each equal in width to $\frac{1}{10}$ of the height of the door; and the top

rail, and all others, of the same width as the stiles. The moulding on the panel should be equal in width to $\frac{1}{4}$ of the width of the stile.

301.—Trimmings.—*Fig.* 179 shows a method of trimming doors: *a* is the door-stud; *b*, the lath and plaster; *c*, the *ground*; *d*, the jamb; *e*, the *stop*; *f* and *g*, architrave casings; and *h*, the door-stile. It is customary in ordinary work to form the stop for the door by *rebating* the jamb. But when the door is thick and heavy, a better plan is to nail on a piece as at *e* in the figure. This piece can be fitted to the door and put on after the door is hung; so, should the door be a trifle *winding*, this will correct the evil, and the door be made to shut solid.

Fig. 180 is an elevation of a door and trimmings suitable for the best rooms of a dwelling. (For trimmings generally, see Sect. V.) The number of panels into which a door should be divided may be fixed at pleasure; yet the present style of finishing requires that the number be as small as a proper regard for strength will admit. In some of our best dwellings, doors have been made having only two upright panels. A few years' experience, however, has proved that the omission of the lock-rail is at the expense of the strength and durability of the door; a four-panel door, therefore, is the best that can be made.

302.—Hanging Doors.—Doors should all be hung so as to open into the principal rooms; and, in general, no door should be hung to open into the hall, or passage. As to the proper edge of the door on which to affix the hinges, no general rule can be assigned.

WINDOWS.

303.—Requirements for Light.—A window should be of such dimensions, and in such a position, as to admit a sufficiency of light to that part of the apartment for which it is designed. No definite rule for the size can well be given that will answer in all cases; yet, as an approxima-

tion, the following has been used for general purpos Multiply together the length and the breadth in feet of th apartment to be lighted, and the product by the height i

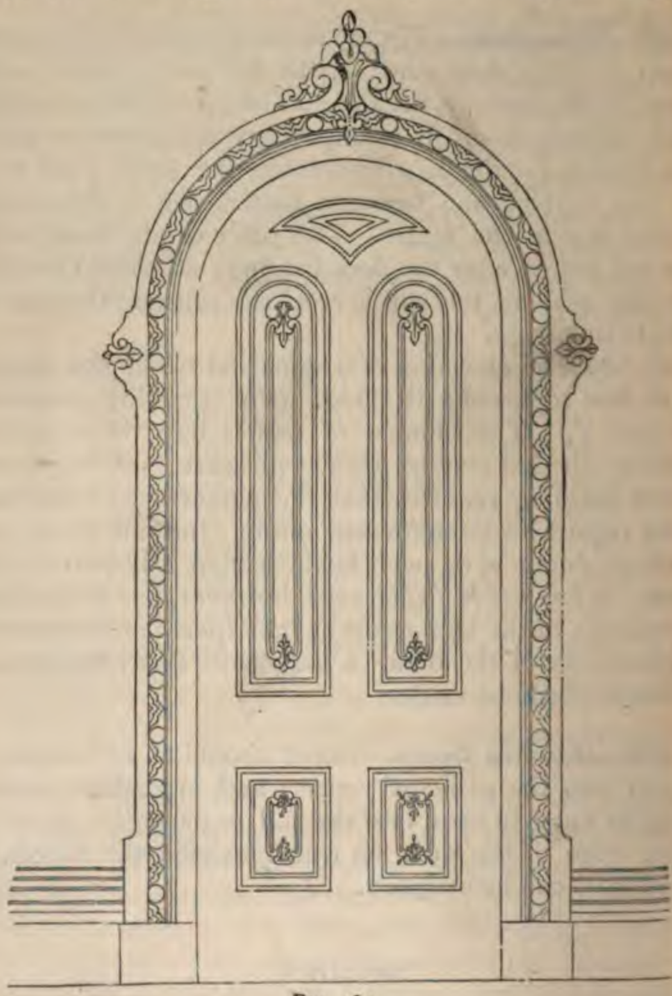


FIG. 180.

feet; then the square root of this product will show t required number of square feet of glass.

304.—Window-Frames.—For the size of window-fram add $4\frac{1}{2}$ inches to the width of the glass for their width, a

$6\frac{1}{2}$ inches to the height of the glass for their height. These give the dimensions, in the clear, of ordinary frames for 12-light windows; the height being taken at the inside edge of the sill. In a brick wall, the width of the opening is 8 inches more than the width of the glass—$4\frac{1}{2}$ for the stiles of the sash, and $3\frac{1}{2}$ for hanging stiles—and the height between the stone sill and lintel is about $10\frac{1}{2}$ inches more than the height of the glass, it being varied according to the thickness of the sill of the frame.

305.—Inside Shutters.—Inside shutters folding into *boxes* require to have the box-shutter about one inch wider than the flap, in order that the flap may not interfere when both are folded into the box. The usual margin shown between the face of the shutter when folded into the box and the quirk of the stop-bead, or edge of the casing, is half an inch; and, in the usual method of letting the *whole* of the thickness of the butt hinge into the edge of the box-shutter, it is necessary to make allowance for the *throw* of the hinge. This may, in general, be estimated at $\frac{1}{4}$ of an inch at each hinging; which being added to the margin, the entire width of the shutters will be $1\frac{1}{2}$ inches more than the width of the frame in the clear. Then, to ascertain the width of the box-shutter, add $1\frac{1}{2}$ inches to the width of the frame in the clear, between the pulley-stiles; divide this product by 4, and add half an inch to the quotient, and the last product will be the required width. For example, suppose the window to have 3 lights in width, 11 inches each. Then, 3 times 11 is 33, and $4\frac{1}{2}$ added for the wood of the sash gives $37\frac{1}{2}$; $37\frac{1}{2}$ and $1\frac{1}{2}$ is 39, and 39 divided by 4 gives $9\frac{3}{4}$; to which add half an inch, and the result will be $10\frac{1}{4}$ inches, the width required for the box-shutter.

306.—Proportion: Width and Height.—In disposing and locating windows in the walls of a building, the rules of architectural taste require that they be of different heights in different stories, but generally of the same width. The windows of the upper stories should all range perpendicularly over those of the first, or principal, story; and they

should be disposed so as to exhibit a balance of parts throughout the front of the building. To aid in this it is always proper to place the front door in the middle of the front of the building; and, where the size of the house will admit of it, this plan should be adopted. (See the latter part of *Art.* 50.) The proportion that the height should bear to the width may be, in accordance with general usage, as follows:

The height	of	basement windows,		$1\frac{1}{3}$	of the width.
"	"	principal-story	"	$2\frac{1}{8}$	"
"	"	second-story	"	$1\frac{7}{8}$	"
"	"	third-story	"	$1\frac{3}{4}$	"
"	"	fourth-story	"	$1\frac{1}{2}$	"
"	"	attic-story	"	the same as the width.	

But, in determining the height of the windows for the several stories, it is necessary to take into consideration the height of the story in which the window is to be placed. For, in addition to the height from the floor, which is generally required to be from 28 to 30 inches, room is wanted above the head of the window for the window-trimming and the cornice of the room, besides some respectable space which there ought to be between these.

307.—Circular Heads.—Doors and windows usually terminate in a horizontal line at top. These require no special directions for their trimmings. But circular-headed doors and windows are more difficult of execution, and require some attention. If the jambs of a door or window be placed at right angles to the face of the wall, the edges of the *soffit*, or surface of the head, would be straight, and its length be found by getting the stretch-out of the circle (*Art.* 524); but when the jambs are placed obliquely to the face of the wall, occasioned by the demand for light in an oblique direction, the form of the soffit will be obtained by the following article; as also when the face of the wall is circular, as shown in the succeeding figure.

308.—Form of Soffit for Circular Window-Heads.—When the light is received in an oblique direction, let *a b c d* (*Fig.* 181) be the ground-plan of a given window, and *e f a* a vertical section taken at right angles to the face of the jambs.

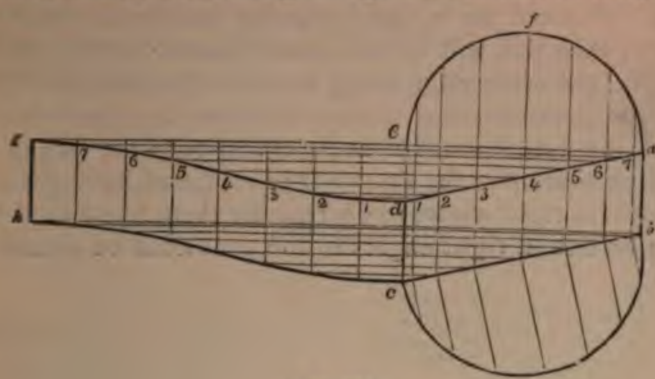


FIG. 181.

From *a*, through *e*, draw *a g* at right angles to *a b*; obtain the stretch-out of *e f a*, and make *e g* equal to it; divide *e g* and *e f a* each into a like number of equal parts, and drop perpendiculars from the points of division in each; from the points of intersection, 1, 2, 3, etc., in the line *a d*,

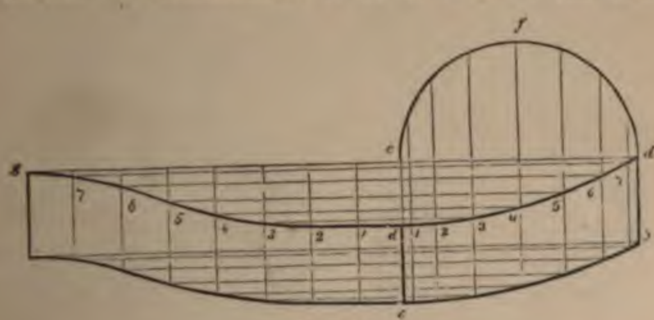


FIG. 182.

draw horizontal lines to meet corresponding perpendiculars from *e g*; then those points of intersection will give the curve line *d g*, which will be the one required for the edge of the soffit. The other edge, *c h*, is found in the same manner.

For the form of the soffit for circular window-heads, when the face of the wall is curved, let *a b c d* (*Fig.* 182) be the ground-plan of a given window, and *e f a* a vertical section of the head taken at right angles to the face of the jambs. Proceed as in the foregoing article to obtain the line *d g*; then that will be the curve required for the edge of the soffit, the other edge being found in the same manner.

If the given vertical section be taken in a line with the face of the wall, instead of at right angles to the face of the jambs, place it upon the line *c b* (*Fig.* 181), and, having drawn ordinates at right angles to *c b*, transfer them to *e f a*; in this way a section at right angles to the jambs can be obtained.

INTERIOR OF ST. PETER'S, ROME.

SECTION V.—MOULDINGS AND CORNICES.

MOULDINGS.

309.—Mouldings: are so called because they are of the ne determinate shape throughout their length, as though : whole had been cast in the same mould or form. The ;ular mouldings, as found in remains of classic architec- :e, are eight in number, and are known by the following mes:

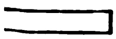


FIG. 183. Annulet, band, cincture, fillet, listel or square.

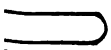


FIG. 184. Astragal or bead.

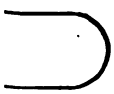


FIG. 185. Torus or tore.

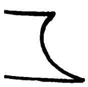


FIG. 186. Scotia, trochilus or mouth.

FIG. 187. Ovolo, quarter-round or echinus.

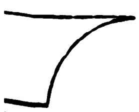


FIG. 188. Cavetto, cove or hollow.

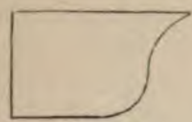


FIG. 189.

Cymatium, or cyma-recta.

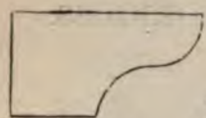


FIG. 190.

Inverted cymatium, or cyma-reversa.

Ogee.

Some of the terms are derived thus: Fillet, from the French word *fil*, thread. Astragal, from *astragalos*, a bone of the heel—or the curvature of the heel. Bead, because this moulding, when properly carved, resembles a string of beads. Torus, or tore, the Greek for *rope*, which it resembles when on the base of a column. Scotia, from *skotia*, darkness, because of the strong shadow which its depth produces, and which is increased by the projection of the torus above it. Ovolo, from *ovum*, an egg, which this member resembles, when carved, as in the Ionic capital. Cavetto, from *cavus*, hollow. Cymatium, from *kumaton*, a wave.

310.—Characteristics of Mouldings.—Neither of these mouldings is peculiar to any one of the orders of architecture; and although each has its appropriate use, yet it is by no means confined to any certain position in an assemblage of mouldings. The use of the fillet is to bind the parts, as also that of the astragal and torus, which resemble ropes. The ovolo and cyma-reversa are strong at their upper extremities, and are therefore used to support projecting parts above them. The cyma-recta and cavetto, being weak at their upper extremities, are not used as supporters, but are placed uppermost to cover and shelter the other parts. The scotia is introduced in the base of a column to separate the upper and lower torus, and to produce a pleasing variety and relief. The form of the bead and that of the torus is the same; the reasons for giving distinct names to them are that the torus, in every order, is always considerably larger than the bead, and is placed among the base mouldings,

as the bead is never placed there, but on the capital or ature; the torus, also, is seldom carved, whereas the s; and while the torus among the Greeks is frequently al in its form, the bead retains its circular shape. While otia is the reverse of the torus, the cavetto is the re- of the ovolo, and the cyma-recta and cyma-reversa are nations of the ovolo and cavetto.

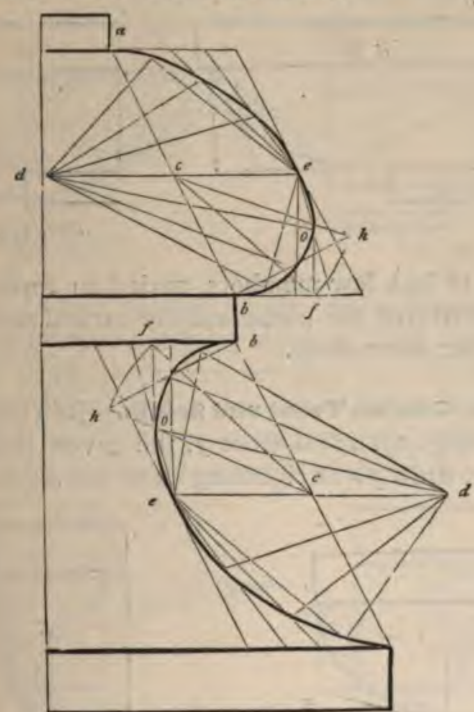


FIG. 191.

e curves of mouldings, in Roman architecture, were generally composed of parts of circles; while those of reeks were almost always elliptical, or of some one of nic sections, but rarely circular, except in the case of ead, which was always, among both Greeks and Ro- of the form of a semicircle. Sections of the cone af- greater variety of forms than those of the sphere; and ps this is one reason why the Grecian architecture so

much excels the Roman. The quick turnings of the ovolo and cyma-reversa, in particular, when exposed to a bright sun, cause those narrow, well-defined streaks of light which give life and splendor to the whole.

311.—A Profile: is an assemblage of essential parts and mouldings. That profile produces the happiest effect which

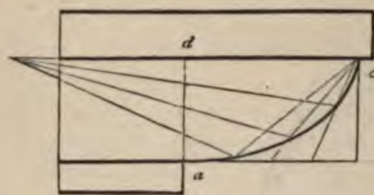


FIG. 192.

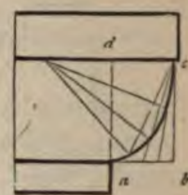


FIG. 193.

is composed of but few members, varied in form and size, and arranged so that the plane and the curved surfaces succeed each other alternately.

312.—The Grecian Torus and Scotia.—Join the extremities a and b (*Fig.* 191), and from f, the given projection of the moulding, draw $f o$ at right angles to the fillets; from b

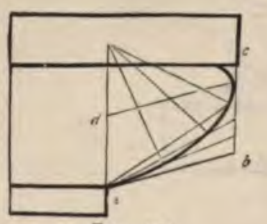


FIG. 194.

FIG. 195.

draw $b h$ at right angles to $a b$; bisect $a b$ in c; join f and c, and upon c, with the radius $c f$, describe the arc $f h$, cutting $b h$ in h; through c draw $d e$ parallel with the fillets; make $d c$ and $c e$ each equal to $b h$; then $d e$ and $a b$ will be conjugate diameters of the required ellipse. To describe the curve by intersection of lines, proceed as directed at *Art.*

51 and *note;* by a trammel, see *Art.* 549; and to find the ›ci, in order to describe it with a string, see *Art.* 548.

313.—**The Grecian Echinus.**—*Figs.* 192 to 199 exhibit, va- ously modified, the Grecian ovolo, or echinus. *Figs.* 192 to 96 are elliptical, *a b* and *b c* being given tangents to the curve; arallel to which the semi-conjugate diameters, *a d* and *d c*,

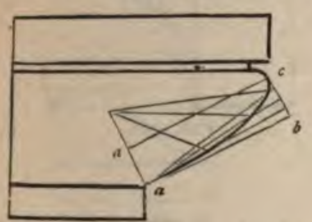


FIG. 196.

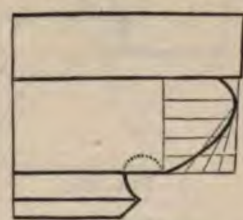


FIG. 197.

e drawn. In *Figs.* 192 and 193 the lines *a d* and *d c* are semi- es, the tangents, *a b* and *b c*, being at right angles to each her. To draw the curve, see *Art.* 551. In *Fig.* 197 the irve is parabolical, and is drawn according to *Art.* 560. In igs. 198 and 199 the curve is hyperbolical, being described cording to *Art.* 561. The length of the transverse axis, *a b*,

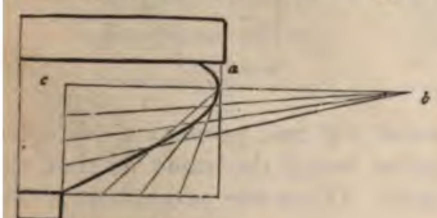


FIG. 198.

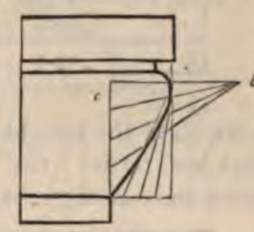


FIG. 199.

ing taken at pleasure in order to flatten the curve, *a b* ould be made short in proportion to *a c*.

314.—**The Grecian Cavetto.**—In order to describe this, igs. 200 and 201, having the height and projection given, e *Art.* 551.

315.—**The Grecian Cyma-Recta.**—When the projection more than the height, as at *Fig.* 202, make *a b* equal to the

height, and divide *a b c d* into four equal parallelograms; then proceed as directed in note to *Art.* 551. When the projection is less than the height, draw *d a* (*Fig.* 203) at right angles

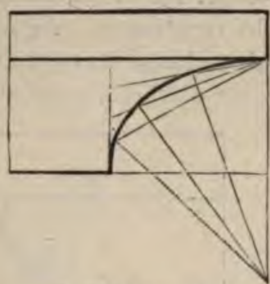


FIG. 200.

FIG. 201.

to *a b*; complete the rectangle, *a b c d*; divide this into four equal rectangles, and proceed according to *Art.* 551.

316.—The Grecian Cyma-Reversa.—When the projection

FIG. 202.

FIG. 203.

is more than the height, as at *Fig.* 204, proceed as directed for the last figure; the curve being the same as that, the position only being changed. When the projection is less

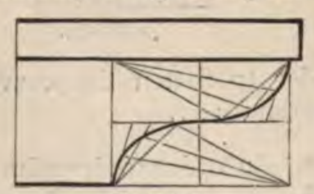


FIG. 204.

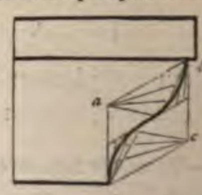


FIG. 205.

than the height, draw *a d* (*Fig.* 205) at right angles to the fillet; make *a d* equal to the projection of the moulding; then proceed as directed for *Fig.* 202.

317.—Roman Mouldings: are composed of parts of circles, and have, therefore, less beauty of form than the Grecian. The bead and torus are of the form of the semicircle, and the scotia, also, in some instances; but the latter is often composed of two quadrants, having different radii, as at *Figs*. 206 and 207, which resemble the elliptical curve. The ovolo and ca-

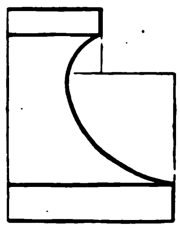


FIG. 206.

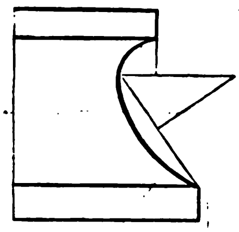


FIG. 207.

vetto are generally a quadrant, but often less. When they are less, as at *Fig*. 210, the centre is found thus: join the extremities, *a* and *b*, and bisect *a b* in *c*; from *c*, and at right angles to *a b*, draw *c d*, cutting a level line drawn from *a* in *d*; then *d* will be the centre. This moulding projects less than its height. When the projection is more than the height, as at *Fig*. 212, extend the line from *c* until it cuts a perpendicular

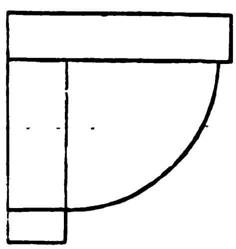


FIG. 208.

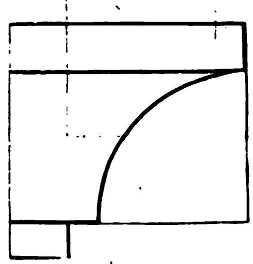


FIG. 209.

drawn from *a*, as at *d*; and that will be the centre of the curve. In a similar manner, the centres are found for the mouldings at *Figs*. 207, 211, 213, 216, 217, 218, and 219. The centres for the curves at *Figs*. 220 and 221 are found thus: bisect the line *a b* at *c*; upon *a*, *c* and *b* successively, with *a c* or *c b* for radius, describe arcs intersecting at *d* and *d*; then those intersections will be the centres.

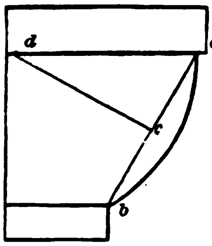


FIG. 210.

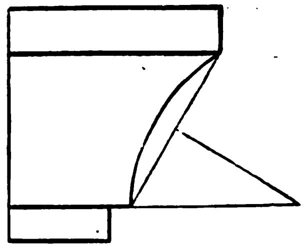


FIG. 211.

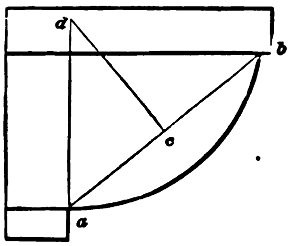


FIG. 212.

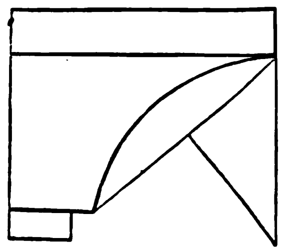


FIG. 213.

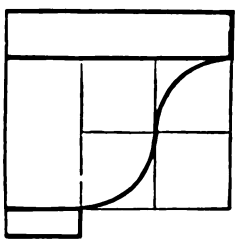


FIG. 214.

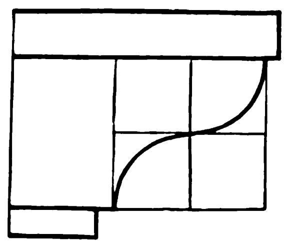


FIG. 215

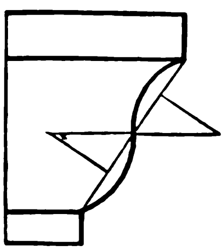


FIG. 216.

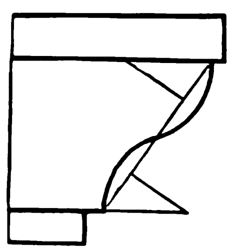


FIG. 217.

318.—**Modern Mouldings:** are represented in *Figs.* 222
29. They have been quite extensively and successfully
l in inside finishing. *Fig.* 222 is appropriate for a bed-
ılding under a low projecting shelf, and is frequently
l under mantel-shelves. The tangent *i h* is found thus:
ct the line *a b* at *c*, and *b c* at *d*; from *d* draw *d e* at
ıt angles to *e b*; from *b* draw *b f* parallel to *e d*; upon *b*,

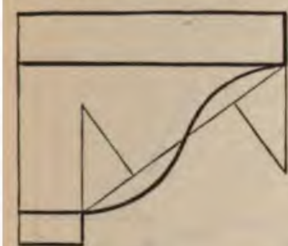


FIG. 218.

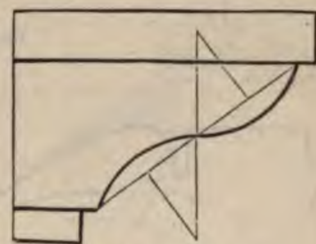


FIG. 219.

h *b d* for radius, describe the arc *d f*; divide this arc
7 equal parts, and set one of the parts from *s*, the limit
he projection, to *o*; make *o h* equal to *o e*; from *h*, through
raw the tangent *h i*; divide *b h*, *h c*, *c i*, and *i a* each into
ke number of equal parts, and draw the intersecting lines
lirected at *Art.* 521. If a bolder form is desired, draw
tangent, *i h*, nearer horizontal, and describe an elliptic

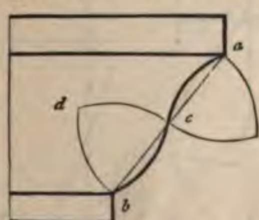


FIG. 220.

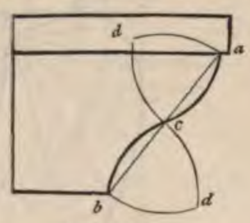


FIG. 221.

ve as shown in *Figs.* 191 and 224. *Fig.* 223 is much used
ase, or skirting, of rooms, and in deep panelling. The
ve is found in the same manner as that of *Fig.* 222. In
case, however, where the moulding has so little projec-
in comparison with its height, the point *e* being found
the last figure, *h s* may be made equal to *s e*, instead of
s in the last figure. *Fig.* 224 is appropriate for a crown

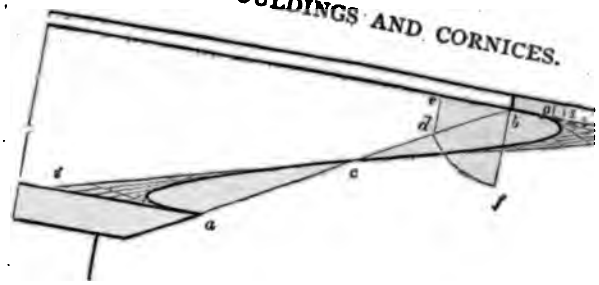


FIG. 222.

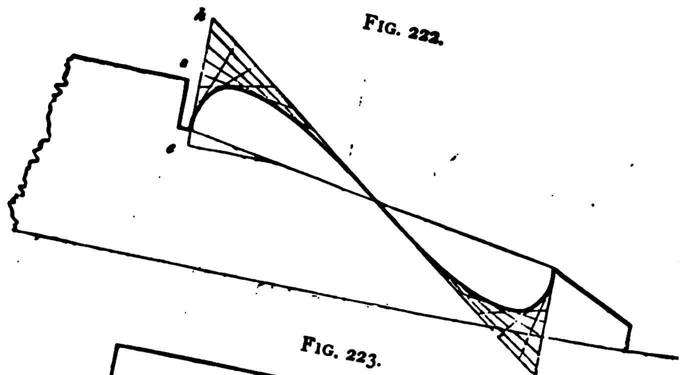


FIG. 223.

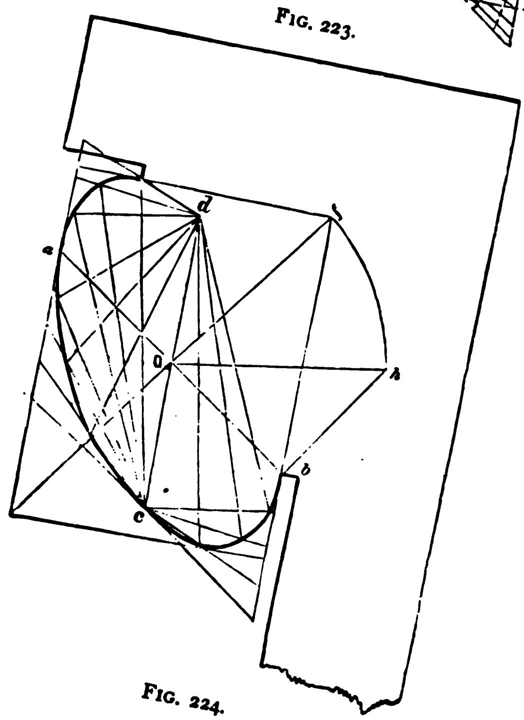


FIG. 224.

moulding of a cornice. In this figure the height and projection are given; the direction of the diameter, *a b*, drawn

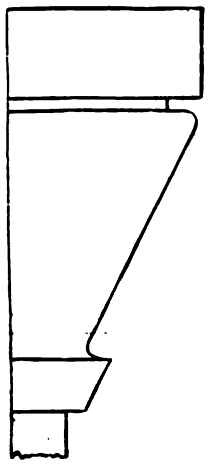


FIG. 225.

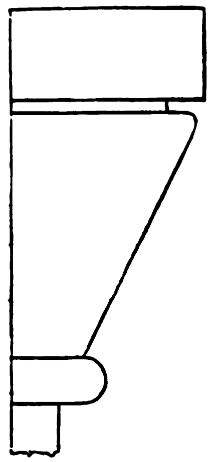


FIG. 226.

through the middle of the diagonal, *e f*, is taken at pleasure; and *d c* is parallel to *a c*. To find the length of *d c*, draw *b h*

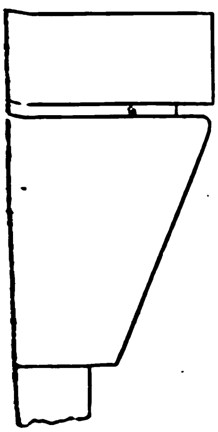


FIG. 227.

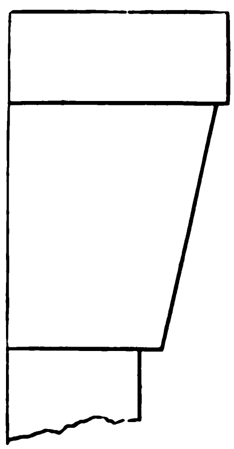


FIG. 228.

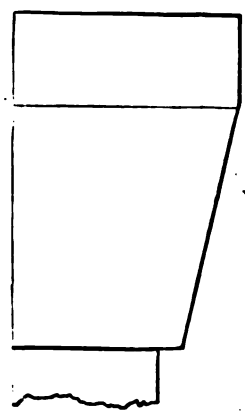


FIG. 229.

at right angles to *a b*; upon *o*, with *o f* for radius, describe the arc, *f h*, cutting *b h* in *h*; then make *o c* and *o d* each

equal to $b\,h$.* To draw the curve, see note to *Art.* 551. *Figs.* 225 to 229 are peculiarly distinct from ancient mouldings, being composed principally of straight lines; the few curves they possess are quite short and quick.

Figs. 230 and 231 are designs for antæ caps. The diameter of the antæ is divided into 20 equal parts, and the height and projection of the members are regulated in accordance with those parts, as denoted under *H* and *P*, height and projection. The projection is measured from the middle of the antæ. These will be found appropriate for porticos, doorways, mantelpieces, door and window trimmings,

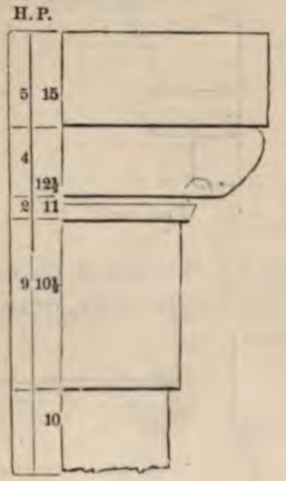


FIG. 230.

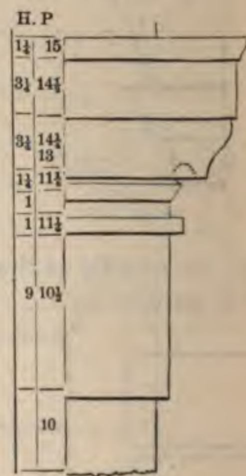


FIG. 231.

etc. The height of the antæ for mantelpieces should be from 5 to 6 diameters, having an entablature of from 2 to 2½ diameters. This is a good proportion, it being similar to the Doric order. But for a portico these proportions are

* The manner of ascertaining the length of the conjugate diameter, $d\,c$, in this figure, and also in *Figs.* 191, 241, and 242 is new, and is important in this application. It is founded upon well-known mathematical principles, viz.: All the parallelograms that may be circumscribed about an ellipsis are equal to one another, and consequently any one is equal to the rectangle of the two axes. And again: The sum of the squares of every pair of conjugate diameters is equal to the sum of the squares of the two axes.

much too heavy: an antæ 15 diameters high and an entablature of 3 diameters will have a better appearance.

CORNICES.

319.—Designs for Cornices.—*Figs.* 232 to 240 are designs for eave cornices, and *Figs.* 241 and 242 are for stucco cornices for the inside finish of rooms. In some of these the projection of the uppermost member from the facia is divided into twenty equal parts, and the various members

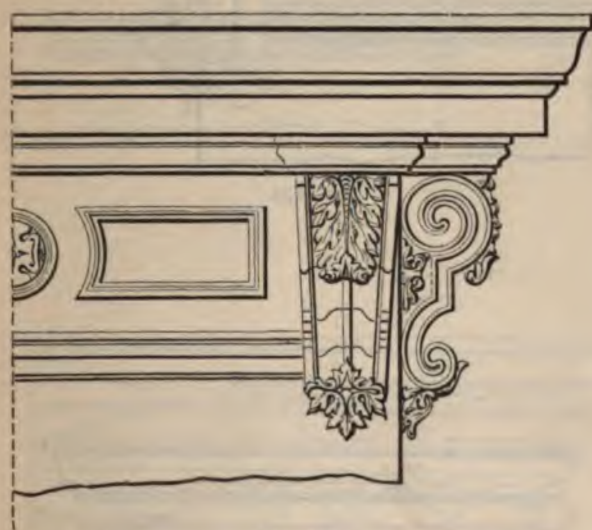


FIG. 232.

are proportioned according to those parts, as figured under *H* and *P*.

320.—Eave Cornices Proportioned to Height of Building.—Draw the line *a c* (*Fig.* 243), and make *b c* and *b a* each equal to 36 inches; from *b* draw *b d* at right angles to *a c*, and equal in length to $\frac{3}{4}$ of *a c*; bisect *b d* in *e*, and from *a*, through *e*, draw *a f*; upon *a*, with *a c* for radius, describe the arc *c f*, and upon *e*, with *e f* for radius, describe the arc *f d*; divide the curve *d f c*, into 7 equal parts, as at 10, 20, 30, etc., and from these points of division draw lines to *b c*

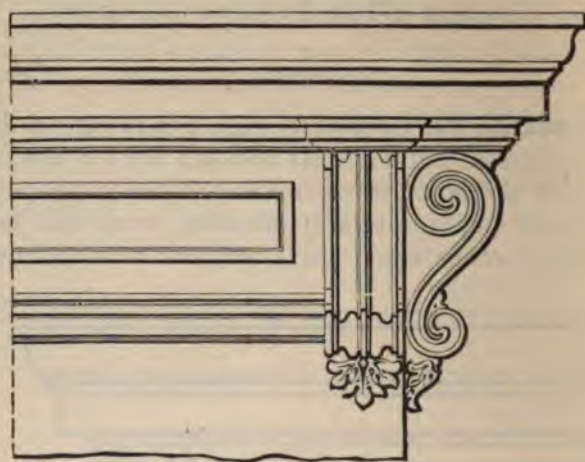


FIG. 233.

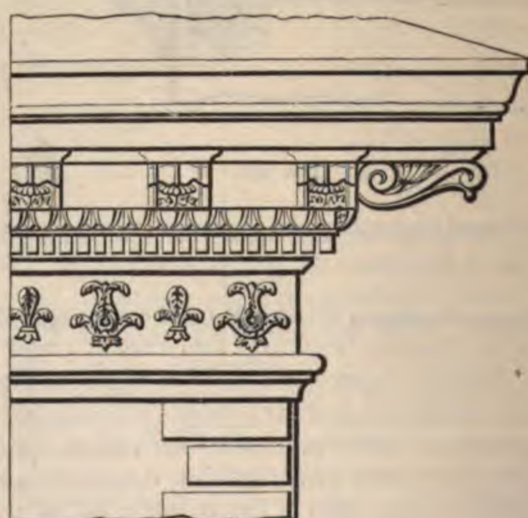


FIG. 234.

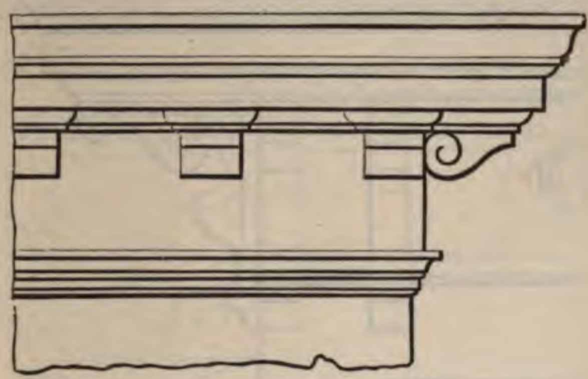


FIG. 235.

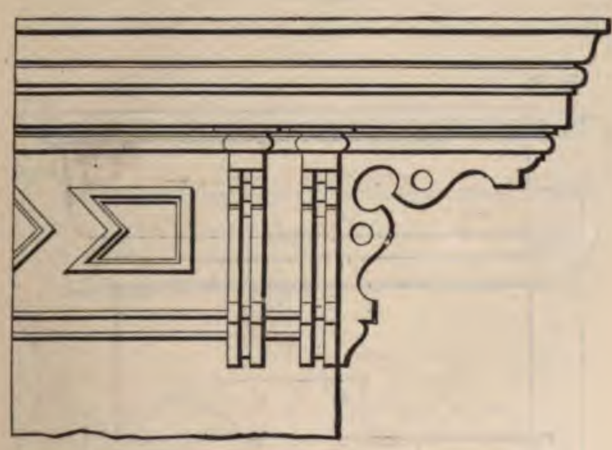


FIG. 236.

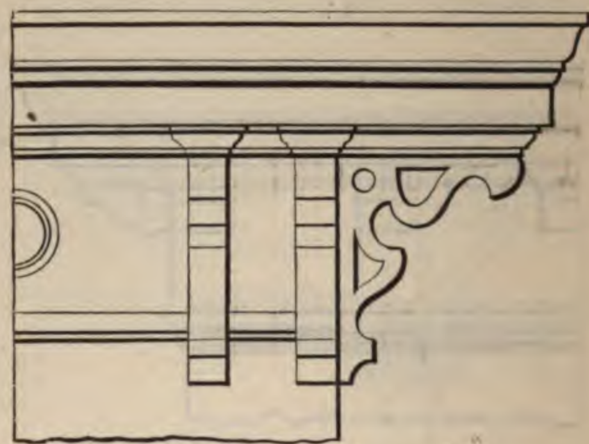


FIG. 237.

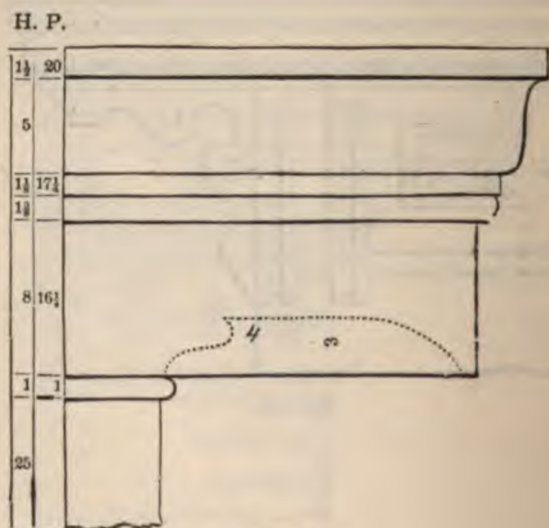


FIG. 238.

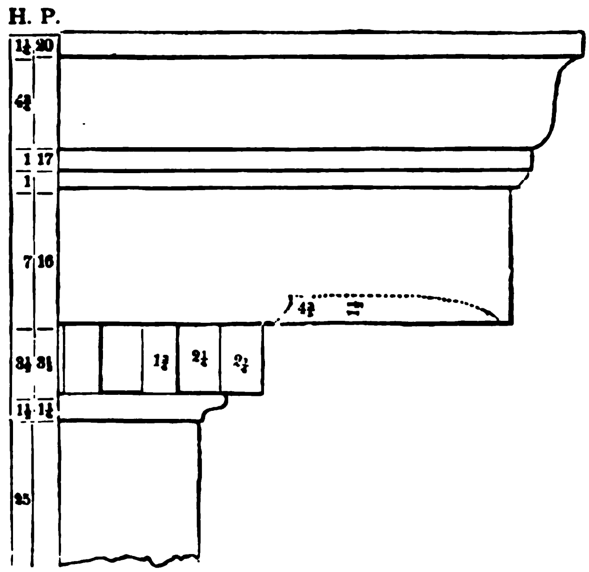


FIG. 239.

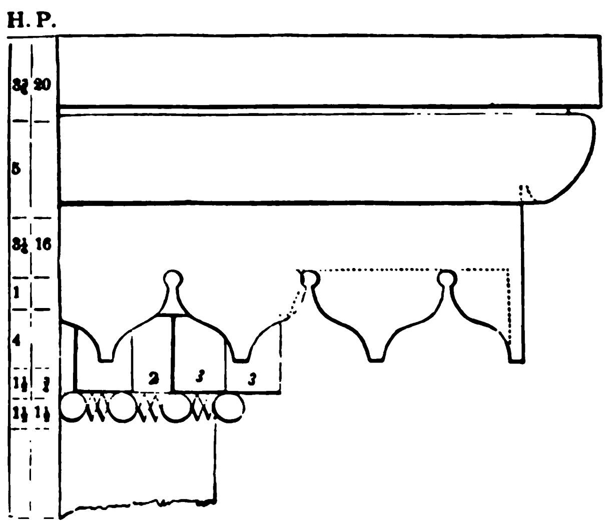


FIG. 240.

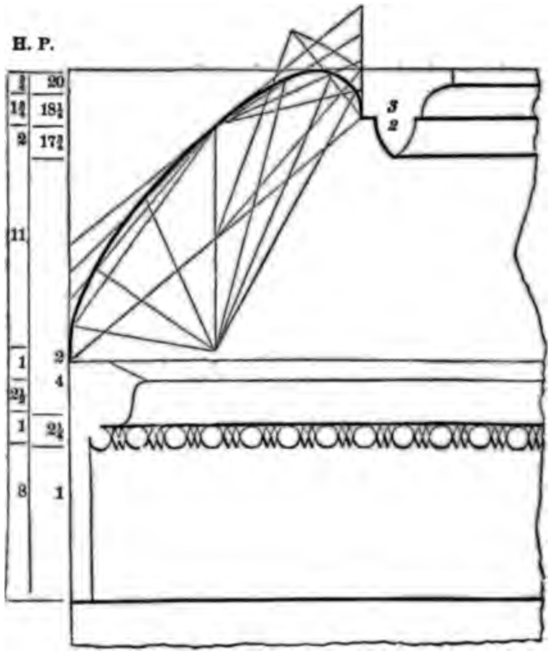


FIG. 241.

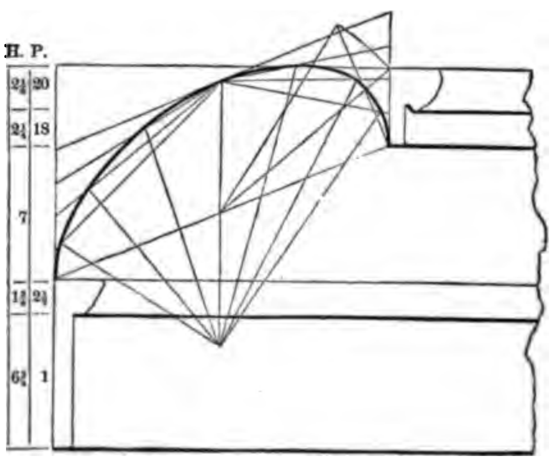


FIG. 242.

el to $d b$; then the distance b 1 is the projection of a
e for a building 10 feet high; b 2, the projection at 20
igh; b 3, the projection at 30 feet, etc. If the projec-
f a cornice for a building 34 feet high is required,
the arc between 30 and 40 into 10 equal parts, and

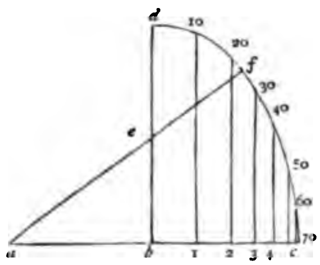


FIG. 243.

he fourth point from 30 draw a line to the base, $b c$,
el with $b d$; then the distance of the point at which
ine cuts the base from b will be the projection re-
l. So proceed for a cornice of any height within 70
The above is based on the supposition that 36 inches

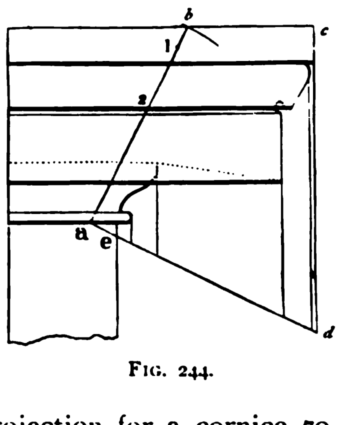


FIG. 244.

proper projection for a cornice 70 feet high. This,
neral purposes, will be found correct; still, the length
line $b c$ may be varied to suit the judgment of those
hink differently.

ving obtained the projection of a cornice, divide it
o equal parts, and apportion the several members

according to its destination—as is shown at *Figs.* 238, 239, and 240.

321.—Cornice Proportioned to a given Cornice.—Let the cornice at *Fig.* 244 be the given cornice. Upon any point in the lowest line of the lowest member, as at *a*, with the height of the required cornice for radius, describe an intersecting arc across the uppermost line, as at *b*; join *a* and *b*; then *b* 1 will be the perpendicular height of the upper fillet for the proposed cornice, 1 2 the height of the crown moulding—and so of all the members requiring to be enlarged to the sizes indicated on this line. For the projection of the

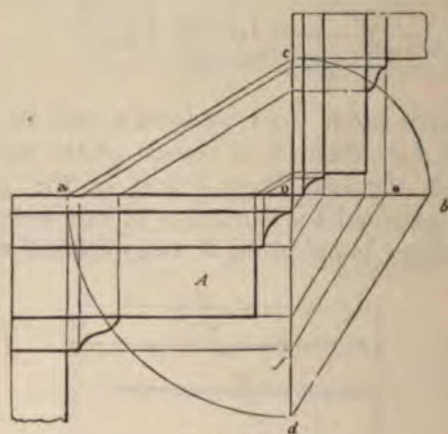


FIG. 245.

proposed cornice, draw *a d* at right angles to *a b*, and *c d* at right angles to *b c*; parallel with *c d* draw lines from each projection of the given cornice to the line *a d*; then *e d* will be the required projection for the proposed cornice, and the perpendicular lines falling upon *e d* will indicate the proper projection for the members.

To proportion a cornice according to a larger given cornice, let *A* (*Fig.* 245) be the given cornice. Extend *a o* to *b*, and draw *c d* at right angles to *a b*; extend the horizontal lines of the cornice, *A*, until they touch *o d*; place the height of the proposed cornice from *o* to *e*, and join *f* and *e*; upon *o*, with the projection of the given cornice, *o a*, for radius,

describe the quadrant *a d*; from *d* draw *d b* parallel to *f e*; upon *o*, with *o b* for radius, describe the quadrant *b c*; then *o c* will be the proper projection for the proposed cornice. Join *a* and *c*; draw lines from the projection of the different members of the given cornice to *a o* parallel to *o d*; from these divisions on the line *a o* draw lines to the line *o c* parallel to *a c*; from the divisions on the line *o f* draw lines to the line *o e* parallel to the line *f e*; then the divisions on the lines *o e* and *o c* will indicate the proper height and projection for the different members of the proposed cornice. In this process, we have assumed the height, *o e*, of the proposed cornice to be given; but if the projection, *o c*, alone

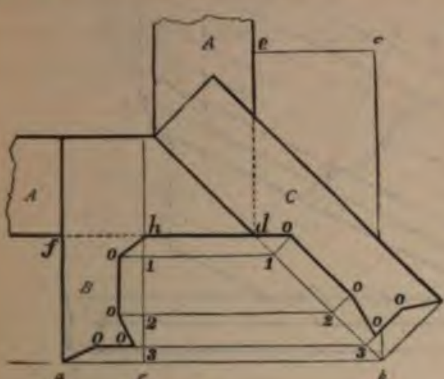


FIG. 246.

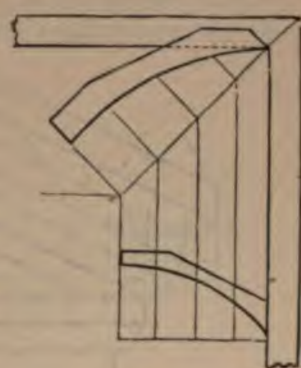


FIG. 247.

be given, we can obtain the same result by a different process. Thus: upon *o*, with *o c* for radius, describe the quadrant *c b*; upon *o*, with *o a* for radius, describe the quadrant *a d*; join *d* and *b*; from *f* draw *f e* parallel to *d b*; then *o e* will be the proper height for the proposed cornice, and the height and projection of the different members can be obtained by the above directions. By this problem, a cornice can be proportioned according to a *smaller* given one as well as to a *larger*; but the method described in the previous article is much more simple for that purpose.

322.—Angle Bracket in a Built Cornice.—Let *A* (*Fig.* 246) be the wall of the building, and *B* the given bracket,

which, for the present purpose, is turned down horizontally. The angle-bracket, *C*, is obtained thus: through the extremity, *a*, and parallel with the wall, *f d*, draw the line *a b*; make *e c* equal *a f*, and through *c* draw *c b* parallel with *e d*; join *d* and *b*, and from the several angular points in *B* draw ordinates to cut *d b* in 1, 2, and 3; at those points erect lines perpendicular to *d b*; from *h* draw *h g* parallel to *f a*; take

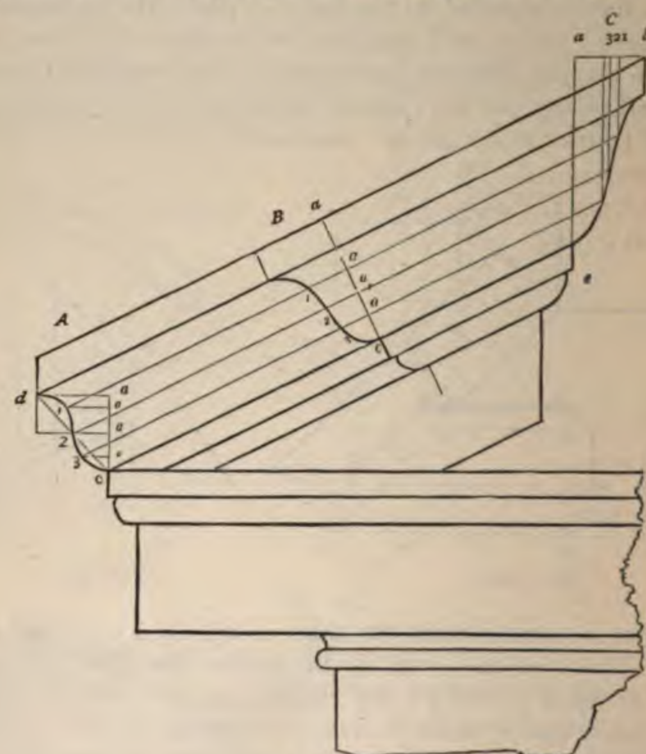


FIG. 248.

the ordinates, 1 *o*, 2 *o*, etc., at *B*, and transfer them to *C*, and the angle-bracket, *C*, will be defined. In the same manner, the angle-bracket for an internal cornice, or the angle-rib of a coved ceiling, or of groins, as at *Fig.* 247, can be found.

323.—Raking Mouldings matched with Level Returns.— Let *A* (*Fig.* 248) be the given moulding, and *A b* the rake of

roof. Divide the curve of the given moulding into any nber of parts, equal or unequal, as at 1, 2, and 3; from se points draw horizontal lines to a perpendicular erected n *c*; at any convenient place on the rake, as at *B*, draw at right angles to *A b*; also from *b* draw the horizontal *b a*; place the thickness, *d a*, of the moulding at *A* from *a*, and from *a* draw the perpendicular line *a e*; from points 1, 2, 3, at *A*, draw lines to *C* parallel to *A b*; ce *a* 1, *a* 2, and *a* 3, at *B*, and at *C*, equal to *a* 1, etc., at *A*; ough the points, 1, 2, and 3, at *B*, trace the curve—this be the proper form for the raking moulding, From 1, nd 3, at *C*, drop perpendiculars to the corresponding inates from 1, 2, and 3, at *A*; through the points of intertion, trace the curve—this will be the proper form for the rn at the top.

PART II.

SECTION VI.—GEOMETRY.

324.—Mathematics Essential.—In this and the following Sections, which will constitute Part II., there are treated of certain matters which may be considered as *elementary*. They are all very necessary to be understood and acquired by the builder, and are here compactly presented in a shape which, it is believed, will aid him in his studies, and at the same time prove to be a great convenience as a matter of reference.

The many geometrical forms which enter into the composition of a building suggest a knowledge of Elementary Geometry as essential to an intelligent comprehension of its plan and purpose. One of the prime requisites of a building is stability, a quality which depends upon a proper distribution of the material of which the building is constructed; hence a knowledge of the laws of pressure and the strength of materials is essential; and as these are based upon the laws of proportion and are expressed more concisely in algebraic language, a knowledge of Proportion and of Algebra are likewise indispensable to a comprehensive understanding of the subject. There will be found in this work, however, only so much of these parts of mathematics as have been deemed of the most obvious utility in the Science of Building. For a more exhaustive treatment of the subjects named, the reader is referred to the many able works, readily accessible, which make these subjects their specialties.

325.—Elementary Geometry.—In all reasoning definitions are necessary, in order to insure in the minds of the

proponent and respondent identity of ideas. A *corollary* is an inference deduced from a previous course of reasoning. An *axiom* is a proposition evident at first sight. In the following demonstrations there are many axioms taken for granted (such as, things equal to the same thing are equal to one another, etc.); these it was thought not necessary to introduce in form.

326.—Definition.—If a straight line, as $A\ B$ (*Fig.* 249), stand upon another straight line, as $C\ D$, so that the two

FIG. 249.

angles made at the point B are equal—$A\ B\ C$ to $A\ B\ D$ (*Art.* 499, *obtuse angle*)—then each of the two angles is called a *right angle*.

327.—Definition.—The circumference* of every circle is supposed to be divided into 360 equal parts, called *degrees;* hence a semicircle contains 180 degrees, a quadrant 90, etc.

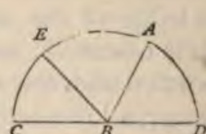


FIG. 250.

328.—Definition.—The *measure of an angle* is the number of degrees contained between its two sides, using the angular point as a centre upon which to describe the arc. Thus the arc $C\ E$ (*Fig.* 250) is the measure of the angle $C\ B\ E$, $E\ A$ of the angle $E\ B\ A$, and $A\ D$ of the angle $A\ B\ D$.

329.—Corollary.—As the two angles at B (*Fig.* 249) are right angles, and as the semicircle, $C\ A\ D$, contains 180 degrees (*Art.* 327), the measure of two right angles, therefore, is

180 degrees; of one right angle, 90 degrees; of half a right angle, 45; of one third of a right angle, 30, etc.

330.—Definition.—In measuring an angle (*Art.* 328), no regard is to be had to the length of its sides, but only to the degree of their inclination. Hence *equal angles* are such as have the same degree of inclination, without regard to the length of their sides.

331.—Axiom.—If two straight lines parallel to one another, as *A B* and *C D* (*Fig.* 251), stand upon another straight line, as *E F*, the angles *A B F* and *C D F* are equal, and the angle *A B E* is equal to the angle *C D E*.

332.—Definition.—If a straight line, as *A B* (*Fig.* 250), stand obliquely upon another straight line, as *C D*, then one

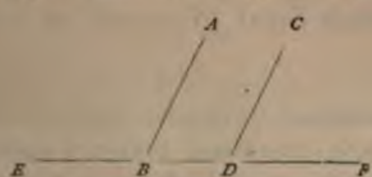


FIG. 251.

of the angles, as *A B C*, is called *an obtuse angle*, and the other, as *A B D*, *an acute angle*.

333.—Axiom.—The two angles *A B D* and *A B C* (*Fig.* 250) are together equal to two right angles (*Arts.* 326, 329); also, the three angles *A B D*, *E B A*, and *C B E* are together equal to two right angles.

334.—Corollary.—Hence all the angles that can be made upon one side of a line, meeting in a point in that line, are together equal to two right angles.

335.—Corollary.—Hence all the angles that can be made on both sides of a line, at a point in that line, or all the angles that can be made about a point, are together equal to four right angles.

336.—Proposition.—If to each of two equal angles a third angle be added, their sums will be equal. Let $A B C$ and $D E F$ (*Fig.* 252) be equal angles, and the angle $I J K$ the one to be added. Make the angles $G B A$ and $H E D$ each equal to the given angle $I J K$; then the angle $G B C$ will be equal to the angle $H E F$; for if $A B C$ and $D E F$ be angles

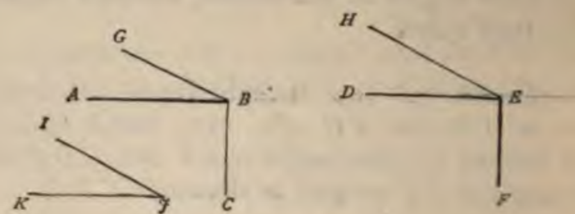


FIG. 252.

of 90 degrees, and $I J K$ 30, then the angles $G B C$ and $H E F$ will be each equal to 90 and 30 added, viz., 120 degrees.

337.—Proposition.—Triangles that have two of their sides and the angle contained between them respectively equal, have also their third sides and the two remaining

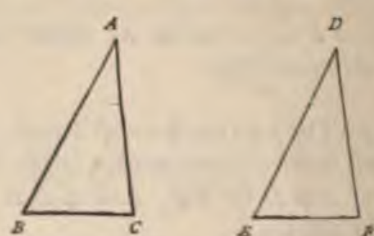


FIG. 253.

angles equal; and consequently one triangle will every way equal the other. Let $A B C$ (*Fig.* 253) and $D E F$ be two given triangles, having the angle at A equal to the angle at D, the side $A B$ equal to the side $D E$, and the side $A C$ equal to the side $D F$; then the third side of one, $B C$, is equal to the third side of the other, $E F$; the angle at B is equal to the angle at E, and the angle at C is equal to the angle at

P. For if one triangle be applied to the other, the three points B, A, C, coinciding with the three points E, D, F, the line $B C$ must coincide with the line $E F$; the angle at B with the angle at E; the angle at C with the angle at F; and the triangle $B A C$ be every way equal to the triangle $E D F$.

338.—Proposition.—The two angles at the base of an isosceles triangle are equal. Let $A B C$ (*Fig.* 254) be an

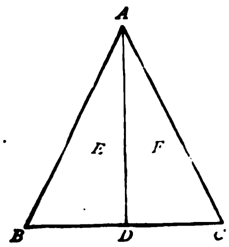


FIG. 254.

isosceles triangle, of which the sides, $A B$ and $A C$, are equal. Bisect the angle (*Art.* 506) $B A C$ by the line $A D$. Then, the line $B A$ being equal to the line $A C$, the line $A D$ of the triangle E being equal to the line $A D$ of the triangle F (being common to each), the angle $B A D$ being equal to the angle $D A C$,—the line $B D$ must, according to *Art.* 337, be

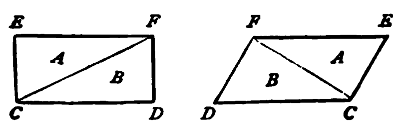


FIG. 255.

equal to the line $D C$, and the angle at B must be equal to the angle at C.

339.—Proposition.—A diagonal crossing a parallelogram divides it into two equal triangles. Let $C D E F$ (*Fig.* 255) be a given parallelogram, and $C F$ a line crossing it diagonally. Then, as $E C$ is equal to $F D$, and $E F$ to $C D$, the angle at E to the angle at D, the triangle A must, according to *Art.* 337, be equal to the triangle B.

340.—Proposition.—Let $J\,K\,L\,M$ (*Fig.* 256) be a given parallelogram, and $K\,L$ a diagonal. At any distance between $J\,K$ and $L\,M$ draw $N\,P$ parallel to $J\,K$; through the point G, the intersection of the lines $K\,L$ and $N\,P$, draw $H\,I$ parallel to $K\,M$. In every parallelogram thus divided, the parallelogram A is equal to the parallelogram B. For, according to *Art.* 339, the triangle $J\,K\,L$ is equal to the triangle $K\,L\,M$, the triangle C to the triangle D, and E to F;

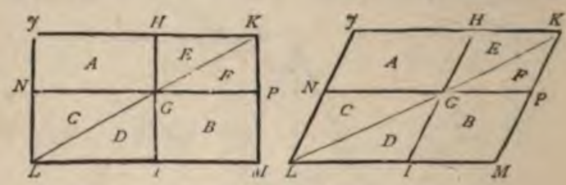


Fig. 256.

this being the case, take D and F from the triangle $K\,L\,M$, and C and E from the triangle $J\,K\,L$, and what remains in one must be equal to what remains in the other; therefore, the parallelogram A is equal to the parallelogram B.

341.—Proposition.—Parallelograms standing upon the same base and between the same parallels are equal. Let $A\,B\,C\,D$ and $E\,F\,C\,D$ (*Fig.* 257) be given parallelograms

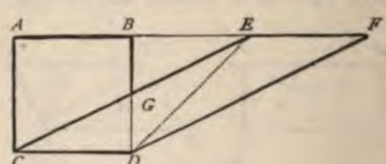


Fig. 257.

standing upon the same base, $C\,D$, and between the same parallels, $A\,F$ and $C\,D$. Then $A\,B$ and $E\,F$, being equal to $C\,D$, are equal to one another; $B\,E$ being added to both $A\,B$ and $E\,F$, $A\,E$ equals $B\,F$; the line $A\,C$ being equal to $B\,D$, and $A\,E$ to $B\,F$, and the angle $C\,A\,E$ being equal (*Art.* 331) to the angle $D\,B\,F$, the triangle $A\,E\,C$ must be equal (*Art.* 337) to the triangle $B\,F\,D$; these two triangles being equal, take the same amount, the triangle $B\,E\,G$, from each,

and what remains in one, *A B G C*, must be equal to what remains in the other, *E F D G*; these two quadrangles being equal, add the same amount, the triangle *C G D*, to each, and they must still be equal; therefore, the parallelogram *A B C D* is equal to the parallelogram *E F C D*.

342.—Corollary.—Hence, if a parallelogram and triangle stand upon the same base and between the same parallels,

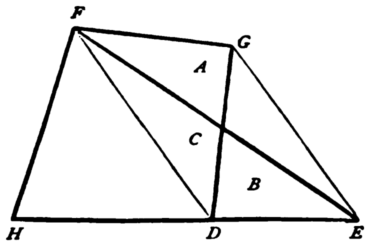


FIG. 258.

the parallelogram will be equal to double the triangle. Thus, the parallelogram *A D* (*Fig.* 257) is double (*Art.* 339) the triangle *C E D*.

343.—Proposition.—Let *F G H D* (*Fig.* 258) be a given quadrangle with the diagonal *F D*. From *G* draw *G E*

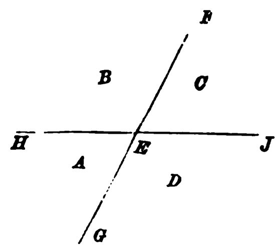


FIG. 259.

parallel to *F D*; extend *H D* to *E*; join *F* and *E*; then the triangle *F E H* will be equal in area to the quadrangle *F G H D*. For since the triangles *F D G* and *F D E* stand upon the same base, *F D*, and between the same parallels, *F D* and *G E*, they are therefore equal (*Arts.* 341, 342); and since the triangle *C* is common to both, the remaining tri-

angles, A and B, are therefore equal; then, B being equal to A, the triangle $F E H$ is equal to the quadrangle $F G H D$.

344.—Proposition.—If two straight lines cut each other, as $F G$ and $H J$ (*Fig.* 259), the vertical, or opposite angles, A and C, are equal. Thus, $F E$, standing upon $H J$, forms the angles B and C, which together amount (*Art.* 333) to two right angles; in the same manner, the angles A and B form two right angles; since the angles A and B are equal to B and C, take the same amount, the angle B, from each pair, and what remains of one pair is equal to what remains of the other; therefore, the angle A is equal to the angle C. The same can be proved of the opposite angles B and D.

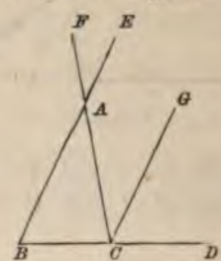


FIG. 260.

345.—Proposition.—The three angles of any triangle are equal to two right angles. Let $A B C$ (*Fig.* 260) be a given triangle, with its sides extended to F, E and D, and the line $C G$ drawn parallel to $B E$. As $G C$ is parallel to $E B$, the angle at H is equal (*Art.* 331) to the angle at L; as the lines $F C$ and $B E$ cut one another at A, the opposite angles at M and N are equal (*Art.* 334); as the angle at N is equal (*Art.* 331) to the angle at J, the angle at J is equal to the angle at M; therefore, the three angles meeting at C are equal to the three angles of the triangle $A B C$; and since the three angles at C are equal (*Art.* 333) to two right angles, the three angles of the triangle $A B C$ must likewise be equal to two right angles. Any triangle can be subjected to the same proof.

346.—Corollary.—Hence, if one angle of a triangle be a right angle, the other two angles amount to just one right angle.

347.—Corollary.—If one angle of a triangle be a right angle and the two remaining angles are equal to one another, these are each equal to half a right angle.

348.—Corollary.—If any two angles of a triangle amount to a right angle, the remaining one is a right angle.

349.—Corollary.—If any two angles of a triangle are together equal to the remaining angle, that remaining angle is a right angle.

350.—Corollary.—If any two angles of a triangle are each equal to two thirds of a right angle, the remaining angle is also equal to two thirds of a right angle.

351.—Corollary.—Hence, the angles of an equilateral triangle are each equal to two thirds of a right angle.

FIG. 261.

352.—Proposition.—If from the extremities of the diameter of a semicircle two straight lines be drawn to any point in the circumference, the angle formed by them at that point will be a right angle. Let *A B C* (*Fig.* 261) be a given semicircle; and *A B* and *B C* lines drawn from the extremities of the diameter *A C* to the given point *B*; the angle formed at that point by these lines is a right angle. Join the point *B* and the centre *D*; the lines *D A*, *D B*, and *D C*, being radii of the same circle, are equal; the angle at *A* is, therefore, equal (*Art.* 338) to the angle at *E*; also, the angle at *C* is, for the same reason, equal to the angle at *F*; the angle *A B C*, being equal to the angles at *A* and *C* taken together, must, therefore (*Art.* 349), be a right angle.

353.—Proposition.—The square on the hypothenuse of a right-angled triangle is equal to the squares on the two re-

maining sides. Let *A B C* (*Fig.* 262) be a given right-angled triangle, having a square formed on each of its sides; then the square *B E* is equal to the squares *H C* and *G B* taken together. This can be proved by showing that the parallelogram *B L* is equal to the square *G B*; and that the parallelogram *C L* is equal to the square *H C*. The angle *C B D* is a right angle, and the angle *A B F* is a right angle; add to each of these the angle *A B C*; then the angle *F B C* will evidently be equal (*Art.* 336) to the angle *A B D*; the triangle *F B C* and the square *G B*, being both upon the same base, *F B*, and between the same parallels, *F B* and *G C*, the square *G B* is equal (*Art.* 342) to twice the triangle *F B C*; the triangle *A B D* and the parallelogram *B L*, being both upon the same

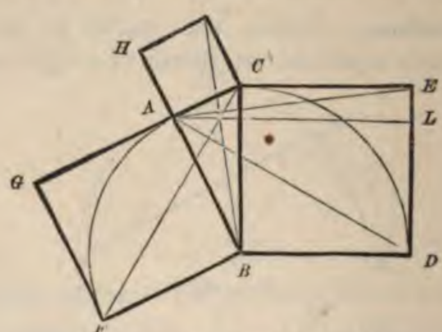


FIG. 262.

base, *B D*, and between the same parallels, *B D* and *A L*, the parallelogram *B L* is equal to twice the triangle *A B D*; the triangles, *F B C* and *A B D*, being equal to one another (*Art.* 337), the square *G B* is equal to the parallelogram *B L*, either being equal to twice the triangle *F B C* or *A B D*. The method of proving *H C* equal to *C L* is exactly similar—thus proving the square *B E* equal to the squares *H C* and *G B*, taken together.

This problem, which is the 47th of the First Book of Euclid, is said to have been demonstrated first by Pythagoras. It is stated (but the story is of doubtful authority) that as a thank-offering for its discovery he sacrificed a hundred oxen to the gods. From this circumstance it is sometimes called the *hecatomb* problem. It is of great value in

the exact sciences, more especially in Mensuration and Astronomy, in which many otherwise intricate calculations are by it made easy of solution.

354.—Proposition.—In an equilateral octagon the semi-diagonal of a circumscribed square, having its sides coincident with four of the sides of the octagon, equals the distance along a side of the square from its corner to the more remote angle of the octagon occurring on that side of the square. Let *Fig.* 263 represent the square referred to; in which O is the centre of each; then $A\,O$ equals $A\,D$. To prove this, it need only be shown that the triangle $A\,O\,D$ is an isosceles triangle having its sides $A\,O$ and $A\,D$ equal. The

FIG. 263.

octagon being equilateral, it is also equiangular, therefore the angles $B\,C\,O$, $E\,C\,O$, $A\,D\,O$, etc., are all equal. Of the right-angled triangle $F\,E\,C$, $F\,C$ and $F\,E$ being equal, the two angles $F\,E\,C$ and $F\,C\,E$, are equal (*Art.* 338), and are therefore (*Art.* 347) each equal to half a right angle. In like manner it may be shown that $F\,A\,B$ and $F\,B\,A$ are also each equal to half a right angle. And since $F\,E\,C$ and $F\,A\,B$ are equal angles, therefore the lines $E\,C$ and $A\,B$ are parallel (*Art.* 331,) and hence the angles $E\,C\,O$ and $A\,O\,D$ are equal. These being equal, and the angles $E\,C\,O$ and $A\,D\,O$ being equal by construction, as before shown, therefore the angles $A\,O\,D$ and $A\,D\,O$ are equal, and consequently the lines $A\,O$ and $A\,D$ are equal. (*Art.* 338.)

355.—Proposition.—An angle at the circumference of a circle is measured by half the arc that subtends it; that is, the angle $A\,B\,C$ (*Fig.* 264) is equal to half the angle $A\,D\,C$. Through the centre D draw the diameter $B\,E$. The triangle $A\,B\,D$ is an isosceles triangle, $A\,D$ and $B\,D$ being radii, and therefore equal; hence, the two angles at F and G are equal (*Art.* 338), and the sum of these two angles is equal to the angle at H (*Art.* 345), and therefore one of them, G, is equal to the half of H. The angles at H and at G (or $A\,B\,E$) are both subtended by the arc $A\,E$. Now, since the angle

FIG. 264.

at H is measured by the arc $A\,E$, which subtends it, therefore the half of the angle at H would be measured by the half of the arc $A\,E$; and since G is equal to the half of H, therefore G or $A\,B\,E$ is measured by the half of the arc $A\,E$. It may be shown in like manner that the angle $E\,B\,C$ is measured by half the arc $E\,C$, and hence it follows that the angle $A\,B\,C$ is measured by half the arc, $A\,C$, that subtends it.

356.—Proposition.—In a circle all the inscribed angles, A, B, and C (*Fig.* 265), which stand upon the same side of the

chord DE are equal. For each angle is measured by half the arc DFE (*Art.* 355). Hence the angles are all equal.

357.—Corollary.—Equal chords, in the same circle, subtend equal angles.

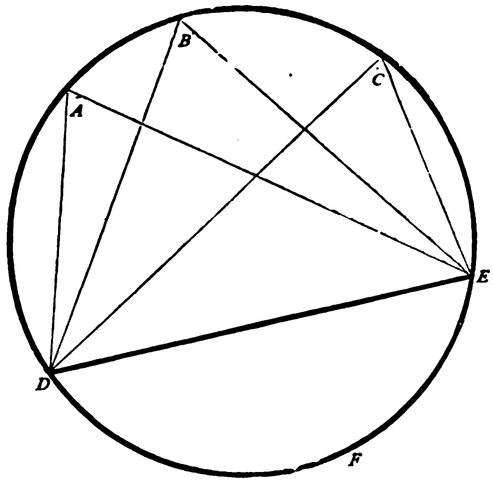


Fig. 265.

358.—Proposition.—The angle formed by a chord and angent is equal to any inscribed angle in the opposite seg-

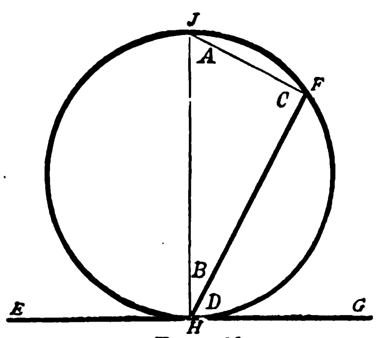


Fig. 266.

ıent of the circle; that is, the angle D (*Fig.* 266) equals the ngle A. Let HF be the chord, and EG the tangent; draw ıe diameter JH; then JHG is a right angle, also JFH is

a right angle. (*Art.* 352.) The angles A and B together equal a right angle (*Art.* 346); also the angles B and D together equal a right angle (equal to the angle JHG); therefore, the sum of A and B equals the sum of B and D. From each of these two equals, taking the like quantity B, the remainders A and D are equal. Thus, it is proved for the angle at A; it is also true for any other angle; for, since all other inscribed angles on that side of the chord line HF equal the angle A (*Art.* 356), therefore the angle formed by a chord and tangent equals any angle in the opposite segment of the circle. This being proved for the acute angle D, it is also true for the obtuse angle EHF; for, from any point, K (*Fig.* 267) in the arc HKF, draw lines to J, F and H; now, if it can

FIG. 267.

be proved that the angle EHF equals the angle FKH, the entire proposition is proved, for the angle FKH equals any of all the inscribed angles that can be drawn on that side of the chord. (*Art.* 356.) To prove, then, that EHF equals HKF: the angle EHF equals the sum of the angles A and B; also the angle HKF equals the sum of the angles C and D. The angles B and D, being inscribed angles on the same chord, JF, are equal. The angles C and A, being right angles (*Art.* 352), are likewise equal. Now, since A equals C and B equals D, therefore the sum of A and B equals the sum of C and D—or the angle EHF equals the angle HKF.

359. — Proposition. — The areas of parallelograms of equal altitude are to each other as the bases of the parallelo-

grams. In *Fig.* 268 the areas of the rectangles $A\,B\,C\,D$ and $B\,E\,D\,F$ are to each other as the bases $C\,D$ and $D\,F$. For, putting the two bases in form of a fraction and reducing this fraction to its lowest terms, then the numerator and denominator of the reduced fraction will be the numbers of equal parts into which the two bases respectively may be divided. For example, let the two given bases be 12 and 9 feet respectively, then $\frac{12}{9} = \frac{4}{3}$, and this gives four parts for the larger base and three parts for the smaller one. So, in *Fig.* 268, divide the base $C\,D$ into four equal parts, and the base $D\,F$ into three equal parts; then the length of any one of the parts in $C\,D$ will equal the length of any one of the parts in $D\,F$. Now, parallel with $A\,C$, draw lines from each point of division to the line $A\,E$. These lines will evidently divide the whole figure into seven equal parts, four of them occupy-

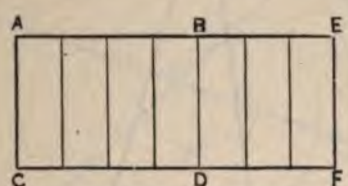


FIG. 268.

ng the area $A\,B\,C\,D$, and three of them occupying the area $B\,E\,D\,F$. Now it is evident that the areas of the two rectangles are in proportion as the number of parts respectively nto which the base-lines are divided, or that—

$$A\,B\,C\,D : B\,E\,D\,F :: C\,D : D\,F.$$

The areas in this particular case are as 4 to 3. But in general the proportion will be as the lengths of the bases. Thus the proposition is proved in regard to rectangles, but t has been shown (*Art.* 341) that all parallelograms of equal base and altitude are equal. Therefore the proposition is proved in regard to parallelograms generally, including rectangles.

360.—Proposition.—Triangles of equal altitude are to each other as their bases. It has been shown (*Art.* 359) that

parallelograms of equal altitude are in proportion as their bases, and it has also been shown (*Art.* 342) that of a triangle and parallelogram, when of equal base and altitude, the parallelogram is equal to double the triangle. Therefore triangles of equal altitude are to each other as their bases.

361.—Proposition.—Homologous triangles have their corresponding sides in proportion. Let the line CD (*Fig.* 269) be drawn parallel with AB. Then the angles ECD and EAB are equal (*Art.* 331), also the angles EDC and EBA are equal. Therefore the triangles ECD and EAB are homologous, or have their corresponding angles equal.

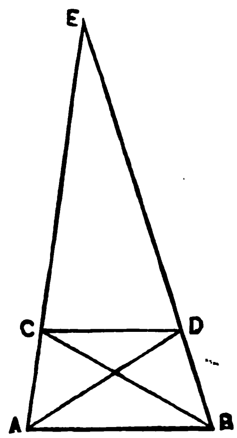


FIG. 269.

For, join C to B, and A to D, then the triangles ACD and BCD, standing on the same base, CD, and between the same parallels, CD and AB, are equal in area. To each of these equals join the common area CDE, and the sums ADE and BCE will be equal. The triangles CDE and ADE, having the same altitude, are to each other as their bases CE and AE (*Art.* 360), or—

$$CDE : ADE :: CE : AE.$$

Also the triangles CDE and BCE, having the same altitude, are to each other as their bases DE and BE, or—

$$CDE : BCE :: DE : BE.$$

And, since the triangles ADE and BCE are equal, as before shown, therefore, substituting in the last proportion ADE for BCE, we have—

$$CDE : ADE :: DE : BE.$$

The first two factors here being identical with the first two in the first proportion above, we have, comparing the two proportions—

$$CE : AE :: DE : BE;$$

or, we have the corresponding sides of one triangle, CDE, in proportion to the corresponding sides of the other, ABE.

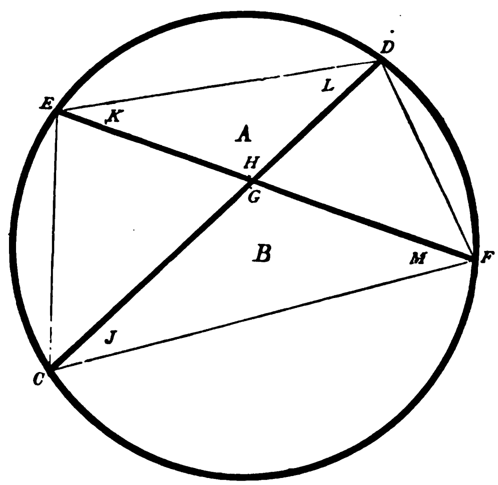


FIG. 270.

362.—Proposition.—Two chords, EF and CD (*Fig.* 270), tersecting, the parallelogram or rectangle formed by the vo parts of one is equal to the rectangle formed by the vo parts of the other. That is, the product of CG multi-ied by GD is equal to the product of EG multiplied by F. The triangle A is similar to the triangle B, because it is corresponding angles. The angle H equals the angle G *lrt.* 344); the angle at J equals the angle at K, because ey stand upon the same chord, DF (*Art.* 356); for the same

reason the angle M equals the angle L, for each stands upon the same chord, $E C$. Therefore, the triangle A having the same angles as the triangle B, the length of the sides of one are in like proportion as the length of the sides in the other (*Art.* 361). So—

$$C G : E G :: G F : G D.$$

Hence, as the product of the means equals the product of the extremes (*Art.* 373), $E G$ multiplied by $G F$ is equal to $C G$ multiplied by $G D$.

363.—Proposition.—If the sides of a quadrangle are bisected, and lines drawn joining the points of bisection in the adjacent sides, these lines will form a parallelogram.

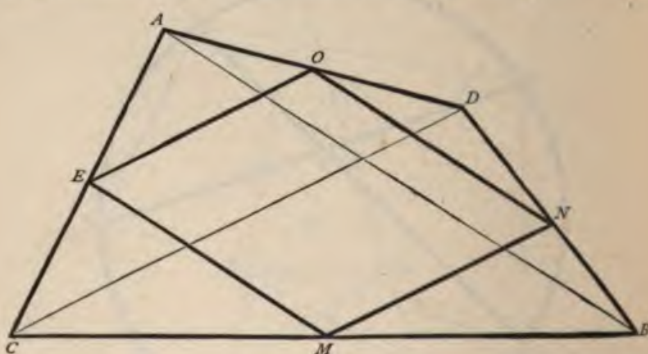


FIG. 271.

Draw the diagonals $A B$ and $C D$ (*Fig.* 271). It will here be perceived that the two triangles $A E O$ and $A C D$ are homologous, having like angles and proportionate sides. Two of the sides of one triangle lie coincident with the two corresponding sides of the other triangle, therefore the contained angles between these sides in each triangle are identical. By construction, these corresponding sides are proportionate; $A C$ being equal to twice $A E$, and $A D$ being equal to twice $A O$; therefore the remaining sides are proportionate, $C D$ being equal to twice $E O$, hence the remaining corresponding angles are equal. Since, then, the angles $A E O$ and $A C D$ are equal, therefore the line $E O$ is parallel with

liagonal *C D*—so, likewise, the line *M N* is parallel to the diagonal, *C D*. If, therefore, these two lines, *E O* and are parallel to the same line, *C D*, they must be parallel ch other. In the same manner the lines *O N* and *E M* proved parallel to the diagonal *A B*, and to each other; fore the inscribed figure *M E O N* is a parallelogram. ay be remarked, also, that the parallelogram so formed contain just one half the area of the circumscribing rangle.

SECTION VII.—RATIO, OR PROPORTION.

364.—Merchandise.—A carpenter buys 9 pounds of nails for 45 cents. He afterwards buys 87 pounds at the same rate. How much did he pay for them?

An answer to this question is readily found by multiplying the 87 pounds by 45 cents, the price of the 9 pounds, and dividing the product, 3915, by 9; the quotient, 435 cents, is the answer to the question.

365.—The "Rule of Three."—The process by which this problem is solved is known as the Rule of Three, or Proportion.

In cases of this kind there are three quantities given to find a fourth. Previous to working the question it is usual to make a statement, placing the three given quantities in such order that the quantity which is of like kind with the answer shall occupy the second place; the quantity upon which this depends for its value is put in the first place, and the remaining quantity, which is of like kind with that in the first place, is assigned to the third place.

When thus arranged, the second and third quantities are multiplied together and the product is divided by the first quantity; the quotient, the answer to the question, is a fourth quantity. These four quantities are related to each other in this manner, namely: the first is in proportion to the second as the third is to the fourth; or, taking the quantities of the given example, and putting them in a formal statement with the customary marks between them, we have—

$$9 : 45 :: 87 : 435,$$

which is read: 9 is to 45 as 87 is to 435; or, 9 is in proportion to 45 as 87 is to 435; or, 9 bears the same relation to 45 as 87 does to 435.

366.—Couples: Antecedent, Consequent.—These four uantities are termed *Proportionals*, and may be divided into wo *couples;* the first and second quantities forming one ouple, and the third and fourth the other couple. Of each ouple the first quantity is termed the *antecedent*, and the ast the *consequent*. Thus 9 is an antecedent and 45 its conequent; so, also, 87 is an antecedent and 435 its consequent.

367.—Equal Couples: an Equation.—These four quantities may be put in form thus:

$$\frac{45}{9} = \frac{435}{87}$$

ach couple is here stated as a fraction: each has its antedent beneath its consequent, and the two couples are parated by a sign, two short parallel lines, signifying uality. This is an *equation*, and is read thus: 45 divided y 9 is equal to 435 divided by 87; or, as ordinary fractions: 5 ninths are equal to 435 eighty-sevenths.

368.—Equality of Ratios.—Each couple is also termed a *atio*, and the two the Equality of Ratios. Thus the ratio $\frac{5}{9}$ is equal to the ratio $\frac{435}{87}$. If the division indicated in ese two ratios be actually performed, the equality between e two will at once be apparent, for the quotient in each ase is 5. The resolution of each couple into its simplest rm by actual division is shown thus:

$$\frac{45}{9} = 5;$$

$$\frac{435}{87} = 5.$$

hese are read: 45 divided by 9 equals 5; and 435 divided y 87 equals 5.

369.—Equals Multiplied by Equals Give Equals.—If two qual quantities be each multiplied by a given quantity, the

two products will be equal. For example, the fractions $\frac{2}{4}$ and $\frac{3}{6}$ are each equal to $\frac{1}{2}$, and are therefore equal to each other. If these two equal quantities be each multiplied by any given number, say, for example, by 4, we shall have 4 times $\frac{2}{4}$ equals $\frac{8}{4}$, and 4 times $\frac{3}{6}$ equals $\frac{12}{6}$; these products, $\frac{8}{4}$ and $\frac{12}{6}$ are each equal to 2, and therefore equal to each other.

370.—Multiplying an Equation.—The quantity on each side of the sign = is called a *member* of the equation. If each member be multiplied by the same quantity, the equality of the two members is not thereby disturbed (*Art.* 369); therefore, if the two members of the equation $\frac{45}{9} = \frac{435}{87}$ (*Art.* 367) be each multiplied by 87, or be modified thus:

$$\frac{45 \times 87}{9} = \frac{435 \times 87}{87};$$

in which ×, the sign for multiplication, indicates that the quantities between which it is placed are to be multiplied together; this ddition to each member of the equation does not destroy the equality; the members are still equal, though considerably enlarged. The equality may be easily tested by performing the operations indicated in the equation. For example: for the first member, we have 45 times 87 equals 3915, and this divided by 9 equals 435. Again, for the second member we have 435 times 87 equals 37845, and this divided by 87 equals 435, the same result as that for the first member. Thus the multiplication has not interfered with the equality of the members.

371.—Multiplying and Dividing one Member of an Equation: Cancelling.—If a quantity be multiplied by a given number, and the product be divided by the same given number, the quotient will equal the original quantity. For example: if 8 be multiplied by 3, the product will be 24; then if this product be divided by 3, the quotient will be 8, the original quantity. Thus the value of a quantity is not

changed by multiplying it by a number, provided it be also divided by the same number.

From this, also, we learn that the value of a quantity which is required to be multiplied and divided by the same number will not be changed if the multiplication and division be both omitted; one cancels the other. Therefore the number 87, appearing in the second member of the equation in the last article both as a multiplier and a divisor, may be omitted without destroying the equality of the two members. The equation thus treated will be reduced to—

$$\frac{45 \times 87}{9} = 435.$$

This expression is read: the product of 45 times 87 divided by 9 equals 435. It will be observed that we have here the four terms of the problem in *Art.* 365, three of them in the first member, and the fourth, the answer to the problem, in the second member.

372.—Transferring a Factor.—Each of the four quantities in the aforesaid equation is termed a *factor*. Comparing the equation of the last article with that of *Art.* 43, it appears that the two are alike excepting that the factor 87 has been transferred from one member of the equation to the other, and that, whereas it was before a divisor, it has now become a multiplier. From this we learn that a factor may be transferred from one member of an equation to the other, provided that in the transfer its relative position to the horizontal line above or below it be also changed; that is, if, before the transfer, it be below the line, it must be put above the line in the other member; or, if above the line, it must be put below, in the other member. For example: in the equation of the last article let the factor 9 be removed to the second member of the equation. It stands as a divisor in the first member; therefore, by the rule, it must appear as a multiplier after the transfer; or—

$$45 \times 87 = 9 \times 435;$$

which is read, 45 times 87 equals 9 times 435. By actually performing the operations here indicated, we find that each member gives the same product, 3915; thus proving that the equality of the two members was not interfered with by the transfer.

373.—Equality of Products: Means and Extremes.—In *Art.* 366, the four factors are put in the usual form of four proportionals. A comparison of these with the four factors as they appear in the equation in the last article, shows that the first member contains the second and third of the four proportionals, and the second member contains the first and the fourth; or, the first contains what are termed the *means*, and the second, the *extremes*. From this we learn that in any set of four proportionals, the product of the means equals the product of the extremes. As for example, $\frac{3}{2} = 1\frac{1}{2}$; so, also, $\frac{6}{4} = 1\frac{1}{2}$, an equality of ratios: hence the four factors, 2, 3, 4, 6, are four proportionals, and may be put thus:

Extreme, mean, mean, extreme.

$$2 : 3 :: 4 : 6$$

and, as above stated, the product of the means $(3 \times 4 =)$ 12, equals the product of the extremes $(2 \times 6 =)$ 12.

374.—Homologous Triangles Proportionate.—The discussion of the subject of Ratios has thus far been confined to its relations with the mercantile problem of *Art.* 364. The rules of proportion or the equality of ratios apply equally to questions other than those of a mercantile character. They apply alike to all questions in which quantities of any kind are comparable. For example, in geometry, lines, surfaces, and solids bear a certain fixed relation to one another, and are, therefore, fit subjects for the rules of proportion. It is shown, in *Art.* 361, that the corresponding sides of homologous triangles are in proportion to one another. Hence, when, of two similar triangles, two corresponding sides and one other side are given, then by the equality of ratios the side corresponding to this other side

may be computed. For example: in two triangles, such as $E\,C\,D$ and $E\,A\,B$ (*Fig.* 269), having their corresponding angles equal, let the side $E\,C$, in the triangle $E\,C\,D$, equal 12 feet, and the corresponding side $E\,A$, in the triangle $E\,A\,B$, equal 16 feet, and the side $E\,D$, of triangle $E\,C\,D$, equal 14 feet. Now, having these three sides given, how can we find the fourth? Putting them in proportion, we have, as in *Art.* 361—

$$CE : AE :: DE : BE;$$

and, substituting for the known sides, their dimensions, we have—

$$12 : 16 :: 14 : BE;$$

and, by *Art.* 373—

$$12 \times BE = 16 \times 14.$$

Dividing each member by 12, gives—

$$BE = \frac{16 \times 14}{12}.$$

Performing the multiplication and division indicated, we have—

$$BE = \frac{224}{12} = 18\tfrac{2}{3}.$$

Thus we have the fourth side equal to $18\frac{2}{3}$ feet.

375.—The Steelyard.—An example of four proportionals may also be found in the relation existing between the arms of a lever and the weights suspended at their ends. A familiar example of a lever is seen in the common *steelyard* used by merchants in weighing goods. This is a bar, $A\,B$, of steel, arranged as in *Fig.* 272, with hooks and links, and a suspended platform to carry R, the article to be weighed; and with a weight P, suspended by a link at B, from the bar $A\,B$, along which the weight P is movable.

The entire load is sustained by links attached to the fulcrum, or point of suspension C. The apparatus is in equilibrium without R and P. In weighing any article, R, the

weight P is moved along the bar $B\,C$ until the weight just balances the load, or until the bar $A\,B$ will remain in a horizontal position. If the weight P be too far from the fulcrum C the end of the bar B will fall, but if it be too near it will rise.

376.—The Lever Exemplified by the Steelyard.—To exemplify the principle of the lever, let the bar $A\,B$ (*Fig.* 272) be balanced accurately with the scale platform, but without the weights R and P. Then, placing the article R upon the platform, move the weight P along the beam until there is an equilibrium. Suppose the distances $A\,C$ and $B\,C$ are found to be 2 and 40 inches respectively, and suppose

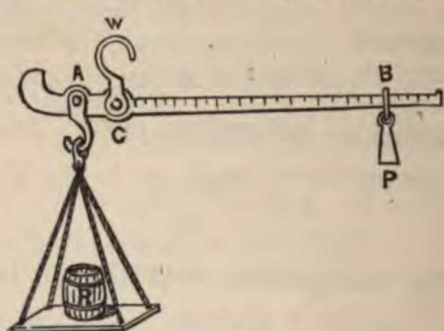


FIG. 272.

the weight P to equal 5 pounds, what at this point will be the weight of R? By trial we shall find that $R = 100$ pounds. Again, if a portion of R be removed, then the weight P would have to be moved along the bar $B\,C$ to produce an equilibrium; suppose it be moved until its distance from C be found to be 20 inches, then the weight of R would be found to be 50 pounds, or—

$$R = 50 \text{ pounds.}$$

Again, suppose a part of the weight taken from R be restored, and the weight P, on being moved to a point required for equilibrium, be found to measure 30 inches from C, then we shall find that—

$$R = 75 \text{ pounds.}$$

us when—

$$BC = 40,\ R = 100;\ \text{or},\ \frac{100}{40} = 2\cdot5;$$

$$BC = 30,\ R = 75;\ \text{or},\ \frac{75}{30} = 2\cdot5;$$

$$BC = 20,\ R = 50;\ \text{or},\ \frac{50}{20} = 2\cdot5;$$

showing an equality of ratios; or, in general, BC is in proportion to R, or—

$$BC : R.$$

If, instead of moving P along $B.C$, its position be permanent, and the weight P be reduced as needed to produce equilibrium with the various articles, R, which in turn may be put upon the scale; then we shall find that if when the weight P equals 5 pounds the article R equals 100, and there is an equilibrium, then when—

$$P = \frac{9}{10} \times 5 = 4\cdot5,\ R \text{ will equal } \frac{9}{10} \times 100 = 90;$$

$$P = \frac{8}{10} \times 5 = 4,\ R \text{ will equal } \frac{8}{10} \times 100 = 80;$$

$$P = \frac{7}{10} \times 5 = 3\cdot5,\ R \text{ will equal } \frac{7}{10} \times 100 = 70;$$

and so on for other proportions, and in every case we shall have the ratio $\frac{R}{P}$ equal 20, thus—

$$\frac{R}{P} = \frac{90}{4\cdot5} = 20\cdot$$

$$\frac{R}{P} = \frac{80}{4} = 20;$$

$$\frac{R}{P} = \frac{70}{3\cdot5} = 20.$$

Thus we have an equality of ratios in comparing the weights.

Again, if the weight P and the article R be permanent in weight, and the distances $A\,C$, $B\,C$ be made to vary, then if there be an equilibrium when $A\,C$ is 2 and $B\,C$ is 40, we shall find that when—

$$A\,C = \frac{8}{10} \times 2 = 1 \cdot 6;\ B\,C \text{ will equal } \frac{8}{10} \times 40 = 32,$$

$$A\,C = \frac{6}{10} \times 2 = 1 \cdot 2;\ B\,C \text{ will equal } \frac{6}{10} \times 40 = 24;$$

$$A\,C = \frac{4}{10} \times 2 = 0 \cdot 8;\ B\,C \text{ will equal } \frac{4}{10} \times 40 = 16;$$

and so on for other proportions, and in every case we shall have the ratio $\frac{B\,C}{A\,C} = 20$; thus—

$$\frac{B\,C}{A\,C} = \frac{32}{1 \cdot 6} = 20;$$

$$\frac{B\,C}{A\,C} = \frac{24}{1 \cdot 2} = 20;$$

$$\frac{B\,C}{A\,C} = \frac{16}{0 \cdot 8} = 20;$$

producing thus an equality of ratios in comparing the arms of the lever. From these experiments we have found, in comparing the article weighed with an arm of the lever, the constant ratio $B\,C : R$, and when comparing the weights we have found the constant ratio $P : R$. Again, in comparing the arms of the lever, we find the constant ratio $A\,C : B\,C$. Putting two of these couples in proportion, we have—

$$A\,C : B\,C :: P : R.$$

Hence (*Art.* 373) we have—

$$A\,C \times R = B\,C \times P.$$

Dividing both members by $A\ C$, we have—

$$R = \frac{B\ C \times P}{A\ C}.$$

In a steelyard the short arm, $A\ C$, and the weight, or poise, P, are unvarying; therefore we have—

$$R = B\ C \times \frac{P}{A\ C};$$

or, when $\frac{P}{A\ C}$ is constant, we have—

$$R : B\ C.$$

377.—The Lever Principle Demonstrated.—The relation between the weights and their arms of leverage may be demonstrated as follows: *

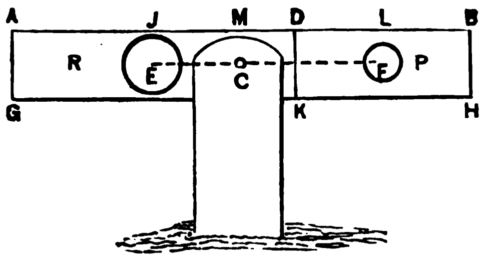


FIG. 273.

Let $A\,B\,G\,H$, *Fig.* 273, represent a beam of homogeneous material, of equal sectional area throughout, and suspended upon an axle or pin at C, its centre. This beam is evidently in a state of equilibrium. Of the part of the beam $A\,D\,G\,K$, let E be the centre of gravity; and of the remaining part, $D\,B\,K\,H$, let F be the centre of gravity.

If the weight of the material in $A\,D\,G\,K$ be concentrated at E, its centre of gravity, and the weight of the material in

* The principle upon which this demonstration is based may be found in an article written by the author and published in the *Mathematical Monthly*, Cambridge, U. S., for 1858, p. 77.

$DBKH$ be concentrated in F, its centre of gravity, the state of equilibrium will not be interfered with. Therefore let the ball R be equal in weight to the part $ADGK$, and the ball P equal to the weight of the part $DBKH$; and let these two balls be connected by the rod EF. Then these two balls and rod, supported at C, will evidently be in a state of equilibrium (the rod EF being supposed to be without weight).

Now, it is proposed to show that R is to P as CF is to CE. This can be proved; for, since R equals the area $ADGK$ and P equals the area $DBKH$, therefore R is in proportion to AD, as P is to DB (*Art.* 359); or, taking the halves of these lines, R is in proportion to AJ as P is to LB.

Also, JL equals half the length of the beam; for JD is the half of AD, and DL is the half of DB; thus these two parts $(JD+DL)$ equal the half of the two parts $(AD+DB)$: or, JL equals the half of AB; or, we have—

$$JD = \frac{AD}{2};\ DL = \frac{DB}{2}.$$

Adding these two equations together, we have—

$$JD + DL = \frac{AD}{2} + \frac{DB}{2} = \frac{AD+DB}{2}.$$

Now, $JD + DL = JL$, and $AD + DB = AB$; therefore, $JL = \frac{AB}{2}$.

Thus we have $AM = JL$. From each of these equals take JM, common to both, then the remainders, AJ and ML, will be equal; therefore, $AJ = CF$.

We have also $MB = JL$. From each of these equals take ML, common to both, and the remainders, JM and LB, will be equal; therefore, $LB = EC$. As was above shown—

$$R : AJ :: P : LB.$$

Substituting for $A J$ and $L B$ their values, as just found, we have—

$$R : CF :: P : EC;$$

from which we have (*Art.* 373)—

$$P \times CF = R \times EC.$$

Thus it is demonstrated that the product of one weight into its arm of leverage, is equal to the product of the other weight into its arm of leverage: a proposition which is known as the law of the lever.

378.—Any One of Four Proportionals may be Found.—Any three of four proportionals being given, the fourth may be found; for either one of the four factors may be made to stand alone; thus, taking the equation of the last article, if we divide both members by CF (*Art.* 371), we have—

$$\frac{P \times CF}{CF} = \frac{R \times EC}{CF}.$$

In the first member CF, in both numerator and denominator, cancel each other (*Art.* 371), therefore—

$$P = \frac{R \times EC}{CF};$$

so likewise we may obtain—

$$R = \frac{P \times CF}{EC}$$

$$CF = \frac{R \times EC}{P}$$

$$EC = \frac{P \times CF}{R}.$$

SECTION VIII.—FRACTIONS.

379.—A Fraction Defined.—As a fracture is a break or division into parts, so a fraction is literally a piece broken off; a part of the whole.

The figures which are generally used to express a fraction show what portion of the whole, or of an integer, the fraction is: for example, let the line $A\,B$, (*Fig.* 274), be divided into five equal parts, then the line $A\,C$, containing three of those parts, will be three fifths of the whole line $A\,B$, and may be expressed by the figures 3 and 5, placed thus, $\frac{3}{5}$, which is known as a fraction and is read, *three fifths*. The number 5 *below* the line denotes the number of parts into which an integer or unit, $A\,B$, is supposed to be divided: it

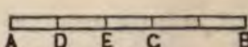


FIG. 274.

is therefore called the *denominator*, and expresses the denomination or kind, whether fifths, sixths, ninths, or any number, into which a unit is supposed to be divided. The number 3 above the line, denoting the number of parts contained in the fraction, is termed the *numerator*, and expresses the number of parts taken, as 2, 3, 4, or any other number.

380.—Graphical Representation of Fractions: Effect of Multiplication.—In *Fig.* 275, let the line $A\,B$ be divided into three equal parts; the line CD into six equal parts; the line $E\,F$ into nine equal parts; the line $G\,H$ into twelve equal parts, and the line $J\,K$ into fifteen equal parts. The lines $A\,B$, CD, $E\,F$, $G\,H$, and $J\,K$, being all of equal length.

Then the parts of these lines, AL, CM, EN, etc., may be expressed respectively by the fractions $\frac{1}{3}$, $\frac{2}{6}$, $\frac{3}{9}$, $\frac{4}{12}$ and $\frac{5}{15}$. In each case the figure below the line, as, 3, 6, 9, 12, or 15, expresses the number of parts into which the whole is divided, and the figure above the line, as 1, 2, 3, 4, or 5, the

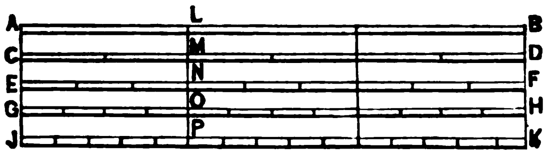


FIG. 275.

number of the parts taken; and, as the lines AL, CM, EN, etc., are all equal to each other, therefore these fractions are all equal to each other. If the numerator and denominator of the first fraction be each multiplied by 2, the products will equal the numerator and denominator of the second fraction; thus—

$$\frac{1 \times 2}{3 \times 2} = \frac{2}{6};$$

so, also,

$$\frac{1 \times 3}{3 \times 3} = \frac{3}{9};$$

and

$$\frac{1 \times 4}{3 \times 4} = \frac{4}{12};$$

and

$$\frac{1 \times 5}{3 \times 5} = \frac{5}{15}.$$

Thus it is shown that when the numerator and denominator of a fraction are each multiplied by the same factor, the product forms a new fraction which is of equal value with the original.

In like manner we have, $\frac{2}{8}$, $\frac{3}{12}$, $\frac{4}{16}$, $\frac{5}{20}$, etc., each equal to one fourth; and which may be found by multiplying the numerator and denominator of $\frac{1}{4}$ successively by 2, 3, 4, 5, etc.

381.—Form of Fraction Changed by Division.—By an operation the reverse of that in the last article, we may reduce several equal fractions to *one* of equal value. Thus, if in each we divide the numerator and denominator by the same number, we reduce it to a fraction of equal value, but with smaller factors.

For example, taking the fractions of the last article, $\frac{2}{6}$, $\frac{3}{9}$, $\frac{4}{12}$, $\frac{5}{15}$, let each be divided by a number which will divide both numerator and denominator without a remainder.*

Thus, $$\frac{2 \div 2}{6 \div 2} = \frac{1}{3}, \qquad \frac{3 \div 3}{9 \div 3} = \frac{1}{3}$$

$$\frac{4 \div 4}{12 \div 4} = \frac{1}{3}, \qquad \frac{5 \div 5}{15 \div 5} = \frac{1}{3}.$$

As these fractions are shown (*Art* 380) to be equal, and as the operation of dividing each factor by a common number produces quotients which in each case form the same fraction, $\frac{1}{3}$, we therefore conclude that the numerator and denominator of a fraction may be divided by a common number without changing the value of the fraction.

382.—Improper Fractions.—The fractions $\frac{9}{3}$, $\frac{17}{5}$, $\frac{24}{3}$, etc., all fractions which have the numerator larger than the denominator are termed *improper* fractions. They are not improper arithmetically, but they are so named because it is an improper use of language to call that a *part* which is greater than the whole.

As expressions of this kind, however, are subject to the same rules as those which are fractions proper, it is customary to include them all under the technical term of *fractions*. Expressions like these—all expressions in which one number is separated by a horizontal line from another number below it, or one set of numbers is thus separated from another set below it—may be called fractions, and are always to be understood as indicating division, or that the quantity above the line is to be divided by the quantity below the line.

* Division is indicated by this sign ÷, which is read "divided by."

Thus, $\frac{9}{3}$, $\frac{17}{5}$, $\frac{24}{3}$, $\frac{3 \times 8 \times 4}{2 \times 12}$, $\frac{17 \times 82}{125}$, etc., are all fractions, technically, although each may be greater than unity. And it is understood in each case that the operation of division is required. Thus, $\frac{9}{3} = 3$, $\frac{24}{3} = 8$, $\frac{3 \times 8 \times 4}{2 \times 12} = 4$. When the division cannot be made without a remainder, then the fraction, by cutting the numerator into two, may be separated into two parts, one of which may be exactly divided, and the other will be a fraction proper. Thus, the fraction $\frac{17}{5}$ is equal to $\frac{15}{5} + \frac{2}{5}$ (for $15 + 2 = 17$); and since $\frac{15}{5}$ equals 3, therefore, $\frac{17}{5} = \frac{15}{5} + \frac{2}{5} = 3 + \frac{2}{5} = 3\frac{2}{5}$. So, likewise, the fraction

$$\frac{17 \times 82}{125} = \frac{1394}{125} = \frac{1375}{125} + \frac{19}{125} = 11 + \frac{19}{125} = 11\frac{19}{125}.$$

383.—Reduction of Mixed Numbers to Fractions.—By an operation the reverse of that in the last article, a given mixed number (a whole number and fraction) can be put into the form of an improper fraction.

This is done by multiplying the whole number by the denominator of the fraction, the product being the numerator of a fraction equal in value to the whole number; the denominator of this fraction being the same as that of the given fraction. The numerator of this fraction being added to the numerator of the given fraction, the sum will be the numerator of the required improper fraction, the denominator of which is the same as that of the given fraction. For example, the required numerator for—

$2\frac{1}{3}$, is $2 \times 3 + 1 = 7$. So $2\frac{1}{3} = \frac{7}{3}$.
$2\frac{1}{4}$, is $2 \times 4 + 1 = 9$. So $2\frac{1}{4} = \frac{9}{4}$.
$3\frac{2}{5}$, is $3 \times 5 + 2 = 17$. So $3\frac{2}{5} = \frac{17}{5}$.

384.—Division Indicated by the Factors put as a Fraction.—Factors placed in the form of a fraction as $\frac{3}{5}$, $\frac{5}{3}$, $\frac{120}{75}$ or

$\frac{820}{41}$ indicate division (*Art.* 382); the denominator (the factor below the line) being the divisor, and the numerator (the factor above the line) the dividend, while the value of the fraction is the quotient. Thus of the fraction, $\frac{820}{41} = 20$. 41 is the divisor, 820 the dividend, and 20 the quotient. From this we learn that division may always be indicated by placing the factors in the form of a fraction, so that the divisor shall form the denominator and the dividend the numerator.

385.—Addition of Fractions having Like Denominators.—Let it be required to add the fractions $\frac{1}{5}$ and $\frac{2}{5}$. By referring to *Art.* 379 we see that $A\,D$ (*Fig.* 274), is one of the five parts into which the whole line $A\,B$ is divided; it is, therefore, $\frac{1}{5}$. We also see that $D\,C$ contains two of the five parts; it is, therefore, $\frac{2}{5}$. We also see that $A\,D + D\,C = A\,C$, which contains three of the five parts, or $A\,C = \frac{3}{5}$ of $A\,B$. We therefore conclude that $\frac{1}{5} + \frac{2}{5} = \frac{3}{5}$. In this operation it is seen that the denominator is not changed, and that the resultant fraction has for a numerator a number equal to the sum of the numerators of the fractions which were required to be added.

By this it is shown that to *add fractions* we simply *take the sum of the numerators for the new numerator*, making *the denominator of the resultant fraction the same as that of the fractions to be added.* For example: What is the sum of the fractions $\frac{1}{9}$, $\frac{3}{9}$ and $\frac{4}{9}$? Here we have $1 + 3 + 4 = 8$ for the numerator, therefore—

$$\frac{1}{9} + \frac{3}{9} + \frac{4}{9} = \frac{8}{9}.$$

386.—Subtraction of Fractions of Like Denominators.—ubtraction is the reverse of addition; therefore, to subract fractions a reverse operation is required to that had in 1e process of addition; or simply to subtract instead of dding.

For example, if $\frac{2}{5}$ be required to be subtracted from $\frac{3}{5}$ re have—

$$\frac{3}{5} - \frac{2}{5} = \frac{1}{5}$$

ly reference to *Fig.* 274 an exemplification of this will be een where we have $A\,C = \frac{3}{5}$, $A\,E = \frac{2}{5}$, and $E\,C = \frac{1}{5}$, and re have—

$$A\,C - A\,E = E\,C.$$

$$\frac{3}{5} - \frac{2}{5} = \frac{1}{5}.$$

We therefore have this rule for the substraction of fracions: *Subtract the less from the greater numerator; the remain'er will be the numerator of the required fraction. The denomnator to be the same as that of the given fractions.*

387.—Dissimilar Denominators Equalized.—The rules ust given for the addition and subtraction of fractions reuire that the given fractions have like denominators. Vhen the denominators are unlike it is required, before addng or substracting, that the fractions be modified so as to nake the denominators equal. For example: Let it be reuired to find the sum of $\frac{2}{3}$ and $\frac{2}{9}$. By reference to *Fig.* 75, we find that $\frac{2}{3}$ on line $A\,B$ is equal to $\frac{6}{9}$ on line $E\,F$. 'hese being equal, we may therefore substitute $\frac{6}{9}$ for $\frac{2}{3}$. 'hen we have—

$$\frac{6}{9} + \frac{2}{9} = \frac{8}{9}$$

Now, it will be seen that the fraction $\frac{6}{9}$ may be had by multiplying both numerator and denominator of the given fraction $\frac{2}{3}$ by 3, for $\frac{2 \times 3 = 6}{3 \times 3 = 9}$;
and we have seen (*Art.* 380) that this operation does not change the value of the fraction. From this we learn that *the denominators may be made equal by multiplying the smaller denominator and its numerator by any number which will effect such a result.*

For example: $\frac{1}{3} + \frac{7}{15} = \frac{5}{15} + \frac{7}{15} = \frac{12}{15}$;

and $\frac{2}{5} + \frac{7}{35} = \frac{14}{35} + \frac{7}{35} = \frac{21}{35}$;

and $\frac{3}{4} + \frac{3}{12} + \frac{7}{16} = \frac{12}{16} + \frac{4}{16} + \frac{7}{16} = \frac{23}{16} = 1\frac{7}{16}$.

In this example the second fraction is changed by multiplying by $1\frac{1}{3}$.

388.—Reduction of Fractions to their Lowest Terms.—The process resorted to in the last article to equalize the denominators, is not always successful. What is needed for a common denominator is to find the smallest number which shall be divisible by each of the given denominators. Before seeking this number, let each given fraction be reduced to its lowest terms, by dividing each factor by a common number. For example: $\frac{5}{15}$ may, by dividing by 5, be reduced to $\frac{1}{3}$, which is its equivalent. So, also, $\frac{21}{28}$, by dividing by 7, is reduced to $\frac{3}{4}$, its lowest terms.

389.—Least Common Denominator.—*To find the least common denominator*, place the several fractions in the order of their denominators, increasing toward the right. If the largest denominator be not divisible by each of the others, double it; if the division cannot now be performed, treble

t, and so proceed until it is multiplied by some number vhich will make it divisible by each of the other denomina- ors. *This number multiplied by the largest denominator will be he least common denominator.* To raise the denominator of ach fraction to this, *divide the common denominator by the de- ominator of one of the fractions,* the quotient will be the umber by which that fraction is to be multiplied, both umerator and denominator, and so proceed with each frac- ion. For example: What is the sum of the fractions $\frac{1}{2}, \frac{3}{4}, \frac{10}{12}, \frac{7}{8}$? One of these, $\frac{10}{12}$, may be reduced, by divid- ng by 2, to $\frac{5}{6}$. Therefore, the series is $\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}$. On trial ve find that 8, the largest denominator, is divisible by the irst and by the second, but not by the third, therefore the argest denominator is to be doubled: $2 \times 8 = 16$. This is ot yet divisible by the third; therefore $3 \times 8 = 24$. This ow is divisible by the third as well as by the first and the econd; 24 is therefore the least common denominator.

Now dividing 24 by 2, the first denominator, the quotient 12 is the factor by which the terms of the first fraction are to be raised, or, $\frac{1 \times 12 = 12}{2 \times 12 = 24}$. For the second we have $24 \div 4 = 6$, and $\frac{3 \times 6 = 18}{4 \times 6 = 24}$. For the third we have $24 \div 6 =$ 4, and $\frac{5 \times 4 = 20}{6 \times 4 = 24}$; and for the fourth, $24 \div 8 = 3$, and $\frac{7 \times 3 = 21}{8 \times 3 = 24}$. Thus the fractions in their reduced form are:

$$\frac{12}{24} + \frac{18}{24} + \frac{20}{24} + \frac{21}{24} = \frac{71}{24} = 2\frac{23}{24}.$$

390.—Least Common Denominator Again.—When the lenominators are not divisible by one another, then to ob- tain a common denominator, it is requisite to *multiply to- gether all of the denominators which will not divide any of the other denominators.* For example: What is the sum of the ractions $\frac{1}{3}, \frac{2}{5}, \frac{3}{7}$, and $\frac{4}{9}$?

In this case the first denominator will divide the last, but the others are prime to each other. Therefore, for the common denominator, multiply together all but the first; or—

$$5 \times 7 \times 9 = 315 \text{ the common denominator;}$$

and—

$315 \div 3 = 105$, common factor for the first fraction;
$315 \div 5 = 63$, common factor for the second fraction;
$315 \div 7 = 45$, common factor for the third;
$315 \div 9 = 35$, common factor for the fourth.

And, then—

$$\frac{1 \times 105}{3 \times 105} = \frac{105}{315}; \ \frac{2 \times 63}{5 \times 63} = \frac{126}{315}; \ \frac{3 \times 45}{7 \times 45} = \frac{135}{315}; \ \frac{4 \times 35}{9 \times 35} = \frac{140}{315};$$

$$\frac{105}{315} + \frac{126}{315} + \frac{135}{315} + \frac{140}{315} = \frac{506}{315} = 1\frac{191}{315}.$$

391.—Fractions Multiplied Graphically.—Let $A\,B\,C\,D$ (*Fig.* 276) be a rectangle of equal sides, or $A\,B$ equal $A\,C$ and each equal one foot. Then $A\,B$ multiplied by $A\,C$ will

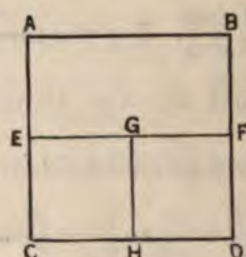


FIG. 276.

equal the area $A\,B\,C\,D$, or $1 \times 1 = 1$ square foot. Let the line $E\,F$ be parallel with $A\,B$, and midway between $A\,B$ and $C\,D$. Then $A\,B \times A\,E$ equals half the area of $A\,B\,C\,D$, or $1 \times \frac{1}{2} = \frac{1}{2}$. Again; let $G\,H$ be parallel with $E\,C$, and midway between $E\,C$ and $F\,D$. Then $E\,G \times E\,C = \frac{1}{2} \times \frac{1}{2}$ equals the area $E\,G\,C\,H$, which is equal to a quarter of the area

A B C D; or $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$; which is a quarter of the superficial area.

The product here obtained is less than either of the factors producing it. It must be remembered, however, that while the factors represent *lines*, the product represents superficial area. The correctness of the result may be recognized by an inspection of the diagram.

392.—Fractions Multiplied Graphically.—In *Fig.* 277 let *A B* equal 8 feet and *A C* equal 5 feet; then the rect-

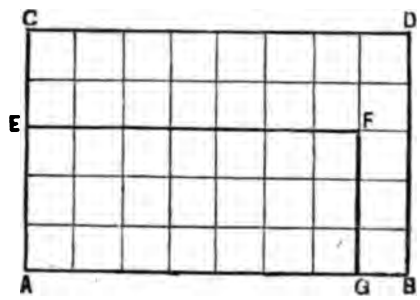


FIG. 277.

angle *A B C D* contains $5 \times 8 = 40$ feet. The interior lines divide the space included within *A B C D* into 40 equal squares of one foot each. Let *A E* equal 3 feet or $\frac{3}{5}$ of *A C*. Let *A G* equal 7 feet or $\frac{7}{8}$ of *A B*. Then the rectangle *E F A G* contains $\frac{3}{5} \times \frac{7}{8} = \frac{21}{40}$, or twenty-one fortieths of the whole area *A B C D*. Thus, while the factor fractions $\frac{3}{5}$ and $\frac{7}{8}$ represent *lines*, it is shown that the product fraction $\frac{21}{40}$ represents surface. Thus $\frac{21}{40}$ is a fraction, *E F A G*, of the whole surface, *C D A B*.

393.—Rule for Mutiplication of Fractions, and Example.—In the example given in the last article it will be ob-

served that the product of the denominators of the two given fractions equals the area of the whole figure ($A\ B\ C\ D$), while the product of the numerators equals the area of the rectangle ($E\ F\ A\ G$), the sides of which are equal respectively to the given fractions. From this we obtain for the product of fractions this—

RULE.—*Multiply together the denominators for the new denominator, and the numerators for a new numerator.*

For example: what is the product of $\frac{13}{21}$ and $\frac{7}{20}$? Here we have $20 \times 21 = 420$ for the new denominator, and $7 \times 13 = 91$ for the new numerator; therefore the product of—

$$\frac{13}{21} \times \frac{7}{20} = \frac{91}{420};$$

or, of a rectangular area divided one way into 20 parts and the other way into 21 parts, thus containing 420 rectangles, the product of the two fractions $\frac{13}{21}$ and $\frac{7}{20}$ is equal to 91 of these rectangles, or $\frac{91}{420}$ of the whole.

394.—Fractions Divided Graphically.—Division is the reverse of multiplication; or, while multiplication requires the product of two given factors, division requires one of the factors when the other and the product are given. Or (referring to *Fig.* 277) in division we have the area of the rectangle, $E\,F\,A\,G$, and one side, $E\,A$, given, to find the other side, $A\,G$.

Now it is required to find the number of times $E\,A$ is contained in $E\,F\,A\,G$. By inspection of the figure we perceive the answer to be, $A\,G$ times; for $E\,A \times A\,G = E\,F\,A\,G$, the given area. Or, when $E\,A\,F\,G$ is given as $\frac{21}{40}$ and $E\,A$ as $\frac{3}{5}$, we have as the given problem—

$$\frac{21}{40} \div \frac{3}{5}.$$

Since division is the reverse of multiplication, instead of multiplying we divide the factors, and have—

$$\frac{21 \div 3}{40 \div 5} = \frac{7}{8}.$$

Thus, to divide one fraction by another, for the numerator of the required factor, *divide the numerator of the product by the numerator of the given factor*, and for the denominator of the required factor *divide the denominator of the product by the denominator of the given factor.* For example:

Divide $\frac{10}{63}$ by $\frac{2}{9}$. Answer, $\frac{5}{7}$.

Divide $\frac{28}{27}$ by $\frac{4}{9}$. Answer, $\frac{7}{3}$.

395.—Rule for Division of Fractions.—The rule just given does not work well when the factors are not commensurable. For example, if it be required to divide $\frac{5}{7}$ by $\frac{2}{9}$ we have by the above rule—

$$\frac{5 \div 2}{7 \div 9} = \frac{\frac{5}{2}}{\frac{7}{9}}.$$

Producing fractional numerators and denominators for the resulting fraction, which require modification in order to reach those composed only of whole numbers. If the numerators, 5 and 7, of this compound fraction be multiplied by 9 (the denominator of the denominator fraction), or the compound fraction by 9, we shall have—

$$\frac{\frac{5}{2}}{\frac{7}{9}} \times 9 = \frac{\frac{5 \times 9}{2}}{\frac{7 \times 9}{9}}$$

And, if these be again multiplied by 2 (the denominator of the numerator fraction), we shall have—

$$\frac{\frac{5\times 9}{2}}{\frac{7\times 9}{9}}\times 2=\frac{\frac{5\times 9\times 2}{2}}{\frac{7\times 9\times 2}{9}}.$$

Like figures above and below in each fraction cancel each other (*Art.* 371), therefore, the result reduces to—

$$\frac{5\times 9}{7\times 2},$$

in which we find the factors of the two original fractions. In one fraction $\frac{5}{7}$ we have the factors in position as given, but in the other $\frac{2}{9}$ they are inverted. The fraction in which the factors are inverted is the divisor. Hence, for division of fractions, we have this—

RULE.—*Invert the factors of the divisor, and then, as in multiplication, multiply the numerators together for the numerator of the required fraction, and the denominators for the denominator of the required fraction.*

Thus, as before, if $\frac{5}{7}$ is required to be divided by $\frac{2}{9}$, we have—

$$\frac{5\times 9}{7\times 2}=\frac{45}{14}.$$

And, to divide $\frac{23}{47}$ by $\frac{7}{9}$, we have—

$$\frac{23\times 9}{47\times 7}=\frac{207}{329}.$$

Again, to divide $\frac{25}{45}$ by $\frac{8}{9}$, we have—

$$\frac{25\times 9}{45\times 8}=\frac{225}{360}=\frac{25}{40}=\frac{5}{8}.$$

This last example has two factors, 9 and 45, one of which measures the other; also, the first fraction $\frac{25}{45}$ is not in its lowest terms; when reduced it is $\frac{5}{9}$. The question, therefore, may be stated thus:

$$\frac{5 \times 9}{9 \times 8} = \frac{5}{8};$$

for the two 9's cancel each other.

SECTION IX.—ALGEBRA.

396.—Algebra Defined.—It occurs sometimes that a student familiar only with computation by numerals is needlessly puzzled, in approaching the subject of Algebra, to comprehend how it is possible to multiply *letters* together, or to divide them. To remove this difficulty, it may be sufficient for them to learn that their perplexity arises from a misunderstanding in supposing the *letters* themselves are ever multiplied or divided. It is true that in treatises on the subject it is usual to speak as though these operations were actually performed upon the letters. It is always understood, however, that it is not the letters, but the *quantities* represented by the letters, which are to be multiplied or divided.

For example, in *Art.* 361 it is shown, in comparing similar sides of homologous triangles, that the bases of the two triangles are to each other as the corresponding sides, or, referring to *Fig.* 269, we have $CE : AE :: DE : BE$. Now, let the two bases CE and AE be represented respectively by a and b, and the two corresponding sides DE and BE by c and d respectively; or, for—

$$CE : AE :: DE : BE,$$

put—

$$a : b :: c : d;$$

and, by *Art.* 373, we have—

$$b \times c = a \times d,$$

which may be written—

$$bc = ad;$$

for ×, the sign for multiplication, is not needed between letters, as it is between numeral factors. The operation of

multiplication is always understood when letters are placed side by side.

Now, here we have an equation in which, as usually read, we have the product of b and c equal to the product of a and d. But the meaning is that the product of the *quantities* represented by b and c is equal to the product of the *quantities* represented by a and d, and that this equation is intended to represent the relation subsisting between the four proportionals, CE, AE, DE, and BE, of *Fig.* 269. In order to secure greater conciseness and clearness, the four small letters are substituted for the four pair of capital letters, which are used to indicate the lines of the figures referred to.

397.—Example: Application.—It was shown in the last article that the four letters a, b, c, and d represent the corresponding sides of the two triangles of *Fig.* 269, and that—

$$bc = ad.$$

Now, let each member of this equation be divided by a, then (*Art.* 371)—

$$\frac{bc}{a} = d.$$

If now the dimensions of the three sides represented by a, b, and c are known, and it is required to ascertain from these the length of the side represented by d, let the three given dimensions be severally substituted for the letters representing them. For example, let $a = 40$ feet; $b = 52$ feet, and $c = 45$ feet; then—

$$d = \frac{bc}{a} = \frac{52 \times 45}{40} = \frac{2340}{40} = 58 \cdot 5 \text{ feet.}$$

The quantities being here substituted for the letters; we have but to perform the arithmetical processes indicated to obtain the arithmetical value of d. From this example it is seen that before any practical use can be made of an algebraical formula in computing dimensions, it is requisite to substitute numerals for the letters and actually perform arithmetically such operations as are only indicated by the letters.

398.—Algebra Useful in Constructing Rules.—In all problems to be solved there are certain conditions or quantities given, by means of which an unknown quantity is to be evolved. For example, in the problem in *Art.* 397, there were three certain lines given to find a fourth, based upon the condition that the four lines were four proportionals. Now, it has been found that the relation between quantities and the conditions of a question can better be stated by letters than by numerals; and it is the office of algebra to present by letters a concise statement of a question, and by certain processes of comparison, substitution and elimination, to condense the statement to its smallest compass, and at last to present it in a *formula* or rule, which exhibits the known quantities on one side as equal to the unknown on the other side. Here algebra ends, at the completion of the rule. To *use* the rule is the office of arithmetic. For, in using the rule, each quantity in numerals must be substituted for the letter representing it, and the arithmetical processes indicated performed, as was done in *Art.* 397.

399.—Algebraic Rules are General.—One advantage derived from algebra is that the rules made are general in their application, For example, the rule of *Art.* 397, $\frac{bc}{a} = d$, is applicable to all cases of homologous triangles, however they may differ in size or shape from those given in *Fig.* 269—and not only this, but it is also applicable in all cases where four quantities are in proportion so as to constitute four proportionals. For example, the case of the four proportionals constituting the arms of a lever and the weights attached (*Arts.* 375–378). For, taking the relation as expressed in *Art.* 377—

$$P \times CF = R \times EC,$$

we may substitute for CF the letter n, and for EC the letter m, then m will represent the arm of the lever EC (*Fig.* 262), and n the arm of the lever FC. Then we have—

$$Pn = Rm,$$

and from this, dividing by n (*Art.* 372), we have—

$$P = R\frac{m}{n}; \qquad (110.)$$

or, dividing by m, we have—

$$R = \frac{Pn}{m}; \qquad (111.)$$

which is a rule for computing the weight of R, when P and the two arms of leverage, m and n, are known. For example, let the weight represented by P be 1200 pounds, the length of the arm m be 4 feet, and that of n be 8 feet, then we have—

$$R = \frac{Pn}{m} = \frac{1200 \times 8}{4} = 2400 \text{ pounds.}$$

This rule, $R = \frac{Pn}{m}$, is precisely like that in *Art.* 397—$\frac{bc}{a} = d$—in which three quantities are given to find a fourth, the four constituting a set of four proportionals.

400.—Symbols Chosen at Pleasure.—The particular letter assigned to represent a particular quantity is a matter of no consequence. Any letter at will may be taken; but when taken, it must be firmly adhered to to represent that particular quantity, throughout all the modifications which may be requisite in condensing the statement into which it enters into a formula for use. For example, the two rules named in *Art.* 399 are precisely alike—three quantities given to find a fourth—yet they are represented by different letters. In one, R and P represent the two weights, and m and n the arms of leverage at which they act; while in the other the letters a, b, c, and d represent severally the four lines which constitute two similar sides of two homologous triangles. The two rules are alike in working, and they might have been constituted with the same letters. And instead of the letters chosen any others might have been taken, which convenience or mere caprice might have dictated. In some

questions it is usual to put the first letters, as a, b, c, etc., to represent known quantities, and the last letters, as x, y, z, for the quantities sought. In works on the strength of materials it is customary to represent weights by capital letters, as P, R, U, W, etc., and lines or linear dimensions by the small letters, as b, d, l, for the breadth, depth, and length, respectively, of a beam. Any other letters may be put to represent these quantities, although the initial letter of the word serves to assist the memory in recognizing the particular dimensions intended.

401.—Arithmetical Processes Indicated by Signs.—In algebra, the four processes of addition, subtraction, multiplication, and division, are frequently required; and when the required process cannot be actually performed upon the letters themselves, a certain method has been adopted by which the process is indicated. For example, in additon, when it is required to add a to b, the two letters cannot be intermingled as numerals may be, and their sum presented; but the process of addition is simply indicated by placing between the two letters this sign, +, which is called plus, meaning *added to*; therefore, to add a to b we have—

$$a + b,$$

which is read a plus b, or the sum of a and b. When the quantities represented by a and b are substituted for them—and not till then—they can be condensed into one sum. For example, let a equal 4 and b equal 3, then for—

$$a + b$$

we have—

$$4 + 3;$$

and we may at once write their sum 7, instead of $4 + 3$.

So, likewise, in the process of subtraction, one letter cannot be taken from another letter so as to show how much of this other letter there will be left as a remainder; but the process of subtraction can be indicated by a sign, as this, −, which is called minus, less, meaning *subtracted from*. For

le, let it be required to subtract b from a. To do
e have—

$$a - b;$$

is read a minus b, and when the values of a and b are uted for them, we have, when a equals 4, and b 3—

$$a - b,$$

$$4 - 3;$$

w, instead of $4 - 3$, we may put the value of the two, is unity, or 1.

e algebraic signs most frequently used are as follows:

s, signifies addition, and that the two quantities beween which it stands are to be added together; as $+ b$, read a added to b.

nus, signifies subtraction, or that of the two quantities etween which it occurs, the latter is to be subtracted rom the former; as $a - b$, read a minus b.

ltiplied by, or the sign of multiplication. It denotes hat the two quantities between which it occurs are to e multiplied together; as $a \times b$, read a multiplied by b, r a times b. This sign is usually omitted between ymbols or letters, and is then understood, as $a\,b$. This as the same meaning as $a \times b$. It is never omitted etween arithmetical numbers; as 9×5, read nine imes five.

ided by, or the sign of division, and denotes that of the wo quantities between which it occurs, the former is o be divided by the latter; as $a \div b$, read a divided by . Division is also represented thus:

the form of a fraction. This signifies that a is to be livided by b. When more than one symbol occurs bove or below the line, or both, as $\frac{a\,n\,r}{c\,m}$, it denotes hat the product of the symbols above the line is to be ivided by the product of those below the line.

$=$, *is equal to*, or sign of equality, and denotes that the quantity or quantities on its left are equal to those on its right; as $a - b = c$, read a minus b is equal to c, or equals c; or, $9 - 5 = 4$, read nine minus five equals four. This sign, together with the symbols on each side of it, when spoken of as a whole, is called an *equation*.

a^2 denotes a squared, or a multiplied by a, or the second power of a, and

a^3 denotes a cubed, or a multiplied by a and again multiplied by a, or the third power of a. The small figure, 2, 3, or 4, etc., is termed the index or exponent of the power. It indicates how many times the symbol is to be taken. Thus, $a^2 = a\,a$, $a^3 = a\,a\,a$, $a^4 = a\,a\,a\,a$.

$\sqrt{\ }$ is the *radical* sign, and denotes that the *square* root of the quantity following it is to be extracted; and

$\sqrt[3]{\ }$ denotes that the *cube* root of the quantity following it is to be extracted. Thus, $\sqrt{9} = 3$, and $\sqrt[3]{27} = 3$. The extraction of roots is also denoted by a fractional index or exponent, thus—

$a^{\frac{1}{2}}$ denotes the square root of a,

$a^{\frac{1}{3}}$ denotes the cube root of a,

$a^{\frac{2}{3}}$ denotes the cube root of the square of a, etc.

402.—Example in Addition and Subtraction: Cancelling.—Let there be some question which requires a statement to represent it, like this—

$$a + d = c - b,$$

which indicates that if the quantity represented by a be added to the quantity represented by d, the sum will be equal to the quantity represented by c, after there has been subtracted from it the quantity represented by b; or, as it is usually read, a plus d equals c minus b; or the sum of a and d equals the difference between c and b. For illustration, take in place of these four letters, in the order they stand, the numerals 4, 2, 9, 3, and we shall have by substitution—

$$a + d = c - b,$$

$$4 + 2 = 9 - 3,$$ or adding

and subtracting— $$6 = 6.$$

If it be required to add to each member of the equation the quantity represented by b, this will not interfere with the equality of the members. For $a+d$ are equal to $c-d$, and if to each of these two equals a common quantity be added, the sums must be equal; therefore—

$$a+d+b=c-b+b,$$

or by numerals—

$$4+2+3=9-3+3,$$

or—

$$9=9.$$

It will be observed that the right hand member contains the quantity $-b$ and $+b$. This shows that the quantity b is to be subtracted and then added. Now, if 3 be subtracted from 9, the remainder will be 6, and then if 3 be added, the sum will be 9, the original quantity. Thus it is seen that *when* in the *same member* of an equation *a symbol appears as a minus quantity and also as a plus quantity, the two cancel each other*, and may be omitted. Therefore, the expression—

$$a+d+b=c-b+b$$

becomes—

$$a+d+b=c.$$

403.—Transferring a Symbol to the Opposite Member.—In comparing, in the last article, the first equation with the last, it will be seen that the same symbols are contained in each, but differently arranged: that while in the first equation b appears in the right hand member and with a minus or *negative* sign, in the last equation it appears in the left hand member and with a plus or *positive* sign. Thus it is seen that in the operation performed b has been made to pass from one member to the other, but in its passage it has been changed. A similar change may be made with another of the symbols. For example, from the last equation, let d be subtracted, or this process indicated, thus—

$$a+d+b-d=c-d.$$

The plus and minus d, in the left hand member cancel each other, therefore—

$$a + b = c - d,$$

or, by numerals—

$$4 + 3 = 9 - 2.$$

Reducing—

$$7 = 7.$$

By this we learn that *any quantity* (connected by + or −) *may be passed from one member of the equation to the other, provided the sign be changed.*

404.—Signs of Symbols to be Changed when they are to be Subtracted.—As an example in subtraction, let the quantities represented by $+b - a - f + c$, be taken from the quantities represented by $+a + b - c - f$. This may be written—

$$(+a + b - c - f) - (+b - a - f + c),$$

an expression showing that the quantities enclosed within the second pair of parentheses are to be subtracted from those included within the first pair. Let the quantities represented in the first pair of parentheses for convenience be represented by A, or, $a + b - c - f = A$. Now, by the terms of the problem, we are required to subtract from A the quantities enclosed within the second pair of parentheses. To do this take first the positive quantity, b, and subtract it or indicate the subtraction, thus—

$$A - b;$$

we will then subtract the positive quantity c, or indicate the subtraction, thus—

$$A - b - c.$$

We have yet to subtract $-a$ and $-f$, two negative quantities.

The method by which this can be accomplished may be discovered by considering the requirements of the problem. The plus quantities b and c, before being subtracted from A, were required to have the two negative quantities a and f de-

ducted from them. It is evident, therefore, that in subtracting b and c, before this deduction was made, too much has been taken from A, and that the excess taken is equal to the sum of a and f. To correct the error, therefore, it is necessary to add just the amount of the excess, or to add the sum of a and f, or annex them by the plus sign, thus—

$$A - b - c + a + f.$$

To test the correctness of the operation as here performed, let numerals be substituted for the symbols; let $a = 2, b = 3, c = 1, f = \frac{1}{2}$; then the given quantities to be subtracted,—

$$(+b - a - f + c),$$

become—

$$(+3 - 2 - \tfrac{1}{2} + 1),$$

which reduces to—

$$(4 - 2\tfrac{1}{2}) = 1\tfrac{1}{2}.$$

Thus the quantity to be substracted equals $1\frac{1}{2}$. Applying the numerals to the above expression—

$$A - b + a + f - c$$

becomes—

$$A - 3 + 2 + \tfrac{1}{2} - 1 = A - 4 + 2\tfrac{1}{2} = A - 1\tfrac{1}{2}.$$

A correct result; it is the same as before. Restoring now the symbols represented by A, we have for the whole expression—

$$+a + b - c - f - b + a + f - c,$$

which, by cancelling (*Art.* 403) and by adding like symbols with like signs, reduces to—

$$2a - 2c.$$

To test this result, let the quantity which was represented by A have the proper numerals substituted, thus:

$$+a + b - c - f,$$

$$+2 + 3 - 1 - \tfrac{1}{2} = 5 - 1\tfrac{1}{2} = 3\tfrac{1}{2}.$$

The sum of the given quantity required to be subtracted was before found to amount to $1\frac{1}{2}$, therefore—

$$A - 1\frac{1}{2}$$

becomes—

$$3\frac{1}{2} - 1\frac{1}{2} = 2.$$

And the result by the symbols as above was—

$$2a - 2c,$$

which becomes—

$$2 \times 2 - 2 \times 1,$$

or—

$$4 - 2 = 2;$$

a result the same as before, proving the work correct. An examination of the signs in the above expression, which denotes the problem performed, will show that the sign of each symbol which was required to be subtracted has been changed in the operation of subtraction. Before subtracting they were—

$$(+b - a - f + c);$$

after subtraction they are—

$$(-b + a + f - c).$$

By this result we learn, that *to subtract a quantity* we have but to *change its sign* and annex it to the quantity from which it was required to be subtracted.

Example: Subtract $a - b$ from $c + d$. Answer, $c + d - a + b$.

If numerals be substituted, say $a = 7$, $b = 4$, $c = 5$, and $d = 9$, then—

$$\begin{array}{lcr} c + d & \text{becomes} & 5 + 9 = 14, \\ a - b & \text{``} & 7 - 4 = \underline{\ 3,} \end{array}$$

$$c + d - (a - b) = 14 - 3 = 11.$$

So, also,—

$$c + d - a + b$$

becomes—

$$5 + 9 - 7 + 4 = 11.$$

405.—Algebraic Fractions: Added and Subtracted.—When algebraic fractions of like denominators are to be added or subtracted, the same rules (*Arts.* 385 and 386) are to be observed as in the addition or subtraction of numerical fractions—namely, add or subtract the numerators for a new numerator, and place beneath the sum or difference the common denominator.

For example, what is the sum of $\frac{a}{b}, \frac{c}{b}, \frac{d}{b}$?

For this we have—

$$\frac{a+c+d}{b}.$$

Subtract $\frac{c}{d}$ from $\frac{b}{d}$. For this we have—

$$\frac{b-c}{d}.$$

What is the algebraical sum of—

$$\frac{b}{d}, \frac{c}{d}, -\frac{n}{d}, \text{ and } -\frac{r}{d}?$$

For these we have—

$$\frac{b+c-n-r}{d}.$$

To exemplify this, let b represent 9, $c = 8$, $n = 2$, $r = 3$, and $d = 12$.

Then, for the algebraic sum, we have—

$$\frac{9+8-2-3}{12} = \frac{12}{12} = 1.$$

Now, taking the positive and negative fractions separately, we have—

$$\frac{9}{12} + \frac{8}{12} = \frac{17}{12};$$

and—

$$\frac{-2}{12} \frac{-3}{12} = \frac{-5}{12}.$$

Together—

$$\frac{17}{12} \frac{-5}{12} = \frac{12}{12} = 1,$$

as before.

406.—The Least Common Denominator.—When the denominators of algebraic fractions differ it is necessary before addition or subtraction can be performed to *harmonize* them, as in the reduction of the denominators of *numerical* fractions (*Arts.* 388–390). For example, add together the fractions $\frac{a}{bc}$, $\frac{e}{b}$, $\frac{r}{ac}$. In these denominators we perceive that they collectively contain the letters a, b and c, and no others. It will be requisite, therefore, that each of the fractions be modified so that its denominator shall have these three factors. To effect this it will be seen that it is necessary to multiply each fraction by that one of these letters which is lacking in its denominator. Thus, in the first, a is lacking, therefore (*Art.* 380) $\frac{a \times a}{bc \times a} = \frac{aa}{abc}$. In the second a and c are lacking, therefore $\frac{e \times ac}{b \times ac} = \frac{ace}{abc}$, and in the third b is lacking, therefore $\frac{r \times b}{ac \times b} = \frac{rb}{abc}$. Placing them now together we have—

$$\frac{aa + ace + br}{abc.} = \frac{a}{bc} + \frac{e}{b} + \frac{r}{ac}.$$

The factor $a\,a$ may be represented thus a^2, which means that a occurs twice, the small figure at the top indicating the number of times the letter occurs; a^2 is called a squared, $a\,a\,a = a^3$, and is called a cubed.

In order to show that the above fraction, resulting as the sum of the three given fractions, is correct, let $a = 2$, $b = 3$, $c = 4$, $e = 5$, and $r = 6$. Then the three given fractions are—

$$\frac{2}{3 \times 4} + \frac{5}{3} + \frac{6}{2 \times 4} = \frac{1}{6} + \frac{5}{3} + \frac{3}{4}.$$

In equalizing these denominators we multiply the second fraction by 2, and the third by $1\frac{1}{2}$, which will give—

$$\frac{5 \times 2 =}{3 \times 2 =} \frac{10}{6}, \frac{3 \times 1\frac{1}{2}}{4 \times 1\frac{1}{2}} = \frac{4\frac{1}{2}}{6};$$

then—

$$\frac{1}{6} + \frac{10}{6} + \frac{4\frac{1}{2}}{6} = \frac{15\frac{1}{2}}{6} = 2\frac{3\frac{1}{2}}{6} = 2\frac{7}{12}.$$

Now the sum of the fractions is—

$$\frac{a^2 + ace + br}{abc};$$

or,

$$\frac{2^2 + \overline{2 \times 4 \times 5} + \overline{3 \times 6}}{2 \times 3 \times 4};$$

or,

$$\frac{4 + 40 + 18}{24} = \frac{62}{24} = 2\frac{14}{24} = 2\frac{7}{12};$$

the same result as before, thus showing that the reduction was rightly made.

407.—Algebraic Fractions Subtracted.—To exemplify the subtraction of fractions, let it be required to find the algebraic sum of $\frac{a}{c} - \frac{b}{d} - \frac{e}{f}$. These denominators all differ. The fractions, therefore, require to be modified, so that each denominator shall contain them all. To accomplish this, the first fraction will need to be thus treated:

$$\frac{a \times df =}{c \times df =} \frac{adf}{cdf};$$

the second—

$$-\frac{b \times cf =}{d \times cf =} - \frac{bcf}{cdf};$$

the third—

$$-\frac{e \times cd =}{f \times cd =} - \frac{cde}{cdf}.$$

The sum of these is—

$$\frac{adf - bcf - cde}{cdf}.$$

That this is a correct answer, let the result be proved by figures; thus, for a put 15; b, 2; c, 3; d, 4; e, 5; f, 6. Then we shall have—

$$\frac{a}{c} - \frac{b}{d} - \frac{e}{f} = \frac{15}{3} - \frac{2}{4} - \frac{5}{6}.$$

It will be observed that these denominators may be equalized by multiplying the first fraction by 2, and the second by $1\frac{1}{2}$, therefore we have—

$$\frac{30}{6} - \frac{3}{6} - \frac{5}{6}.$$

To make the required subtraction we are to deduct from 30 (the numerator of the positive fraction), first 3, then 5; or, the sum of the numerators of the negative fractions; or for the numerator of the new fraction we have $30 - 8 = 22$. The required result, therefore, is—

$$\frac{22}{6} = \frac{11}{3} = 3\frac{2}{3}.$$

To apply this test to the algebraic sum we have—

$$\frac{adf - bcf - cde}{cdf} = \frac{\overline{15 \times 4 \times 6} + \overline{2 \times 3 \times 6} + \overline{3 \times 4 \times 5}}{3 \times 4 \times 6},$$

which by multiplication reduces to—

$$\frac{360 - 36 - 60}{72} = \frac{264}{72} = \frac{22}{6} = \frac{11}{3} = 3\frac{2}{3},$$

a result the same as before, proving the work correct. Another example:

$$\text{From } \frac{a}{n} - \frac{b}{m} \text{ take } \frac{c}{n}, \frac{d}{m} \text{ and } \frac{e}{n};$$

or, find the algebraic sum of—

$$\frac{a}{n} - \frac{b}{m} - \frac{c}{n} - \frac{d}{m} - \frac{e}{n}.$$

The fractions which have the same denominator may be grouped together thus:

$$\frac{a}{n} - \frac{c}{n} - \frac{e}{n} = \frac{a - c - e}{n};$$

and—

$$\frac{b}{m} - \frac{d}{m} = \frac{b - d}{m}.$$

To harmonize these two denominators, m and n, the first fraction must be multiplied by m and the last by n, or—

$$\frac{m(a - c - e)}{mn} + \frac{n(b - d)}{mn} = \frac{m(a - c - e) + n(b - d)}{mn}.$$

In the polynomial factor within the parentheses $(a - c - e)$ we have the positive quantity a, from which is to be taken the two negatives c and e, or their sum is to be taken from a, or $(a - (c + e))$. With this modification we have for the algebraic sum of the five given fractions—

$$\frac{m(a - (c + e)) + n(b - d)}{mn}.$$

To test the accuracy of this result, let the value of the several letters respectively be as follows: $a = 11$, $b = 9$, $c = 3$, $d = 4$, $e = 5$, $m = 10$, and $n = 8$. Then the sum is—

$$\frac{10(11 - (3 + 5)) + 8(9 - 4)}{10 \times 8} = \frac{70}{80} = \frac{7}{8}.$$

Now, taking the fractions separately, we have—

$$\frac{a}{n} - \frac{c}{n} - \frac{e}{n} = \frac{11}{8} - \left(\frac{3}{8} + \frac{5}{8}\right) = \frac{11}{8} - \frac{8}{8} = \frac{3}{8};$$

again,— $$\frac{b}{m} - \frac{d}{m} = \frac{9}{10} - \frac{4}{10} = \frac{5}{10};$$

or, together we have, as the sum of these two results—

$$\frac{3}{8} + \frac{5}{10}.$$

To harmonize these denominators we may multiply the first fraction by 5, and the second by 4, thus:

$$\frac{3 \times 5}{8 \times 5} = \frac{15}{40}, \quad \frac{5 \times 4}{10 \times 4} = \frac{20}{40},$$

and then the sum is—

$$\frac{15}{40} + \frac{20}{40} = \frac{35}{40} = \frac{7}{8};$$

the same result as before, thus the accuracy of the work is established.

408.—Graphical Representation of Multiplication.—In *Fig.* 278, let $A\,B\,C\,D$, a rectangle, have its sides $A\,B$ and

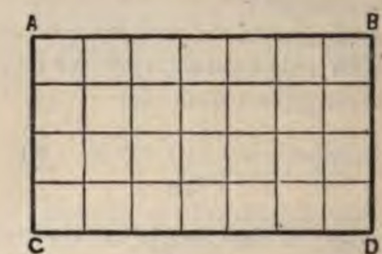


FIG. 278.

$A\,C$ divided into equal parts. Then the area of the figure will be obtained by multiplying one side by the other, or putting a for the side $A\,B$, and b for the side $A\,C$, then the area will be $a \times b$, or $a\,b$. This will be the correct area of the figure, whatever the length of the sides may be. If, as shown, the area be divided into $4 \times 7 = 28$ equal rectangles, then a would equal 7, and b equal 4, and $a\,b = 7 \times 4 = 28$, the area. If $A\,B$ equal 28 and $A\,C$ equal 16, then will $a = 28$, and $b = 16$, and $a\,b = 28 \times 16 = 448$, the area.

409.—Graphical Multiplication: Three Factors.—Let $A\,B\,C\,D\,E\,F\,G$ (*Fig.* 279) represent a rectangular solid which may be supposed divided into numerous small cubes as shown. Now, if a be put for the edge $A\,B$, b for the edge $A\,C$, and c for the edge $C\,D$, then the cubical solidity of the

figure will be represented by $a \times b \times c = a\,b\,c$. If the
$1\,B$ measures 6, the edge $A\,C$ 3, and the edge $C\,D$ 4,
$\imath\,b\,c = 6 \times 3 \times 4 = 72 =$ the cubic contents of the
or the number of small cubes contained in it.

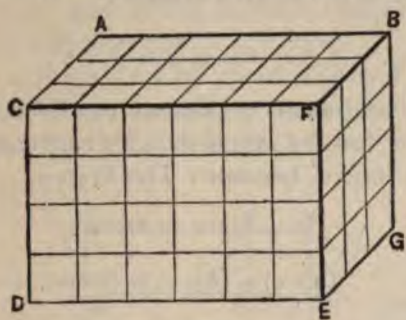


FIG. 279.

).—Graphic Representation: Two and Three Fac-
-*Figs.* 278 and 279 serve to illustrate the algebraic ex-
›ns $a\,b$ and $a\,b\,c$. In the former it is shown that the
lication of two lines produces a rectangular surface,
: if a and b represent lines, then $a\,b$ may represent a
gular surface (*Fig.* 278) having sides respectively
:o a and b. And so if a, b, and c represent three sev-
ıes, then $a\,b\,c$ may represent a rectangular solid (*Fig.*
ıving edges respectively equal to a, b, and c.

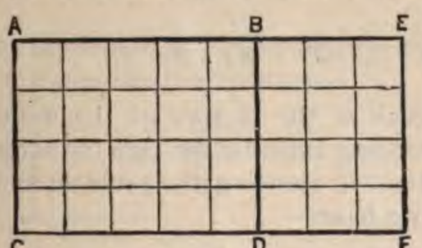


FIG. 280.

.—Graphical Multiplication of a Binomial.—Let
D (*Fig.* 280) be a rectangular surface, and $B\,E\,D\,F$ an-
rectangular surface, adjoining the first. The area of
ıole figure is evidently equal to—

$$(A\,B + B\,E) \times A\,C.$$

The area is also equal to—

$$\overline{AB} \times \overline{AC} + \overline{BE} \times \overline{BD};$$

or, since $AC = BD$, the area equals—

$$\overline{AB} \times \overline{AC} + \overline{BE} \times \overline{AC};$$

or, if symbols be put to represent the lines, say a for AB, b for BE, and c for AC, then the two representatives of the area, as above shown, become: The first—

$$(a + b) \times c = \text{area};$$

and the last—

$$(a \times c) + (b \times c) = \text{area}.$$

Hence we have—

$$(a + b)\,c = a\,c + b\,c.$$

This result exemplifies the algebraic multiplication of a binomial, which is performed thus: Let $a + b$ be multiplied by c.

The problem is stated thus:

$$(a + b)\,c.$$

To perform the multiplication indicated we proceed thus:

$$\begin{array}{c} a + b \\ c \\ \hline a\,c + b\,c \end{array}$$

multiplying each of the factors of the multiplicand separately and annexing them by the sign for addition. Putting the two together, or showing the problem and its answer in an equation, we have—

$$(a + b)\,c = a\,c + b\,c,$$

producing the same result, above shown, as derived from the graphic representation.

412.—Graphical Squaring of a Binomial.—Let $EGCJ$ (*Fig.* 281) be a rectangle of equal sides, and within it draw

› lines, $A\ H$ and $F\ D$, parallel with the lines of the le, and at such a distance from them that the sides, d $B\ D$, of the rectangle, $A\ B\ C\ D$, shall be of equal We then have in this figure the three squares, J, $A\ B\ C\ D$, and $F\ G\ B\ H$, also the two equal rect- $E\ F\ A\ B$ and $B\ H\ D\ J$.

$E\ F$ be represented by a and $F\ G$ by b, then the area $C\ D$ will be $a \times a = a^2$; the area of $F\ G\ B\ H$ will be b^2; the area of $E\ F\ A\ B$ will be $a \times b = a\ b$, and that

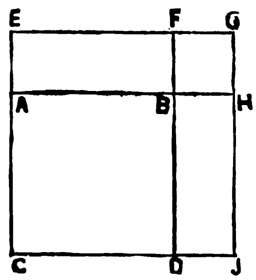


Fig. 281.

$D\ J$ will be the same. Putting these areas together

$$a^2 + 2\,a\,b + b^2,$$

equals the area of the whole figure—equals the prod- $E\ G \times E\ C$—equals the product—

$$(a + b) \times (a + b).$$

refore, we have—

$$(a + b)\ (a + b) = a^2 + 2\,a\,b + b^2; \qquad (112.)$$

;eneral, *the square of a binomial equals the square of* , *plus twice the first by the second, plus the square of the* This result is obtained graphically. The same re- y be obtained by algebraic multiplication, combining

each factor of the multiplier with each factor of the multiplicand and adding the products, thus—

$$\begin{array}{l} a+b \\ a+b \\ \hline a^2+ab \\ \quad\quad ab+b^2 \\ \hline a^2+2ab+b^2. \end{array}$$

The same result as above shown by graphical representation.

413.—Graphical Squaring of the Difference of Two Factors.—Let the line $E\,C$ (*Fig.* 281) be represented by c, and the line $A\,E$ and $A\,C$ as before respectively by b and a, then—

$$E\,C - A\,E = A\,C.$$

$$c - b = a.$$

From this, squaring both sides, we have—

$$(c-b)^2 = a^2.$$

The area of the square $A\,B\,C\,D$ may be obtained thus: From the square $E\,G\,C\,J$ take the rectangle $E\,G \times E\,A$ and the rectangle $F\,G \times D\,J$, minus the square $F\,G\,B\,H$, or from c^2 take the rectangle $c\,b$, and the rectangle $c\,b$, minus the square, b^2, and the remainder will be the square, a^2; or, in proper form—

$$c^2 - c\,b - c\,b + b^2 = a^2$$

In deducting from c^2 the rectangle $c\,b$ twice, we have taken away the small square twice; therefore, to correct this error, we have to add the small square, or b^2. Then, when reduced, the expression becomes—

$$c^2 - 2\,c\,b + b^2 = a^2 = (c-b)^2.$$

This result is obtained graphically. The result by algebraic

process will now be sought. The square of a quantity may be obtained by multiplying the quantity by itself, or—

$$\begin{array}{r} c - b \\ c - b \\ \hline c^2 - bc \\ - bc + b^2 \\ \hline (c - b)^2 = c^2 - 2bc + b^2. \end{array} \qquad (113.)$$

In this process, as before, each factor of the multiplier is combined with each factor of the multiplicand and the several products annexed with their proper signs (*Art.* 415), and thus, by algebraic process, a result is obtained precisely like that obtained graphically. This result is the square of the difference of c and b; and since c and b may represent any quantities whatever, we have this general—

RULE.—*The square of the difference of two quantities is equal to the sum of the squares of the two quantities, minus twice their product.*

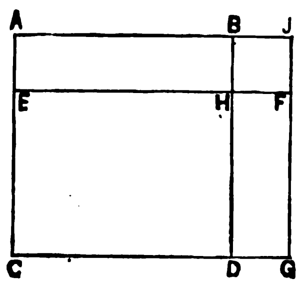


FIG. 282.

414.—Graphical Product of the Sum and Difference of Two Quantities.—Let the rectangle $A\,B\,C\,D$ (*Fig.* 282) have its sides each equal to a. Let the line $E\,F$ be parallel with $A\,B$ and at the distance b from it, also, the line $F\,G$ made parallel with $B\,D$, and at the distance b from it. Then the line $E\,F$ equals $a + b$, and the line $E\,C$ equals $a - b$. Therefore the area of the rectangle $E\,F\,C\,G$ equals $a + b$,

multiplied by $a - b$. From the figure, for the area of this rectangle, we have—

$$ABCD - ABEH + HFDG = EFCG;$$

or, by substitution of the symbols,

$$a^2 - ab + b\,(a - b).$$

Multiply the last quantity thus—

$$\begin{array}{l} a - b \\ b \\ \hline ab - b^2 = b\,(a - b). \end{array}$$

Substituting this in the above we have—

$$a^2 - ab + ab - b^2 = (a + b) \times (a - b).$$

Two of these like quantities, having contrary signs, cancel each other and disappear, reducing the expression to this—

$$a^2 - b^2 = (a + b) \times (a - b).$$

The correctness of this result is made manifest by an inspection of the figure, in which it is seen that the rectangle $EFCG$ is equal to the square $ABCD$ minus the square $BJHF$. For $ABEH$ equals $BJDG$. Now, if from the square $ABCD$ we take away $ABEH$, and place it so as to cover $BJDG$, we shall have the rectangle $EFCG$ plus the square $BJHF$; showing that the square $ABCD$ is equal to the rectangle $EFCG$ plus the square $BJHF$; or—

$$a^2 = (a + b) \times (a - b) + b^2.$$

The last quantity may be transferred to the first member of the equation by changing its sign (*Art.* 403). Therefore—

$$a^2 - b^2 = (a + b) \times (a - b),$$

as was before shown.

The result here obtained is derived from the geometrical figure, or graphically. Precisely the same result may be obtained algebraically; thus—

$$\begin{array}{r l}
 & a + b \\
 & a - b \\
\hline
 & a^2 + ab \\
 & \quad - ab - b^2 \\
\hline
(a + b) \times (a - b) = & a^2 \ldots . - b^2
\end{array} \qquad (114.)$$

Here the two like quantities, having unlike signs, cancel each other and disappear, leaving as the result only the difference of the squares.

The result here obtained is general; hence we have this—

RULE.—*The product of the sum and difference of two quantities equals the difference of their squares.*

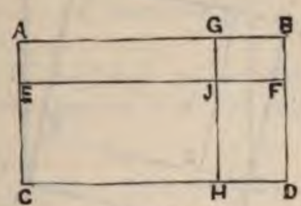


FIG. 283.

415.—Plus and Minus Signs in Multiplication.—In previous articles the signs in multiplication have been given to products in accordance with this rule, namely: *Like signs give plus; unlike signs, minus.* This rule may be illustrated graphically, thus: In the rectangular *Fig.* 283, let it be required to show the area of the rectangle $AGCH$, in terms of the several parts of the whole figure. Thus the area of $AGEJ$ equal $ABEF - GBJF$ and the area of $EJCH$ equals $EFCD - JFHD$. And the areas of $AGEJ + EJCH$ equals the area of $AGCH$. Therefore the sum of the two former expressions equals $AGCH$. Thus $ABEF - GBJF + EFCD - JFHD = AGCH$. Let the several lines now be represented by algebraic symbols; for example,

let $AB = EF = a$; let $GB = JF = b$; let $AE = GJ = c$; and $EC = JH = d$, and let these symbols be substituted for the lines they represent, thus $ABEF - GBJF + EFCD - JFHD = AGCH$.

$$ac - bc + ad - bd = (a - b) \times (c + d).$$

An inspection of the figure shows this to be a correct result. It will now be shown that an algebraical multiplication of the two binomials, allotting the signs in accordance with the rule given, will produce a like result. For example—

$$\begin{array}{c} a - b \\ \underline{c + d} \\ ac - bc + ad - bd. \end{array}$$

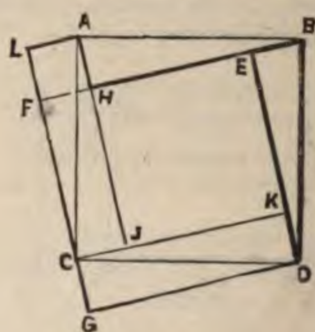


FIG. 284.

416.—Equality of Squares on Hypothenuse and Sides of Right-Angled Triangle.—The truth of this proposition has been proved geometrically in *Art.* 353. It will now be shown graphically and proved algebraically.

Let $ABCD$ (*Fig.* 284) be a rectangle of equal sides, and BED the right-angled triangle, the squares upon the sides of which, it is proposed to consider. Extend the side BE to F; parallel with BF draw DG, CK, and AL. Parallel with ED draw AJ and LG. These lines produce triangles, AHB, ACJ, ALC, CKD, and CGD, each equal to the given triangle BED (*Art.* 337). Now, if from the square

$A\,B\,C\,D$ we take $A\,B\,H$ and place it at $C\,D\,G$; and if we take $B\,E\,D$ and place it at $A\,L\,C$ we will modify the square $A\,B\,C\,D$, so as to produce the figure $L\,G\,D\,E\,H\,A\,L$, which is made up of two squares, namely, the square $D\,E\,F\,G$ and the square $A\,L\,F\,H$, and these two squares are evidently equal to the square $A\,B\,C\,D$. Now, the square $D\,E\,F\,G$ is the square upon $E\,D$, the base of the given right-angled triangle, and the square $A\,L\,F\,H$ is the square upon $A\,H = B\,E$, the perpendicular of the given right-angled triangle, while the square $A\,B\,C\,D$ is the square upon $B\,D$, the hypothenuse of the given right-angled triangle. Thus, graphically, it is shown that *the square upon the hypothenuse of a right-angled triangle is equal to the sum of the squares upon the remaining two sides.*

To show this algebraically, let $B\,E$, the perpendicular of the given right-angled triangle, be represented by a; $E\,D$, the base, by b, and $B\,D$, the hypothenuse, by c. Then it is required to show that—

$$c^2 = a^2 + b^2.$$

Now, since $D\,K = B\,E = a$, therefore, $E\,K = E\,D - D\,K = b - a$, and the square $E\,K\,J\,H$ equals $(b - a)^2$, which (*Art.* 413) equals

$$b^2 - 2\,a\,b + a^2.$$

This is the value of the square $E\,K\,J\,H$ which, with the four triangles surrounding it, make up the area of the square $A\,B\,C\,D$. Placing the triangle $A\,B\,H$ of this square outside of it at $C\,D\,G$, and the triangle $B\,E\,D$ at $A\,L\,C$, we have the four triangles, grouped two and two, and thus forming the two rectangles $C\,G\,D\,K$ and $A\,L\,C\,J$. Each of these rectangles has its shorter side ($A\,L$, $C\,G$) equal to $B\,E = a$, and its longer side $L\,C$, $G\,D$, equal to $E\,D = b$; and the sum of the two rectangles is $a\,b + a\,b = 2\,a\,b$. This represents the area of the two rectangles, which are equal to the four triangles, which, together with the square $E\,K\,J\,H$, equal the square $A\,B\,C\,D$; or—

$$A\,B\,C\,D = E\,K\,J\,H + C\,G\,D\,K + A\,L\,C\,J,$$

or—

$$c^2 = (b - a)^2 + a\,b + a\,b, \text{ or—}$$
$$c^2 = (b - a)^2 + 2\,a\,b.$$

Then, substituting for $(b - a)^2$, its equivalent as above, we have—

$$c^2 = b^2 - 2\,a\,b + a^2 + 2\,a\,b.$$

Remove the two like quantities with unlike signs (*Art.* 402), and we have—

$$c^2 = b^2 + a^2; \qquad (115.)$$

which was to be proved.

417.—Division the Reverse of Multiplication.—As division is the reverse of multiplication, so to divide one quantity by another is but to retrace the steps taken in multiplication. If we have the area $a\,b$ (*Fig.* 278), and one of the factors a given to find the other, we have but to remove from $a\,b$ the factor a, and write the answer b.

If we have the cubic contents of a solid $a\,b\,c$ (*Fig.* 279), and one of the factors a given to find the area represented by the other two, we have but to remove a, and write the others, $b\,c$, as the answer.

If there be given the area represented by $a\,(b+c)$ (see *Art.* 411), and one of the factors a to find the other, we have but to remove a and write the answer $b + c$. Sometimes, however, $a\,(b + c)$ is written $a\,b + a\,c$. Then the given factor is to be removed from each monomial and the answer written $b + c$.

If there be given the area represented by $a^2 + 2\,a\,b + b^2$ to find the factors, then we know by *Art.* 412 that this area is that of a square the sides of which measure $a + b$, and that the area is the product of $a + b$ by $a + b$; or, that $a + b$ is the square root of $a^2 + 2\,a\,b + b^2$.

If there be given the area $a^2 - 2\,a\,b + b^2$ to find its factors, then we know by *Art.* 413 that this area is that of a square whose sides measure $a - b$, or that it is the product of $a - b$ by $a - b$, or the square of $a - b$.

If there be given the difference of the squares of two quantities, or the area represented by $a^2 - b^2$, to find its factors, then we know by *Art.* 414 that this is the area produced by the multiplication of $a - b$ by $a + b$.

418.—Division: Statement of Quotient.—In any case of division the requirement may be represented as a fraction; thus: To divide $c + d - f$ by $a - b$ we write the quotient thus—

$$\frac{c + d - f}{a - b}.$$

For example, to illustrate by numerals, let $a = 7$, $b = 3$, $c = 4$, $d = 5$, and $f = 6$. Then the above becomes—

$$\frac{4 + 5 - 6}{7 - 3} = \frac{3}{4}.$$

419.—Division; Reduction.—When each monomial in either the numerator or denominator contains a common quantity, that quantity may be removed and placed outside of parentheses containing the monomials from which it was taken; thus, in—

$$\frac{2\,a\,b + 4\,a\,c - 8\,a\,d}{f},$$

we have 2 and a factors common to each monomial of the numerator. Therefore the expression may be reduced to

$$\frac{2\,a\,(b + 2\,c - 4\,d)}{f}.$$

To test this arithmetically we willl put $a = 9$, $b = 7$, $c = 5$, $d = 4$, and $f = 6$. Then for the first expression we have—

$$\frac{\overline{2 \times 9 \times 7} + \overline{4 \times 9 \times 5} - \overline{8 \times 9 \times 4}}{6},$$

which equals—

$$\frac{126 + 180 - 288}{6} = 3.$$

And for the second expression—

$$\frac{2 \times 9 (7 + \overline{2 \times 5} - \overline{4 \times 4})}{6},$$

which equals—

$$\frac{18 (17 - 16)}{6} = \frac{18}{6} = 3;$$

the same result as before. It will be observed that in this process of removing all common factors algebra furnishes the means of performing the work arithmetically with many less figures. The reduction is greater when the common factors are found in both numerator and denominator. For example, in the expression—

$$\frac{3 an + 9 bn - 15 cn}{12 dn - 18 fn}$$

we have $3n$ a factor common to each monomial in the numerator and denominator; therefore the expression reduces to

$$\frac{3 n (a + 3 b - 5 c)}{3 n (4 d - 6 f)}.$$

And now, since $3n$ is a factor common to both numerator and denominator, these cancel each other; therefore (*Art.* 371) the expression reduces to—

$$\frac{a + 3 b - 5 c}{4 d - 6 f}.$$

To test these reductions arithmetically, let $a = 9$, $b = 8$, $c = 4$, $d = 6$, $f = 3$, and $n = 5$. Then the first expression becomes—

$$\frac{\overline{3 \times 9 \times 5} + \overline{9 \times 8 \times 5} - \overline{15 \times 4 \times 5}}{12 \times 6 \times 5 - 18 \times 3 \times 5}$$

which equals—

$$\frac{135 + 360 - 300}{360 - 270} = \frac{195}{90} = 2\frac{1}{6};$$

and the second expression becomes—

$$\frac{9+\overline{3\times 8}-\overline{5\times 4}}{4\times 6 - 6\times 3};$$

which equals—

$$\frac{9+24-20}{24-18}=\frac{13}{6}=2\frac{1}{6}.$$

The same result, but with many less figures.

420.—Proportionals: Analysis.—In the formula of the lever (*Art.* 377), $P\times CF = R\times EC$. Let n be put for the arm of leverage CF and m for EC. Then we have—

$$Pn = Rm,$$

from which by division (*Art.* 372) we have (*Art.* 399)—

$$P = R\frac{m}{n}, \qquad (110.)$$

and—

$$R = P\frac{n}{m}, \qquad (111.)$$

Suppose there be a case in which neither R nor P severally are known, but that their sum is known; and it is required from this and the m and n to find R and P. Let—

$$W = R + P,$$

then—

$$W - R = P. \qquad \text{(See } \textit{Art.} \text{ 403.)}$$

The value of P was above found to be—

$$P = R\frac{m}{n}.$$

Since $P = R\frac{m}{n}$ and also equals $W - R$, therefore—

$$W - R = R\frac{m}{n}.$$

Transferring R to the opposite member (*Art.* 403) we have—

$$W = R + R\frac{m}{n}.$$

Here R appears as a common factor and may be separated by division (*Art.* 419); thus—

$$W = R\left(1 + \frac{m}{n}\right).$$

By division the factor $\left(1 + \frac{m}{n}\right)$ may be transferred to the opposite member (*Art.* 371). Thus we have—

$$R = \frac{W}{1 + \frac{m}{n}}, \qquad (116.)$$

by which we find the value of R developed. As an example, let $W = 1000$ pounds, $m = 3$ feet and $n = 7$ feet; then—

$$R = \frac{1000}{1 + \frac{3}{7}} = \frac{1000}{\frac{10}{7}}.$$

Multiplying the numerator and denominator by 7, we get—

$$R = \frac{7 \times 1000}{10} = 700.$$

Since— $R + P = 1000,$
and— $R \quad = 700,$
then— $P \quad = 300.$

But a process similar to the above develops an expression for the value of P, which is—

$$P = \frac{W}{1 + \frac{n}{m}}. \qquad (117.)$$

Putting this to the test of figures, we have—

$$P = \frac{1000}{1 + \frac{7}{3}} = \frac{1000}{\frac{10}{3}} = \frac{3000}{10} = 300.$$

421.—Raising a Quantity to any Power.—When a quantity is required to be multiplied by its equal, the product is called the square of the quantity. Thus $a \times a = a^2$ (*Art.* 412). If the square be multiplied by the original quantity the result is a cube; or, $a^2 \times a = a^3$; or, generally, for—

$$a, a\,a, a\,a\,a, a\,a\,a\,a, a\,a\,a\,a\,a,$$

we put—

$$a, \quad a^2, \quad a^3, \quad a^4, \quad a^5;$$

in which the small number at the upper right-hand corner indicates the number of times the quantity occurs in the expression. Thus, if $a = 2$, then $a^2 = 2 \times 2 = 4$, $a^3 = 4 \times 2 = 8$, $a^4 = 8 \times 2 = 16$, $a^5 = 16 \times 2 = 32$; any term in the series of powers may be found by multiplying the preceding one by a, or by dividing the succeeding one by a. Thus $a^4 \times a = a^5$, and $\frac{a^5}{a} = a^4$.

422.—Quantities with Negative Exponents.—The series of powers, by division, may be extended backward. Thus, if we divide $\frac{a^5}{a} = a^4$; $\frac{a^4}{a} = a^3$; $\frac{a^3}{a} = a^2$; $\frac{a^2}{a} = a^1$; $\frac{a^1}{a} = a^0$; $\frac{a^0}{a} = a^{-1}$; $\frac{a^{-1}}{a} = a^{-2}$; $\frac{a^{-2}}{a} = a^{-3}$, etc.

In this series we have $\frac{a}{a} = a^0$. But a quantity divided by its equal gives unity for quotient, or $\frac{3}{3} = 1$. Therefore, $\frac{a}{a} = 1$, and $a^0 = 1$. This result is remarkable, and holds good regardless of the value of a.

From this and the preceding negative exponents we derive the following:

$$a^0 = \frac{a}{a} = 1,$$

$$a^{-1} = \frac{a^0}{a} = \frac{1}{a},$$

$$a^{-2} = \frac{a^{-1}}{a} = \frac{1}{a\,a} = \frac{1}{a^2},$$

$$a^{-3} = \frac{a^{-2}}{a} = \frac{1}{a^2\,a} = \frac{1}{a^3}, \text{ etc.}$$

Showing that *a quantity with a negative exponent may have substituted for it the same quantity with a positive exponent, but used as a denominator to a fraction having unity for the numerator.*

423.—Addition and Subtraction of Exponential Quantities.—Equal quantities raised to the same power may be added or subtracted; as, $a^2 + 2a^2 = 3a^2$; but expressions in which the powers differ cannot be reduced; thus, $a^2 + a - a^3$ cannot be condensed.

424.—Multiplication of Exponential Quantities.—It will be observed in *Art.* 421 that in the series of powers, the index or exponent increases by unity; thus, a^1, a^2, a^3, a^4, etc.; and that this increase is effected by multiplying by the root, or original quantity. From this we learn that *to multiply two quantities having equal roots we simply add their exponents.*

Thus the product of a, a^2, and a^3 is $a^1 \times a^2 \times a^3 = a^6$.

The product of a^{-2}, a^3, and a^5 is $a^{-2} \times a^3 \times a^5 = a^6$.

The exponents here, are: $-2 + 3 + 5 = 8 - 2 = 6$.

425.—Division of Exponential Quantities.—As division is the reverse of multiplication, *to divide equal quantities raised to various powers,* we need simply to *subtract the exponent of the divisor from that of the dividend.* Thus, to divide a^5 by a^2 we have $a^{5-2} = a^3$. That this is correct is manifest; for the two factors, $a^2 \times a^3$, in their product, a^5, produce the dividend.

To divide a^2 by a^5, we have $a^{2-5} = a^{-3}$, which is equal to $\frac{1}{a^3}$,

(see *Art.* 422). The same result may be had by stating the question in the usual form. Thus, to divide a^2 by a^5 we have $\frac{a^2}{a^5}$, a fraction which is not in the lowest terms, for it may be put thus, $\frac{a^2}{a^2\,a^3} = \frac{a^2}{a^5}$, by which it is seen that it has in both its numerator and denominator the quantity a^2, which cancel each other (*Art.* 371). Therefore, $\frac{a^2}{a^5} = \frac{1}{a^3}$; the same result as before.

426.—Extraction of Radicals.—We have seen that the square of a is $a^1 \times a^1 = a^2$; of $2\,a^3$ is $2\,a^3 \times 2\,a^3 = 4\,a^6$; in each case the square is obtained by doubling the exponent.

To obtain the square root the converse follows, namely, take half of the exponent.

Thus the square root of a^4 is a^2, of a^2 is a, of a^6 is a^3.

The same rule, when the exponent is an odd number, gives a fractional exponent, thus: the square root of a^3 is $a^{\frac{3}{2}}$; or, of a^5, is $a^{\frac{5}{2}}$. So, also, the square root of a, or a^1, is $a^{\frac{1}{2}}$. Therefore, we have $a^{\frac{1}{2}} = \sqrt[2]{a}$, equals the square root of a, and the cube root of $a^1 = a^{\frac{1}{3}} = \sqrt[3]{a}$.

427.—Logarithms.—We have seen in the last article the nature of fractional exponents. Thus the square root of a^5 equals $a^{\frac{5}{2}}$, which may be put $a^{2\frac{1}{2}}$. In this way we may have an exponent of any fraction whatever, as $a^{1\frac{7}{8}}$. Between the exponents 2 and 3, we may have any number of fractional exponents all less than 3 and more than 2. So, also, the same between 3 and 4, or any other two consecutive numbers.

The consideration of fractional exponents or indices has led to the making of a series of decimal numbers called *logarithms*, which are treated in the manner in which exponents are treated; namely—

To multiply numbers add their logarithms.

To divide numbers, subtract the logarithm of the divisor from the logarithm of the dividend.

To raise any number to a given power, multiply its logarithm by the exponent of that power.

To obtain the root of any power, divide the logarithm of the given number by the exponent of the given power.

As an example by which to exemplify the use of logarithms: What is the product of 25 by 375?

We first make this statement:

$$\begin{array}{ll} \text{Log. of } 25\cdot = 1\cdot \\ \quad\text{"}\quad 375\cdot = 2\cdot \end{array}$$

Putting at the left of the decimal point the integer *characteristic*, or whole number of the logarithm at one less than the number of figures in the given number at the left of *its* decimal point.

To find the decimal part of the required logarithm we seek in a book of Logarithms (such as that of Law's, in Weale's Series, London) in the column of numbers for the given number 25, or 250 (which is the same as to the *mantissa*) and opposite to this and in the next column we find 7940 and a place for two other figures, which a few lines above are seen to be 39; annex these and the whole number is 0·397940. These we place as below:

$$\text{Log. of } 25\cdot = 1\cdot397940.$$

Now, to find the logarithm of 375, the other factor, we turn to 375 in the column of numbers and find the figures opposite to it, 4031, which are to be preceded by 57, the two figures found a few lines above, making the whole, ·574031, which are placed as below, and added together.

$$\begin{array}{rl} \text{Log. of } 25\cdot = & 1\cdot397940 \\ \text{"}\quad 375\cdot = & \underline{2\cdot574031} \\ \text{The sum} = & 3\cdot971971 \end{array}$$

This sum is the logarithm of the product. To find the product, we seek in the column of logarithms, headed 0·, for ·971971, the decimal part. We find first 97, the first two

figures, and a little below seeking for 1971, the remaining four figures, we find 1740, those which are the next less, and opposite these, to the left, we find 7, and above 93, or together, 937; these are the first three figures of the required product.

For the fourth figure we seek in the horizontal column opposite 7 and 1740 for 1971, the remaining four figures of the logarithm, and find them in the column headed 5.

This figure 5 is the fourth of the product and completes it, as there are only four figures required when the integer number of the logarithm is 3. The completed statement therefore is—

$$\begin{array}{lrl} \text{Log. of} & 25\cdot = & 1\cdot397940, \\ \text{"}\quad\text{"} & 375\cdot = & 2\cdot574031, \\ \hline \text{"}\quad\text{"} & 9375 = & 3\cdot971971. \end{array}$$

Another example in the use of logarithms. What is the product of 3957 by 94360?

The preliminary statement, as explained in last article, is—

$$\begin{array}{lrl} \text{Log.} & 3957 = & 3. \\ \text{"} & 94360 = & 4. \end{array}$$

In the book of logarithms seek in the column of numbers for 3957. In the first column we find only 395, and opposite to this, in the next column, we find a blank for two figures, above which are found 59. Take these two figures as the first two of the *mantissa*, or decimal part of the required logarithm, thus, 0·59. Again, opposite 395 and in the column headed by 7 (the fourth figure of the given number), we have the four figures 7366. These are to be annexed to (0·59) the first two obtained. The decimal part of the logarithm, therefore, is 0·597366.

To obtain the logarithm for 94360, the other given number, we proceed in a similar manner, and, opposite 943, we find 0·97; then, opposite 943 and in column headed 6, we find 4788, or, together, the logarithm is 0·974788. The whole is now stated thus—

$$\begin{array}{lr}
\text{Log. of} & 3957 = 3 \cdot 597366 \\
\text{" "} & 94360 = 4 \cdot 974788 \\
\hline
\text{" "} & 373382000 = 8 \cdot 572154 = \text{sum of logs.}
\end{array}$$

The two logarithms are here added together, and their sum is the logarithm of the product of the two given factors. The number corresponding to the above resultant logarithm may be found thus: Look in the column headed 0 for 57, the first two numbers of the mantissa, then in the same column, farther down, seek 2154, the other four figures of the mantissa; or, the four (1709) which are the next less than the four sought, and opposite these to the left, in the column of numbers, will be found 373, the first three figures of the product; opposite these, to the right, seek the four figures next less than 2154, the other four figures of the mantissa. These are found in the column headed 3 and are 2058. The 3 at the head of the column is the fourth figure in the product. From 2154, the last four figures of the mantissa, deduct the above 2058, or—

$$\begin{array}{lr}
 & 2154, \\
 & 2058, \\
\hline
\text{Remainder,} & 96.
\end{array}$$

At the bottom of the page, opposite the next less number (3727) to that contained in 3733, the answer already found, seek the number next less to the above remainder, 96. This is 92·8, and is in the column headed 8. Then 8 is the next number in the product. From 96 deduct 92·8, and multiply it by 10, or—

$$\begin{array}{l}
96 \\
92 \cdot 8 \\
\hline
3 \cdot 2 \times 10 = 32.
\end{array}$$

Then, in the same horizontal column, seek for 32 or its next less number. This is 23·2, found in column 2. This 2 is the next figure in the product. Additional figures may be obtained by the table of proportional parts, but they cannot be

depended upon for accuracy beyond two or three figures. We therefore arrest the process here.

The product requires one more figure than the integer of the logarithm indicates; as the integer is 8, there must be nine figures in the product. We have already six; to make the requisite number nine we annex three ciphers, giving the completed product—

$$3957 \times 94360 = 373382000.$$

By actual multiplication we find that the true product in the last article is 373382520. In a book of logarithms, carried to seven places, the required result is found to be 373382500, which is more nearly exact.

The utility of logarithms is more apparent when there are more than two factors to be multiplied, as, in that case, the operation is performed all in one statement. Thus: What is the product of 3·75, 432·95, 1712, and 0·0327?

The statement is as follows:

$$\begin{array}{rrl} \text{Log.} & 3\cdot75 & = 0\cdot574031 \\ & 432\cdot95 & = 2\cdot636438 \\ & 1712\cdot & = 3\cdot233504 \\ & \underline{\cdot0327} & = \underline{8\cdot514548} \\ \text{Product} = & 90891. & = 4\cdot958521 \\ & & \underline{16} \\ & & 5\cdot \end{array}$$

Explanations of working are given more in detail in most of the books of logarithms.

428.—Completing the Square of a Binomial.—We have seen in *Art.* 412 that the square of a binomial $(a + b)$ equals $a^2 + 2\,a\,b + b^2$—a trinomial—the first and last terms of which are each the square of one of the two quantities, while the second term contains the second quantity multiplied by twice the first quantity—

In analytical investigations it frequently occurs that an expression will be obtained which may be reduced to this form:

$$a^2 + m\,a\,b = f, \qquad (118.)$$

in which m is the coefficient of the second term, and a and b are two quantities represented by a and b or any other two symbols.

A comparison of this expression with the square of a binomial (112.) contained in *Art.* 412, shows that the member at the left comprises two out of the three terms of the square of a binomial; as thus—

$$a^2 + 2\,a\,b + b^2,$$

but with a coefficient m instead of 2. It is desirable, as will be seen, to ascertain a proper third term for the given expression; or, as it is termed, "to complete the square." The method by which this is done will now be shown.

A consideration of the above trinomial shows that the third term is equal to the square of the quotient obtained by dividing the second term by twice the square root of the first; or—

$$\left(\frac{2\,a\,b}{2\,a}\right)^2 = b^2.$$

Now a third term to the above binomial, equation (118.), may be obtained by this same rule. For example—

$$\left(\frac{m\,a\,b}{2\,a}\right)^2 = \left(\frac{m\,b}{2}\right)^2.$$

The rule for the third term then is: *Divide the second term by twice the square root of the first, and square the quotient.*

As an example, let it be required to find the third term required to complete the square in the expression—

$$6\,n\,x + 4\,x^2 = f,$$

in which n and f are known quantities and x unknown. Putting it in this form—

$$4\,x^2 + 6\,n\,x = f,$$

and dividing by 4, we have—

$$x^2 + \frac{6}{4} n x = \frac{f}{4},$$

which reduces to—

$$x^2 + \frac{3}{2} n x = \frac{f}{4}$$

Now applying the above rule for finding the third term, we have—

$$\left(\frac{\frac{3}{2} n x}{2 x}\right)^2 = \left(\frac{3}{4} n\right)^2,$$

which is the required third term. To complete the square we add this third term to both members of the above reduced expression, and have—

$$x^2 + \tfrac{3}{2} n x + \left(\frac{3}{4} n\right)^2 = \frac{f}{4} + \left(\frac{3}{4} n\right)^2.$$

The member of this expression at the left is the completed square of a binomial, the two quantities constituting which are the square roots of the first and third terms respectively; or x and $\frac{3}{4} n$, and we therefore have—

$$\sqrt{x^2 + \frac{3}{2} n x + \left(\frac{3}{4} n\right)^2} = x + \frac{3}{4} n,$$

and now taking the square root of both sides of the expression, we have—

$$x + \frac{3}{4} n = \sqrt{\frac{f}{4} + \left(\frac{3}{4} n\right)^2};$$

and, by transferring the second quantity to the right member, we have—

$$x = \sqrt{\frac{f}{4} + \left(\frac{3}{4} n\right)^2} - \frac{3}{4} n;$$

an expression in which x, the unknown quantity, is made to stand alone and equal to known quantities.

The process of completing the square is useful, as has been shown, in developing the value of an unknown quan-

tity where it enters into an expression in two forms, one as the square of the other.

As an example to test the above result, let $f = 256$ and $n = 8$. Then we have by the last expression for the value of x—

$$x = \sqrt{\frac{256}{4} + \left(\frac{3}{4} \times 8\right)^2} - \frac{3}{4} \times 8,$$
$$= \sqrt{64 + 36} - 6,$$
$$= \sqrt{100} - 6,$$
$$x = 10 - 6 = 4.$$

Now this value of x may be tested in the original expression—

$$6\,n\,x + 4\,x^2 = f,$$

for which we have—

$$6 \times 8 \times 4 + 4 \times 4^2 = f,$$
$$192 + 64 = f,$$
$$256 = f;$$

the correct value as above.

PROGRESSION.

429.—Arithmetical Progression.—In a series of numbers, as 1, 3, 5, 7, 9, etc., proceeding in regular order, increasing by a common difference, the series is called an arithmetical progression; the quantity by which one number is increased beyond the preceding one is termed the difference. If d represent the difference and a the first term, then the progression may be stated thus—

Terms—	1,	2,	3,	4,	5,
	a,	$a + d$,	$a + 2\,d$,	$a + 3\,d$,	$a + 4\,d$, etc.

The coefficient of d is equal to the number of terms preceding the one in which it occupies a place. Thus the fifth term is $a + 4\,d$, in which the coefficient 4 equals the number of the preceding terms.

From this we learn the rule by which at once to desig-

nate any term without finding all the preceding terms. For the one hundredth term we should have $a + 99\,d$, or, if the number of terms be represented by n, then the last term would be represented by—

$$l = a + (n - 1)\,d. \qquad (119.)$$

For example, in a progression where a, the first term, equals 1, d the difference, 2, and n, the number of terms, 90, the last term will be—

$$l = a + (n - 1)\,d = 1 + (90 - 1)\,2 = 179.$$

Therefore, to find the last term:

To the first term add the product of the common difference into the number of terms less one.

By a transposition of the terms in the above expression, so as to give it this form—

$$a = l - (n - 1)\,d, \qquad (120.)$$

we have a rule by which to find the first term, which, in words, is—

Multiply the number of terms less one by the common difference, and deduct the product from the last term; the remainder will be the first term.

By a transposition of the terms of the former expression to this form—

$$l - a = (n - 1)\,d,$$

and dividing both members by $(n - 1)$, we have—

$$d = \frac{l - a}{n - 1}; \qquad (121.)$$

which is a rule for the common difference, and which, in words, is—

Subtract the first term from the last, and divide the remainder by the number of terms less one; the quotient will be the common difference.

Multiplying both members of the equation (121.) by $(n - 1)$ and dividing by d, we obtain—

$$n - 1 = \frac{l - a}{d}.$$

Transferring 1 to the second member, we have—

$$n = \frac{l - a}{d} + 1; \qquad (122.)$$

which is a rule for finding the number of terms, and which, in words, is—

Divide the difference between the first and last terms by the common difference; to the quotient add unity, and the sum will be the number of terms.

Thus it has been shown, in equations (119,) (120), (121), and (122), that when, of the four quantities in arithmetical progression, any three are given, the fourth may be found.

The *sum* of the terms of an arithmetical progression may be ascertained by adding them; but it may also be had by a shorter process. If the terms are written in order in a horizontal line, and then repeated in another horizontal line beneath the first, but in reversed order, as follows:

$$\begin{array}{cccccccc} 1, & 3, & 5, & 7, & 9, & 11, & 13, & 15, \\ 15, & 13, & 11, & 9, & 7, & 5, & 3, & 1, \\ \hline 16, & 16, & 16, & 16, & 16, & 16, & 16, & 16, \end{array}$$

and the vertical columns added, the sums will be equal. In this case the sum of each vertical couple is 16, and there are 8 couples; hence the sum of these 8 couples is $8 \times 16 = 128$. And in general the sum will be the product of one of the couples into the number of couples. It will be observed that the first couple contains the first and last terms, 1 and 15; therefore the sum of the double series is equal to the product of the sum of the first and last terms into the number of terms. Or if S be put to represent the sum of the series, we shall have—

$$2\,S = (a + l)\,n,$$

and, dividing both sides by 2—

$$S = (a + l)\frac{n}{2}; \qquad (123.)$$

Or, in words: *The sum of an arithmetical series equals the product of the sum of the first and last terms, into half the number of terms.*

430.—Geometrical Progression.—A series of numbers, such as 1, 2, 4, 8, 16, 32, 64, 128, 256, etc., in which any one of the terms is obtained by multiplying the preceding one by a constant quantity, is termed a *Geometrical Progression.*

The constant quantity is termed the common *Ratio*, and is equal to any term divided by the preceding one. Thus in the above example $\frac{16}{8}$ or $\frac{8}{4}$ or $\frac{4}{2} = 2$, equals the common ratio of the above series. In the series, 1, 3, 9, 27, etc., we have for the ratio—

$$\frac{27}{9} = \frac{9}{3} = \frac{3}{1} = 3;$$

which is the common ratio of this series.

A geometrical series may be put thus:

Terms:	1,	2,	3,	4;
Progress.:	1,	1×3,	$1 \times 3 \times 3$,	$1 \times 3 \times 3 \times 3$;

or thus—

Terms:	1,	2,	3,	4:
Progress.:	1,	1×3,	1×3^2,	1×3^3;

in which the common ratio, in this case 3, appears in each term and with an exponent which is equal to the number of terms preceding that in which it occupies a place.

If the first term be represented by a and the common ratio by r, then the following will represent any geometrical progression—

$$a,\ ar,\ ar^2,\ ar^3,\ ar^4,\ \text{etc.} \qquad (124.)$$

For example, let $a = 2$ and $r = 4$; then the progression will be—

2, 8, 32, 128, 512, etc.

If $r =$ unity, then when $a = 2$ the progression becomes—

$$2,\ 2,\ 2,\ 2,\ 2, \text{ etc.}$$

If r be less than unity, then the progression will be a decreasing one.

For example, let $a = 2$ and $r = \frac{1}{2}$. Then we have for the progression—

$$2,\ 1,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \frac{1}{16}, \text{ etc.}$$

If the number of terms be represented by n, and the last by l, then the last term will be—

$$l = a r^{n-1}. \qquad (125.)$$

For example, let n equal 6, then the progression will be—

Terms:	1,	2,	3,	4,	5,	6;
Progress.:	a,	ar,	ar^2,	ar^3,	ar^4,	ar^5;

in which the exponent of the last term equals $n - 1 = 6 - 1 = 5$.

If S be put for the sum of a geometrical progression, we will have—

$$S = a + ar + ar^2 + \ldots\ldots + ar^{n-2} + ar^{n-1}.$$

Multiply each member by r, then—

$$Sr = ar + ar^2 + \ldots\ldots + ar^{n-2} + ar^{n-1} + ar^n.$$

Subtract the upper line from the lower; then—

$$\begin{array}{l} Sr = \qquad ar + ar^2 + \ldots\ldots + ar^{n-2} + ar^{n-1} + ar^n, \\ \underline{S \ = a + ar + ar^2 + \ldots\ldots + ar^{n-2} + ar^{n-1}} \\ Sr - s = -a \quad * \qquad\qquad * \qquad * \qquad + ar^n, \\ Sr - s = -a + ar^n, \\ \qquad S(r - 1) = -a + ar^n = ar^n - a, \\ \qquad S = \dfrac{ar^n - a}{r - 1}. \end{array}$$

The last term (equation (125.)) equals $l = a r^{n-1}$, and since $a r^n = r \times a r^{n-1} = r l$, therefore—

$$S = \frac{r l - a}{r - 1}. \qquad (126.)$$

Thus, to find the sum of a geometrical progression: *Multiply the last term by the ratio; from the product deduct the first term, and divide the remainder by the ratio less unity.*

For example, the sum of the geometrical progression—

$$S = 1 + 3 + 9 + 27 + 81 + 243 + 729 = 1093$$

by actual addition.

To obtain it by the above rule—

$$S = \frac{rl - a}{r - 1} = \frac{\overline{3 \times 729} - 1}{3 - 1} = 1093,$$

the correct result.

If there be a decreasing geometrical progression, as $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}$, etc., in which the ratio equals $\frac{1}{3}$, the sum will be—

$$S = 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81}, \text{ etc., to infinity.}$$

Multiply this by 3, and subtract the first from the last—

$$3\,S = 3 + 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \text{ to infinity.}$$
$$S = \phantom{3 +{}} 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \text{ to infinity.}$$
$$2\,S = 3, \text{ or } S = 1\tfrac{1}{2}.$$

In a decreasing progression let r, the common ratio, be represented by $\frac{b}{c}$ (b less than c), and the first term by a, then the sum will be—

$$S = a + a\frac{b}{c} + a\frac{b^2}{c^2} + a\frac{b^3}{c^3} +, \text{ etc., to infinity.}$$

Multiply this by $\frac{b}{c}$, and subtract the product from the above—

$$S\frac{b}{c} = a\frac{b}{c} + a\frac{b^2}{c^2} + a\frac{b^3}{c^3} + \text{etc., to infinity.}$$

$$S = a + a\frac{b}{c} + a\frac{b^2}{c^2} + a\frac{b^3}{c^3} + \text{to infinity.}$$

$$S\frac{b}{c} = a\frac{b}{c} + a\frac{b^2}{c^2} + a\frac{b^3}{c^3} + \text{to infinity.}$$

$$S - S\frac{b}{c} = a * \quad * \quad *$$

Or—

$$S\left(1 - \frac{b}{c}\right) = a,$$

$$S = \frac{a}{1 - \frac{b}{c}} \qquad (127.)$$

For example, let the first term of a geometrical progression equal 2, and the ratio equal $\frac{1}{2}$, then the sum will be—

$$S = \frac{a}{1 - \frac{1}{2}} = \frac{2}{\frac{1}{2}} = \frac{4}{1} = 4.$$

From this, therefore, we have this rule for the sum of an infinite geometrical progression, namely : *Divide the first term by unity less the ratio.*

431.—Relation of Sum and Difference of Two Lines.—Let AB and CD (*Fig.* 285) be two given lines; make EH

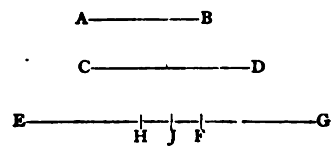


FIG. 285.

equal to AB, and HG equal to CD; then EG equals the sum of the two lines.

Make FG equal to AB, which is equal to EH.

Bisect EG in J; then, also, J bisects HF; for—

$$EJ = JG,$$

and—

$$EH = FG.$$

Subtract the latter from the former; then—

$$EJ - EH = JG - FG;$$

but—

$$EJ - EH = HJ,$$

and—

$$JG - FG = JF;$$

therefore—

$$HJ = JF.$$

Now, EJ is half the sum of the two lines, and HJ is half the difference; and—

$$EJ - HJ = EH = AB.$$

Or: ***Half the sum of two quantities, minus half their difference, equals the smaller of the two quantities.***

Let the shorter line be designated by a, and the longer by b; then the proposition is expressed by—

$$a = \frac{a+b}{2} - \frac{b-a}{2} \qquad (128.)$$

We also have $E J + J F = E F = C D$; or, *half the sum of two quantities, plus half their difference, equals the larger quantity.*

432.—Perpendicular, in Triangle of Known Sides.—Let $A B C$ (*Fig.* 286) be the given triangle, and $C E$ a perpendicular let fall upon $A B$, the base. Let the several lines of

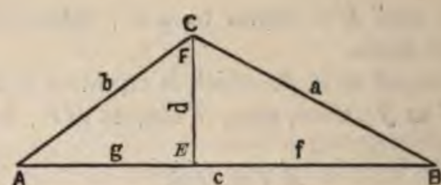


FIG. 286.

the figure be represented by the symbols a, b, c, d, g, and f, as shown. Then, since $A E C$ and $B E C$ are right-angled triangles, we have (*Art.* 416) the following two equations, and, by subtracting one from the other, the third—

$$\begin{aligned} f^2 + d^2 &= a^2, \\ g^2 + d^2 &= b^2, \\ \hline f^2 - g^2 &= a^2 - b^2. \end{aligned}$$

Then (*Art.* 414), by substitution, we have—

$$(f+g)(f-g) = (a+b)(a-b).$$

By division we obtain—

$$f - g = \frac{(a+b)(a-b)}{f+g}.$$

According to *Art.* 431, equation (128.), we have—

$$g = \frac{f+g}{2} - \frac{f-g}{2}.$$

In this expression let the value of $f - g$, as above, be substituted, then we will have—

$$g = \frac{f+g}{2} - \frac{(a+b)(a-b)}{2(f+g)}.$$

Multiply the first fraction by $(f+g)$, then join the two fractions, when we will have—

$$g = \frac{(f+g)^2 - (a+b)(a-b)}{2(f+g)}$$

The lines f and g, in the *figure*, together equal the line c; therefore, by substitution—

$$g = \frac{c^2 - (a+b)(a-b)}{2c}. \qquad (129.)$$

This is the value of the line g.

It may be expressed in words, thus: The shorter of the two parts into which the base of a triangle is divided by a perpendicular let fall from the apex upon the base, *equals the quotient arising from a division by twice the base, of the difference between the square of the base and the product of the sum and difference of the two inclined lines.*

As an example to show the application of this rule, let $a = 9$, $b = 6$, and $c = 12$; then equation (129.) becomes—

$$g = \frac{12^2 - (9+6)(9-6)}{2 \times 12},$$

$$g = \frac{144 - 15 \times 3}{24},$$

$$g = \frac{99}{24} = 4\frac{1}{8}.$$

Now, to obtain the length of d, the perpendicular, by the *figure*, we have—

$$d^2 = b^2 - g^2,$$

and, extracting the square root—

$$d = \sqrt{b^2 - g^2}, \qquad (130.)$$

or, in words: The altitude of a triangle *equals the square root of the difference of the squares of one of the inclined sides and its base.*

As an example, take the same dimensions as before, then equation (130.) becomes—

$$d = \sqrt{6^2 - 4\tfrac{1}{8}^2}.$$

$$\begin{array}{lrl} \text{The square of} & 6 = & 36\cdot \\ \text{"} \quad \text{"} \quad \text{"} & 4\tfrac{1}{8} = & 17\cdot 015625 \\ \hline & 6^2 - 4\tfrac{1}{8}^2 = & 18\cdot 984375, \end{array}$$

the square root of which is 4·44234; therefore—

$$d = \sqrt{6^2 - 4\tfrac{1}{8}^2} = 4\cdot 44234.$$

This may be tested by applying the rule to the other inclined side and its base—

$$\begin{array}{rl} c = & 12 \\ g = & 4\tfrac{1}{8} \\ \hline f = & 7\tfrac{7}{8}. \end{array}$$

Then,

$$d = \sqrt{9^2 - 7\tfrac{7}{8}^2},$$

$$\begin{array}{rl} 9^2 = & 81\cdot \\ 7\tfrac{7}{8}^2 = & 62\cdot 015625 \\ \hline 9^2 - 7\tfrac{7}{8}^2 = & 18\cdot 984375. \end{array}$$

he same result as before, producing for its square root ie same, 4·44234, the value of d; therefore—

$$d = \sqrt{9^2 - 7\tfrac{7}{8}^2} = 4{\cdot}44234.$$

433.—Trigon: Radius of Circumscribed and Inscribed Circles: Area.—Let $A\,B\,C$ (*Fig.* 287) be a given trigon or triangle with its circumscribed and inscribed circles. Draw the lines $A\,D\,F$, $D\,B$ and $D\,C$.

The three triangles, $A\,B\,D$, $A\,C\,D$, and $B\,D\,C$, have their apexes converging at D, and form there the three angles, $A\,D\,B$, $A\,D\,C$, and $B\,D\,C$. These three angles together form

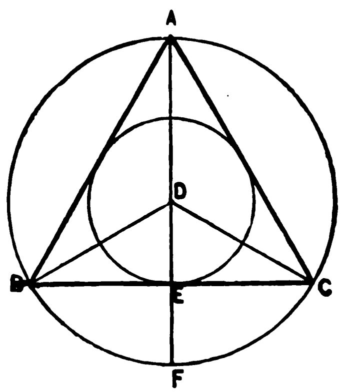


FIG. 287.

ur right angles (*Art.* 335), and each of them, therefore, quals $\frac{4}{3}$ of a right angle.

The angles of the triangle $B\,D\,C$ together equal two ght angles (*Art.* 345). As above, the angle $B\,D\,C$ equals $\frac{4}{3}$ a right angle, hence $2 - \frac{4}{3} = \frac{6-4}{3} = \frac{2}{3}$ of a right angle, uals the sum of the two remaining angles at B and C. ıe triangle $B\,D\,C$ is isoceles (*Art.* 338); for the two sides D and $D\,C$, being radii, are equal; therefore the two angles the base B and C are equal, and as their sum, as above, uals $\frac{2}{3}$ of a right angle, therefore each angle equals $\frac{1}{3}$ of a ght angle. Draw the two lines $F\,C$ and $F\,B$. Now, be-

cause DC and DF are radii, they are equal, hence DFC is an isoceles triangle.

It was before shown that the angle BDC equals $\frac{4}{3}$ of a right angle; now, since the diameter AF bisects the chord BC, the angles BDE and EDC are equal, and each equals the half of the angle BDC; or, $\frac{1}{2}$ of $\frac{4}{3}$ of a right angle equals $\frac{2}{3}$ of a right angle. Deducting this from two right angles (the sum of the three angles of the triangle), or $2 - \frac{2}{3} = 1\frac{1}{3} = \frac{4}{3}$ of a right angle equals the sum of the angles at F and C; hence each equals the half of $\frac{4}{3}$, or $\frac{2}{3}$ of a right angle; therefore the triangle DFC is equilateral. The triangles DBF and DFC are equal. The angles BDC and BFC are equal; the line BC is perpendicular to DF and bisects it, making DE and EF equal; hence DE equals half DF, or DB, radii of the circumscribing circle. Therefore, putting R to represent BD, the radius of the circumscribing circle, and $b = BC$, a side of the triangle ABC, by *Art.* 416, we have—

$$BD^2 = BE^2 + DE^2,$$

$$R^2 = \left(\frac{b}{2}\right)^2 + \left(\frac{R}{2}\right)^2.$$

Transferring and reducing—

$$R^2 - \left(\frac{R}{2}\right)^2 = \left(\frac{b}{2}\right)^2,$$

$$R^2 - \frac{R^2}{4} = \frac{b^2}{4},$$

$$R^2\left(1 - \frac{1}{4}\right) = \frac{1}{4}b^2,$$

$$\frac{3}{4}R^2 = \frac{1}{4}b^2,$$

$$R^2 = \frac{4}{3} \times \frac{1}{4}b^2 = \frac{1}{3}b^2 = \frac{b^2}{3},$$

$$R = \frac{b}{\sqrt{3}}; \qquad (131.)$$

Or, *The Radius of the circumscribing circle of a regular trigon or equilateral triangle, equals a side of the triangle divided by the square root of* 3.

By reference to *Fig.* 287 it will be observed, as was above shown, that $DE = EF = \frac{DF}{2} = \frac{BD}{2}$; or, DE, the radius of the inscribed circle, equals half the radius of the circumscribed circle; or, again, dividing equation (131.) by 2, we have—

$$\frac{R}{2} = \frac{b}{2\sqrt{3}};$$

and, putting r for the radius of the inscribed circle, we have—

$$r = \frac{b}{2\sqrt{3}}. \qquad (132.)$$

Or: *The radius* of the *inscribed circle of a regular trigon equals the half of a side of the trigon divided by the square root of* 3.

To obtain the area of a trigon or equilateral triangle; we have (*Art.* 408) the area of a parallelogram by multiplying its base into its height; and (*Arts.* 341 and 342) the area of a triangle is equal to half that of a parallelogram of equal base and height, therefore, the area of the triangle BDC (*Fig.* 287) is obtained by multiplying BC, the base, into the half of ED, its height. Or, when N is put for the area—

$$N = BC \times \frac{ED}{2},$$

or—

$$N = b \times \frac{R}{4};$$

substituting for R its value (131.)—

$$N = b \times \frac{b}{4\sqrt{3}},$$

$$N = \frac{b^2}{4\sqrt{3}}.$$

This is the area of the triangle BDC.

The triangle ABC is compounded of three equal triangles, one of which is the triangle BDC; therefore the area of the triangle ABC equals three times the area of the triangle BDC; or, when A represents the area—

$$A = \frac{3 b^2}{4 \sqrt{3}}; \qquad (133.)$$

Or: The *area* of a regular *trigon* or equilateral *triangle* equals *three fourths of the square of a side of the triangle divided by the square root of* 3.

434.—Tetragon: Radius of Circumscribed and Inscribed Circles: Area.—Let $A\,B\,C\,D$ (*Fig.* 288) be a given tetragon or square, with its circumscribed and inscribed

FIG. 288.

circles, of which $A\,E$ is the radius of the former and $E\,F$ that of the latter. The point F bisects $A\,B$, the side of the square. $A\,F$ equals $E\,F$ and equals half $A\,B$, a side of the square. Putting R for the radius of the circumscribed circle and b for $A\,B$, we have (*Art.* 416)—

$$A\,E^2 = A\,F^2 + E\,F^2,$$

$$R^2 = \left(\frac{b}{2}\right)^2 + \left(\frac{b}{2}\right)^2 = 2\left(\frac{b}{2}\right)^2 = \frac{2\,b^2}{4} = \frac{b^2}{2}.$$

$$R = \frac{b}{\sqrt{2}}; \qquad (134.)$$

Or: The *radius* of the *circumscribed* circle of a regular tetragon equals *a side of the square divided by the square root of* 2.

By referring to the figure it will be seen that the *radius* of the *inscribed* circle equals *half a side of the square*—

$$r = \frac{b}{2}. \qquad (135.)$$

The *area* of the *square* equals *the square of a side*—

$$A = b^2. \qquad (136.)$$

435.—Hexagon: Radius of Circumscribed and Inscribed Circles: Area.—Let $A\,B\,C\,D\,E\,F$ (*Fig.* 289) be an equilateral hexagon with its circumscribed and inscribed circles, of which $E\,G$ is the radius of the former, and $G\,H$ that of the latter. The three lines, $A\,D$, $B\,E$, and $C\,F$, divide the

Fig. 289.

hexagon into six equal triangles with their apexes converging at G. The six angles thus formed at G are equal, and since their sum about the point G amounts to four right angles (*Art.* 335), therefore each angle equals $\frac{4}{6}$ or $\frac{2}{3}$ of a right angle. The sides of the six triangles radiating from G are the radii of the circle, hence they are equal; therefore, each of the triangles is isosceles (*Art.* 338), having equal angles at the base. In the triangle $E\,G\,D$, the sum of the three angles being equal to two right angles (*Art.* 345), and the angle at G being, as above shown, equal to $\frac{2}{3}$ of a right angle, therefore the sum of the two angles at E and D equals $2 - \frac{2}{3} = \frac{4}{3}$ of a right angle; and, since they equal each other,

therefore each equals $\frac{2}{3}$ of a right angle and equals the angle at G; therefore $E\,G\,D$ is an equilateral triangle. Hence $E\,D$, a *side of a hexagon, equals* $E\,G$, *the radius of the circumscribing circle*—

$$R = b. \qquad (137.)$$

As to the radius of the inscribed circle, represented by $G\,H$, a perpendicular from the centre upon $E\,D$, the base; the point H bisects $E\,D$. Therefore, $E\,H$ equals half of a side of the hexagon, equals half the radius of the circumscribing circle. Let $R =$ this radius, and r the radius of the inscribed circle, while $b =$ a side of the hexagon; then we have (*Arts.* 353 and 416)—

$$G\,H^2 = E\,G^2 - E\,H^2,$$

$$r^2 = R^2 - \left(\frac{R}{2}\right)^2,$$

$$r^2 = R^2 - \tfrac{1}{4}R^2 = \tfrac{3}{4}R^2,$$

$$r = \sqrt{\tfrac{3}{4}}\,R.$$

Now, $R = b$, therefore—

$$r = \sqrt{\tfrac{3}{4}}\,b = \frac{\sqrt{3}}{\sqrt{4}}\,b = \tfrac{1}{2} \times \sqrt{3}\,b,$$

$$r = \sqrt{3}\,\frac{b}{2}. \qquad (138.)$$

Or: The *radius* of the *inscribed* circle of a regular *hexagon* equals *the half of a side of the hexagon, multiplied by the square root of* 3.

As to the area of the hexagon, it will be observed that the six triangles, $A\,B\,G$, $B\,G\,C$, etc., converging at G, the centre, are together equal to the area of the hexagon. The area of $E\,G\,D$, one of these triangles, is equal to the product of $E\,D$, the base, into the half of $G\,H$, the perpendicular; or, when N is put to equal the area—

$$N = E\,D \times \frac{G\,H}{2},$$

$$N = b \times \frac{r}{2},$$

l, since r, as above, equals $\sqrt{3}\,\frac{b}{2}$,

$$N = b \times \frac{\sqrt{3}\frac{b}{2}}{2} = b \times \frac{\sqrt{3}\,b}{4},$$

$$N = \frac{\sqrt{3}\,b}{4}.$$

is is the area of one of the six equal triangles; therefore, en A is put to represent the area of the hexagon, we have—

$$A = 6 \times \sqrt{3}\,\frac{b^2}{4},$$

$$A = \sqrt{3}\,\frac{3\,b^2}{2}. \qquad (139.)$$

: The *area* of a regular *hexagon equals three halves of the are of a side multiplied by the square root of* 3.

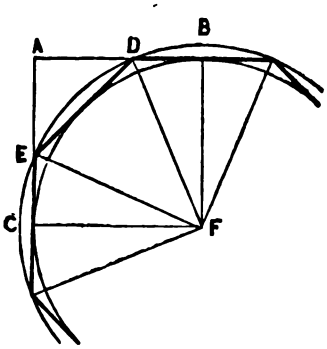


FIG. 290.

436.—Octagon: Radius of Circumscribed and Inibed Circles: Area.—Let $C\,E\,D\,B\,F$ (*Fig.* 290) represent a ırter of a regular octagon, in which F is the centre, $E\,D$ ide, and $C\,E$ and $D\,B$ each half a side, while $C\,F$ and ˀ are radii of the inscribed circle, and $E\,F$ and $D\,F$ are lii of the circumscribed circle.

Let R represent the latter, and r the former; also let b represent $E D$, one of the sides, and n be put for $A D$, and for $A E$. Then we have—

$$A D + D B = C F.$$

$$n + \frac{b}{2} = r,$$

or—

$$n = r - \frac{b}{2}.$$

Since $A D E$ is a right-angled triangle (*Art.* 416), we have—

$$A E^2 + A D^2 = E D^2,$$

$$n^2 + n^2 = b^2,$$

$$2 n^2 = b^2,$$

$$n^2 = \frac{b^2}{2}$$

$$n = \sqrt{\frac{b^2}{2}}.$$

Placing the value of n, equal to the value before found, we have—

$$r - \frac{b}{2} = \sqrt{\frac{b^2}{2}},$$

$$r = \sqrt{\frac{b^2}{2}} + \frac{b}{2} = \frac{\sqrt{b^2}}{\sqrt{2}} + \frac{b}{2},$$

$$r = \frac{b}{\sqrt{2}} + \frac{b}{2} = \left(\frac{1}{\sqrt{2}} + \frac{1}{2}\right)b.$$

This coefficient may be reduced by multiplying the first fraction by $\sqrt{2}$, thus—

$$\frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}\,1,$$

therefore—

$$r = \left(\frac{\sqrt{2}}{2} + \frac{1}{2}\right) b = \frac{\sqrt{2}+1}{2} b,$$

$$r = (\sqrt{2} + 1) \frac{b}{2}. \qquad (140.)$$

Or: The *radius* of the *inscribed* circle of a regular *octagon equals half a side of the octagon multiplied by the sum of unity plus the square root of* 2. In regard to the radius of the circumscribed circle, by *Art.* 416 we have—

$$D F^2 = B F^2 + D B^2,$$

$$R^2 = r^2 + \left(\frac{b}{2}\right)^2.$$

In this expression substituting for r^2, its value as above, we have—

$$R^2 = (\sqrt{2} + 1)^2 \left(\frac{b}{2}\right)^2 + \left(\frac{b}{2}\right)^2,$$

$$R^2 = [(\sqrt{2} + 1)^2 + 1]\left(\frac{b}{2}\right)^2.$$

The square of the coefficient $(\sqrt{2} + 1)$ by *Art.* 412 equals $2 + 2\sqrt{2} + 1 = 2\sqrt{2} + 3$, then—

$$R^2 = [(2\sqrt{2} + 3) + 1]\left(\frac{b}{2}\right)^2.$$

$$R^2 = (2\sqrt{2} + 4)\left(\frac{b}{2}\right)^2.$$

$$R = \sqrt{2\sqrt{2} + 4}\ \frac{b}{2}. \qquad (141.)$$

Or: The *radius* of the *circumscribed* circle of a regular *octagon equals half a side of the octagon multiplied by the square root of the sum of twice the square root of* 2 *plus* 4.

In regard to the area of the octagon, the figure shows that one eighth of it is contained in the triangle $D E F$.

The area of $D E F$, putting it equal to N, is—

$$N = E D \times \frac{B F}{2},$$

$$N = b \times \frac{r}{2},$$

$$N = b \times \frac{(\sqrt{2} + 1)\,\frac{b}{2}}{2}$$

$$N = (\sqrt{2} + 1)\frac{b^2}{4}.$$

This is the area of one eighth of the octagon; the whole area, therefore, is—

$$A = (\sqrt{2} + 1)\,\frac{8\,b^2}{4},$$

$$A = (\sqrt{2} + 1)\,2\,b^2. \qquad (142.)$$

Or: The *area* of a regular *octagon equals twice the square of a side, multiplied by the sum of the square root of* 2 *added to unity.*

When a side of the enclosing square, or diameter of the inscribed circle, is given, a side of the octagon may be found; for from equation (140.), multiplying by two, we have—

$$2\,r = (\sqrt{2} + 1)\,b.$$

Dividing by $\sqrt{2} + 1$, gives—

$$b = \frac{2\,r}{\sqrt{2} + 1}. \qquad (143.)$$

The numerator, $2\,r$, equals the diameter of the inscribed circle, or a side of the enclosing square; therefore:

The *side* of a regular *octagon, equals a side of the enclosing square divided by the sum of the square root of* 2 *added to unity.*

437.—Dodecagon: Radius of Circumscribed and Inscribed Circles: Area.—Let $A B C$ (*Fig.* 291) be an equilat-

eral triangle. Bisect $A B$ in F; draw $C F D$; with radius $A C$ describe the arc $A D B$. Join A and D, also D and B; bisect $A D$ in E; with the radius $E C$ describe the arc $E G$. Then $A D$ and $D B$ are sides of a regular dodecagon, or twelve-sided polygon; of which $A C$, $D C$, and $B C$ are radii of the circumscribing circle, while $E C$ is a radius of the inscribed circle.

The line $A B$ is the side of a regular hexagon (*Art.* 435). Putting R equal to $A C$ the radius of the circumscribing circle; $r, = E C$, the radius of the inscribed circle; $b, = A D$, a side of the dodecagon, and $n = D F$. Then comparing the

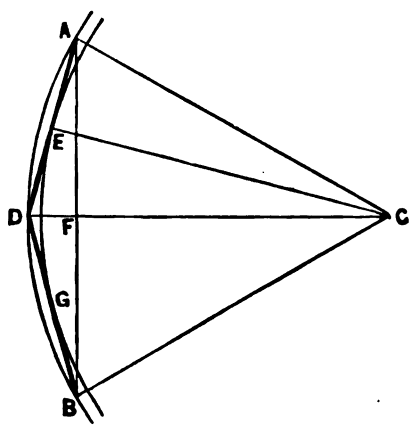


FIG. 291.

homologous triangles, $A D F$ and $A E C$ (the angle $A D F$ equals the angle $E A C$, and the angles $D F A$ and $A E C$ are right angles); therefore, the two remaining angles $D A F$ and $A C E$ must be equal, and the two triangles homologous (*Art.* 345). Thus we have—

$$D F : D A :: A E : A C,$$

$$n : b :: \frac{b}{2} : R,$$

$$R = \frac{b^2}{2 n}.$$

In *Art.* 435 it was shown that FC (*Fig.* 291), or GH of *Fig.* 289, the radius of the inscribed hexagon, equals $\sqrt{3}\,\frac{b}{2}$, and in which its $b = R$; $Fc = \sqrt{3}\,\frac{R}{2}$.

Now (*Fig.* 291)—

$$DF = DC - FC,$$

or—

$$n = R - \sqrt{3}\frac{R}{2} = R(1 - \tfrac{1}{2}\sqrt{3}).$$

Substituting this value of n, in the above expression, we have—

$$R = \frac{b^2}{2R(1 - \frac{1}{2}\sqrt{3})}.$$

Multiplying by R and reducing, we have—

$$R^2 = \frac{b^2}{2 - \sqrt{3}},$$

$$R = \sqrt{\frac{1}{2 - \sqrt{3}}}\, b. \qquad (144.)$$

Or: The *radius* of the *circumscribed* circle of a regular *dodecagon, equals a side of the dodecagon multiplied by the square root of a fraction, having unity for its numerator and for its denominator* 2 *minus the square root of* 3.

Comparing the same triangles, as above, we have—

$$FD : FA :: EA : EC,$$

or—

$$n : \frac{R}{2} :: \frac{b}{2} : r,$$

$$r = \frac{Rb}{4n} = \frac{Rb}{4R(1 - \frac{1}{2}\sqrt{3})}.$$

$$r = \frac{b}{4 - 2\sqrt{3}}. \qquad (145.)$$

Or: The *radius* of the *inscribed* circle of a regular *dodecagon equals a side of the dodecagon divided by the difference between* 4 *and the square root of* 3.

The area of a dodecagon is equal to twelve times the area of the triangle $A\,D\,C$ (*Fig.* 291). The area of this triangle is equal to half the base by its perpendicular; or, $A\,E \times E\,C$; or—

$$\frac{b}{2} \times r,$$

or, where N equals the area—

$$N = \tfrac{1}{2}\,b\,r.$$

Or, for the area of the whole dodecagon—

$$12\,N = 6\,b\,r,$$
$$A = 6\,b\,r.$$

Substituting for r its value as above, we have—

$$A = \frac{6}{4 - 2\sqrt{3}}\,b^2. \qquad (146.)$$

Or: The *area* of a regular *dodecagon equals the square of a side of the dodecagon, multiplied by a fraction having* 6 *for its numerator, and for its denominator,* 4 *minus twice the square root of* 3.

438.—Hecadecagon: Radius of Circumscribed and Inscribed Circles: Area.—Let $A\,B\,C\,D$ (*Fig.* 292) be a square enclosing a quarter of a regular octagon $C\,E\,F\,B$, $E\,F$ being one of its sides, and $C\,E$ and $F\,B$ each half a side, while $F\,D$ is the radius of the circumscribed circle, and $J\,D$ the radius of the inscribed circle of the octagon. Draw the diagonal $A\,D$; with $D\,F$ for radius, describe the circumscribed circle $E\,G\,F$; join G with F and with E; then $E\,G$ and $G\,F$ will each be a side of a regular hecadecagon, or polygon of sixteen sides.

An expression for $F\,D$, the radius of the circumscribed

circle, may be obtained thus: Putting $FD = R$; $HD = r$; $GF = b$; $GJ = n$; and $JF = \frac{s}{2}$ (*Art.* 416), we have—

$$GJ^2 = GF^2 - JF^2,$$
$$n^2 = b^2 - \left(\frac{s}{2}\right)^2.$$

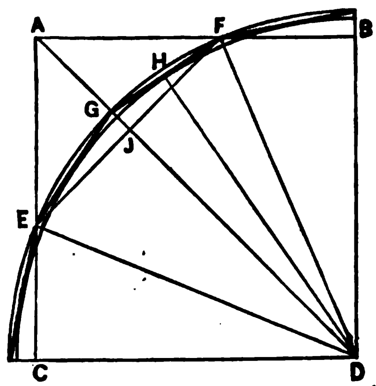


FIG. 292.

Comparing the two homologous (*Art.* 361) triangles, GJF and FHD (*Art.* 374), we have—

$$GJ : GF :: HF : FD,$$
$$n : b :: \frac{b}{2} : R,$$
$$n = \frac{b^2}{2R},$$
$$n^2 = \frac{b^4}{4R^2}.$$

Putting this value of n^2 in an equation against the former value, we have—

$$\frac{b^4}{4R^2} = b^2 - \left(\frac{s}{2}\right)^2.$$

In *Art.* 436, the value of FD, as the radius of the circumscribed circle of a regular octagon, is given in equation (141.) as—

$$R = \sqrt{2\sqrt{2} + 4}\,\frac{b}{2},$$

n which b represents a side of the octagon, or EF, for which we have put s. Substituting s for b and putting the numerical coefficient under the radical, equal to B, we have—

$$R = \sqrt{B}\frac{s}{2}.$$

Squaring each member gives—

$$R^2 = B\left(\frac{s}{2}\right)^2.$$

From which, by transposition, we have—

$$\frac{R^2}{B} = \left(\frac{s}{2}\right)^2.$$

Substituting in the above expression for $\left(\frac{s}{2}\right)^2$, this value of it, gives—

$$\frac{b^4}{4R^2} = b^2 - \frac{R^2}{B}.$$

Transposing, we have—

$$\frac{b^4}{4R^2} + \frac{R^2}{B} = b^2.$$

Multiplying the first term by B, and the second by $4R^2$, we have—

$$\frac{Bb^4}{4BR^2} + \frac{4R^4}{4BR^2} = b^2,$$

$$\frac{Bb^4 + 4R^4}{4BR^2} = b^2,$$

$$Bb^4 + 4R^4 = 4BR^2b^2.$$

Transposing, we have—

$$4R^4 - 4BR^2b^2 = -Bb^4.$$

To complete the square (*Art.* 428) we proceed thus—

$$R^4 - BR^2b^2 = -\tfrac{1}{4}Bb^4,$$

$$R^4 - BR^2b^2 + (\tfrac{1}{2}Bb^2)^2 = (\tfrac{1}{2}Bb^2)^2 - \tfrac{1}{4}Bb^4.$$

Taking the square root, we have—

$$R^2 - \tfrac{1}{2}Bb^2 = \sqrt{\tfrac{1}{4}B^2b^4 - \tfrac{1}{4}Bb^4},$$

$$R^2 = \sqrt{\tfrac{1}{4}B^2b^4 - \tfrac{1}{4}Bb^4} + \tfrac{1}{2}Bb^2,$$

$$R^2 = b^2\sqrt{\tfrac{1}{4}B^2 - \tfrac{1}{4}B} + \tfrac{1}{2}Bb^2,$$

$$R^2 = b^2(\sqrt{\tfrac{1}{4}B^2 - \tfrac{1}{4}B} + \tfrac{1}{2}B),$$

$$R^2 = b^2(\sqrt{\tfrac{1}{4}B(B-1)} + \tfrac{1}{2}B).$$

Restoring B to its value, $2\sqrt{2} + 4$ as above, we have—

$$\tfrac{1}{4}B = \tfrac{1}{2}\sqrt{2} + 1,$$

$$B - 1 = \underline{2\sqrt{2} + 3;}$$

multiply these—

$$2 + 2\sqrt{2},$$

$$\underline{3 + \tfrac{3}{2}\sqrt{2},}$$

$$\tfrac{1}{4}B(B-1) = 5 + \tfrac{7}{2}\sqrt{2},$$

$$\tfrac{1}{2}B = \sqrt{2} + 2.$$

Therefore—

$$R^2 = b^2(\sqrt{5 + \tfrac{7}{2}\sqrt{2}} + \sqrt{2} + 2),$$

$$R = b\sqrt{\sqrt{5 + \tfrac{7}{2}\sqrt{2}} + \sqrt{2} + 2}. \qquad (147.)$$

Or: The *radius* of the *circumscribed circle* of a regular *ecadecagon equals a side of the hecadecagon multiplied by the quare root of the sum of two quantities, one of which is the quare root of* 2 *added to* 2, *and the other is the square root of he sum of seven halves of the square root of* 2 *added to* 5.

To obtain the radius of the inscribed circle we have (*Fig.* 92)—

$$HD^2 = FD^2 - HF^2,$$

$$r^2 = R^2 - \left(\frac{b}{2}\right)^2.$$

Substituting for R^2 its value as above, we have—

$$r^2 = b^2 \left(\sqrt{\tfrac{1}{4} B (B - 1)} + \tfrac{1}{2} B\right) - \left(\frac{b}{2}\right)^2,$$

$$r^2 = b^2 [(\sqrt{\tfrac{1}{4} B (B - 1)} + \tfrac{1}{2} B) - (\tfrac{1}{2})^2],$$

$$r = b \sqrt{\sqrt{\tfrac{1}{4} B (B - 1)} + \tfrac{1}{2} B - \tfrac{1}{4}}.$$

The coefficient of b is the same as in the case above, ex- cept the $-\frac{1}{4}$; therefore its numerical value will be $\frac{1}{4}$ less, or—

$$r = b \sqrt{\sqrt{5 + \tfrac{7}{2} \sqrt{2}} + \sqrt{2} + 1\tfrac{3}{4}}. \qquad (148).$$

Or: The *radius* of the *inscribed circle* of a regular *hecadecagon equals a side of the hecadecagon multiplied by the square root of two quantities, one of which is the square root of* 2 *added to* $1\frac{3}{4}$, *and the other is the square root of the sum of seven halves of the square root of* 2 *added to* 5.

To obtain the area of the hecadecagon it will be observed that the area of the triangle GFD (*Fig.* 292) equals $HD \times HF$, and that this is the $\frac{1}{16}$ part of the polygon; we therefore have—

$$A = 16\, HD \times HF,$$

$$A = 16\, r \frac{b}{2} = 8\, r\, b.$$

The value of r is shown in (148.); therefore we have—

$$A = 8b^2\sqrt{\sqrt{5 + \tfrac{7}{2}\sqrt{2}} + \sqrt{2} + 1\tfrac{3}{4}}. \qquad (149.)$$

Or: The *area* of a regular *hecadecagon equals eight times the square of its side, multiplied by the square root of two quantities, one of which is the square root of* 2 *added to* $1\frac{3}{4}$, *and the other is the square root of the sum of seven halves of the square root of* 2 *added to* 5.

439.—Polygons: Radius of Circumscribed and Inscribed Circles: Area.—In *Arts.* 433 to 438 the relation of the radii to a side in a trigon, tetragon, hexagon, octagon, dodecagon and hecadecagon have been shown by methods based

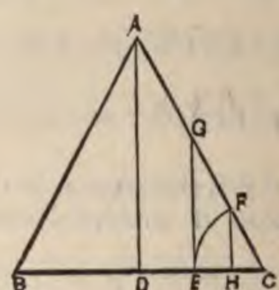


FIG. 293.

upon geometrical proportions. This relation in polygons of seven, nine, ten, eleven, thirteen, fourteen and fifteen sides, cannot be so readily shown by geometry, but can be easily obtained by trigonometry—as also said relation of the parts in a regular polygon of any number of sides. The nature of trigonometrical tables is discussed in *Arts.* 473 and 474. So much as is required for the present purpose will here be stated.

Let $A\,B\,C$ (*Fig.* 293) represent one of the triangles into which any polygon may be divided, in which $B\,C = b =$ a side of the polygon; $A\,C = R =$ the radius of the circumscribed circle; and $A\,D = r =$ the radius of the inscribed circle.

Make EC equal unity; on C as a centre describe the arc EF; draw FH and EG perpendicular to BC, or parallel to AD; then for the uses of trigonometry EG is called the *tangent* of c, or of the angle ACB, and FH is the *sine*, and HC the *cosine* of the same angle.

These trigonometrical quantities for angles varying from zero up to ninety degrees have been computed and are to be found in trigonometrical tables.

Referring now to *Fig.* 293 we have—

$$HC : FC :: DC : AC,$$

$$\cos. c : 1 :: \frac{b}{2} : R,$$

$$R = \frac{b}{2 \cos. c}. \qquad (150.)$$

Again—

$$EC : EG :: DC : AD,$$

$$1 : \tan. c :: \frac{b}{2} : r,$$

$$r = \frac{b}{2} \tan. c. \qquad (151.)$$

These two equations give the required radii of the circumscribed and inscribed circles. They may be stated thus:

The *radius* of the *circumscribed* circle of *any* regular *polygon equals a side of the polygon divided by twice the cosine of the angle formed by a side of the polygon and a radius from one end of the side.*

The *radius* of the *inscribed* circle of *any* regular *polygon equals half of a side of the polygon multiplied by the tangent of the angle formed by a side of the polygon and a radius from one end of the side.*

The area of a polygon equals the area of the triangle ABC (*Fig.* 293), (of which BC is one side of the polygon and A is the centre), multiplied by the number of sides in the polygon; or, if n be put to represent the number of the sides and A the area, then we have—

$$A = B\,n,$$

in which B equals the area of the triangle. The area of A B C (*Fig.* 293) is equal to $A\,D \times B\,D$, or—

$$B = r \times \frac{b}{2}.$$

For r substituting its value, as in equation (151.), we have—

$$B = \frac{b}{2} \tan.\, c \frac{b}{2} = \frac{1}{4} b^2 \tan.\, c.$$

Therefore, by substitution—

$$A = \frac{1}{4} b^2\, n \tan.\, c. \qquad (152.)$$

Or: The *area* of a regular *polygon equals the square of a side of the polygon, multiplied by one fourth of the number of its sides, and by the tangent of the angle formed by a side of the polygon, and a radius from one end of the sides.*

440.—Polygons: Their Angles.—Let a line be drawn from each angle of a regular polygon to its centre, then these lines form with each other angles at the centre, which taken together amount to four right angles, or to 360 degrees (*Arts.* 327, 335).

If this 360 degrees be divided by the number of the sides of the polygon, the quotient will equal the angle at the centre of the polygon, of each triangle formed by a side and two radii drawn from the ends of the side. For example: if A B C (*Fig.* 293) be one of the triangles referred to, having B C one of the sides of the polygon and the point A the centre of the polygon, then the angle B A C will be equal to 360 degrees divided by the number of the sides of the polygon. If the polygon has six sides, then the angle B A C will contain $\frac{360}{6} = 60$ degrees; or if there be 10 sides, then the angle at A, the centre, will contain $\frac{360}{10} = 36$ degrees. The angle

$B\,A\,D$ equals half the angle $B\,A\,C$, or, when n equals the number of sides, the angle $B\,A\,C$ equals—

$$\frac{360}{n},$$

and the triangle $B\,A\,D = \frac{B\,A\,C}{2}$, equals—

$$\frac{360}{2\,n}.$$

Now the angles $B\,A\,D + D\,B\,A$ equal one right angle (*Art.* 346), or 90 degrees. Hence the angle $D\,B\,A = 90° - B\,A\,D$, or the angle c equals—

$$c° = 90° - \frac{360°}{2\,n}. \qquad (153.)$$

For example, if n equal 6, or the polygon have six sides, then—

$$c° = 90° - \frac{360°}{12} = 90 - 30 = 60°.$$

Therefore, the *angle* c, contained in equations (150.), (151.), and (152.), *equals* 90 *degrees, less the quotient derived from a division of* 360 *by twice the number of sides to the polygon.*

441.—Pentagon: Radius of the Circumscribed and Inscribed Circles: Area.—The rules for polygons developed in the two former articles will here be exemplified in their application to the case of a regular pentagon, or polygon of five sides.

To obtain the angle $c°$ (153.), we have $n = 5$, and—

$$c° = 90° - \frac{360}{10} = 90 - 36 = 54°.$$

For the radius of the circumscribed circle, we have (150.)—

$$R = \frac{b}{2\cos.c},$$

$$R = \frac{b}{2 \cos. 54^\circ}.$$

$$R = b \frac{1}{2 \cos. 54^\circ}.$$

Using a table of logarithmic sines and tangents (*Art.* 427), we have—

$$\begin{aligned} \text{Log. } 2 &= 0 \cdot 3010300 \\ \text{Cos. } 54^\circ &= 9 \cdot 7692187 \\ \text{Their sum} &= 0 \cdot 0702487 \; - \text{ subtracted from} \\ \text{Log. } 1 &= 0 \cdot 0000000 \\ 0 \cdot 85065 &= 9 \cdot 9297513 \end{aligned}$$

Therefore—

$$R = 0 \cdot 85065\, b.$$

Or: The *radius* of the *circumscribed* circle of a regular *pentagon equals a side of the pentagon multiplied by the decimal* 0·85065.

For the radius of the inscribed circle, we have (151.)—

$$r = \frac{b}{2} \tan. c,$$

$$r = b \frac{\tan. 54^\circ}{2}.$$

For this we have—

$$\begin{aligned} \text{Log. tan. } 54^\circ &= 0 \cdot 1387390 \\ \text{Log. } 2 &= 0 \cdot 3010300 \\ 0 \cdot 68819 &= 9 \cdot 8377090. \end{aligned}$$

Therefore—

$$r = 0 \cdot 68819\, b.$$

Or: The *radius* of the *inscribed* circle of a regular *pentagon equals a side of the pentagon multiplied by the decimal* 0·68819.

For the area we have (152.)—

$$A = \tfrac{1}{4} b^2 n \tan. c,$$

$$A = \tfrac{1}{4} \times 5 \tan. 54^\circ\, b^2,$$

$$A = \tfrac{5}{4} \tan. 54^\circ\, b^2.$$

For this we have—

$$\begin{aligned} \text{Log. } 5. &= 0 \cdot 6989700 \\ \text{Log. tan. } 54^\circ &= 0 \cdot 1387390 \\ \hline & 0 \cdot 8377090 \\ \text{Log. } 4 &= 0 \cdot 6020600 \\ \hline 1 \cdot 72048 &= 0 \cdot 2356490 \end{aligned}$$

Therefore—

$$A = 1 \cdot 72048\, b^2.$$

Or: The *area* of a regular *pentagon equals the square of its side multiplied by* 1·72048.

442.— Polygons: Table of Constant Multipliers. — To obtain expressions for the radii of the circumscribed and inscribed circles, and for the area for polygons of 7, 9, 10, 11, 13, 14, and 15 sides, a process would be needed such precisely as that just shown in the last article for a pentagon, except in the value of n and c, which are the only factors which require change for each individual case.

No useful purpose, therefore, can be subserved by exhibiting the details of the process required for these several polygons. The values of the constants required for the radii and for the areas of these polygons have been computed, and the results, together with those for the polygons treated in former articles, gathered in the annexed Table of Regular Polygons.

REGULAR POLYGONS.

SIDES.	$\frac{R}{b} =$	$\frac{r}{b} =$	$\frac{A}{b^2} =$
3. Trigon	·57735	·28868	·43301
4. Tetragon	·70711	·50000	1·00000
5. Pentagon	·85065	·68819	1·72048
6. Hexagon	1·00000	·86603	2·59808
7. Heptagon	1·15238	1 03826	3·63391
8. Octagon	1·30656	1·20711	4·82843
9. Nonagon	1·46190	1·37374	6·18182
10. Decagon	1·61803	1·53884	7·69421
11. Undecagon	1·77473	1·70284	9·36564
12. Dodecagon	1·93185	1·86603	11·19615
13. Tredecagon	2·08929	2·02858	13·18577
14. Tetradecagon	2·24698	2·19064	15·33451
15. Pentadecagon	2·40487	2·35231	17·64236
16. Hecadecagon	2·56292	2·51367	20·10936

In this table R represents the radius of the circumscribed circle; r the radius of the inscribed circle; b one of the sides, and A the area of the polygon. By the aid of the constants of this table, R, the radius of the circumscribed circle of any of the polygons named, may be found when a side of the polygon is given. For this purpose, putting m for any constant of the table, we have—

$$R = b\,m. \tag{154.}$$

As an example: let it be required to find R, for a pentagon having each side equal to 5 feet; then the above expression becomes—

$$R = 5 \times 0{\cdot}85065,$$
$$R = 4{\cdot}25325.$$

The radius will be 4 feet 3 inches and a small fraction.

In like manner the radius of the inscribed circle will be—

$$r = b\,m; \tag{155.}$$

and for a pentagon with sides of 5 feet, we have—

$$r = 5 \times 0{\cdot}68819,$$
$$r = 3{\cdot}44095.$$

Or, the radius of the inscribed circle will be 3 ft. $\frac{44}{100}$ and a small fraction. Or, multiplying the decimal by 12, 3 ft. 5 in. $\frac{29}{100}$ and a small fraction.

The area of any polygon of the table may be obtained by this expression—

$$A = b^2 m; \tag{156.}$$

and, applying this to the pentagon as before, we have—

$$A = 5^2 \times 1{\cdot}72048,$$
$$A = 43{\cdot}012.$$

or, the area of a pentagon having its sides equal to 5 feet, is 43 feet and $\frac{12}{1000}$ of a foot.

By the constants of the table a side of any of its polygons may be found, when either of the radii, or the area, are known.

When R is known, we have—

$$b = \frac{R}{m}. \tag{157.}$$

When r is known, we have—

$$b = \frac{r}{m}. \tag{158.}$$

When the area is known, we have—

$$b = \sqrt{\frac{A}{m}}. \tag{159.}$$

SECTION XI.—THE CIRCLE.

443.—Circles: Diameter and Perpendicular: Mean Proportional.—Let $A\,B\,C$ (*Fig.* 294) be a semicircle. From C, any point in the curve, draw a line to A and another to B; then $A\,B\,C$ will be a right-angled triangle (*Art.* 352). Draw the line $C\,D$ perpendicular to the diameter $A\,B$; then $C\,D$ will divide the triangle $A\,B\,C$ into two triangles. $A\,C\,D$ and $C\,B\,D$, which are homologous. For, let the triangle $C\,B\,D$ be revolved on D as a centre until its line $C\,D$ shall come to the position $E\,D$, and the line $D\,B$ occupy the position $D\,F$, each in a position at right angles

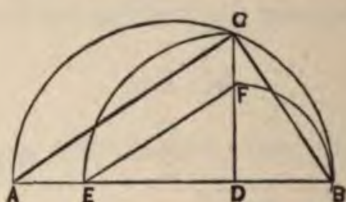


FIG 294.

to its former position, the point B describing the curve $B\,F$, and the point C the curve $C\,E$, and each forming a quadrant or angle of ninety degrees. Since these points have revolved ninety degrees, therefore the three lines of the triangle $C\,B\,D$ have revolved into a position at right angles to that which they before occupied; hence the line $E\,F$ is at right angles to $C\,B$, and (from the fact that $A\,C\,B$ is a right angle) parallel with $A\,C$. Since the triangle $E\,F\,D$ equals the triangle $C\,B\,D$. and since the lines of $E\,F\,D$ are parallel respectively to the corresponding lines of $A\,C\,D$, therefore the triangles $A\,C\,D$ and $C\,B\,D$ are homologous.

Comparing the lines of these triangles and putting $a = A\,B$, $y = C\,D$, and $x = D\,B$, we have—

$$DB : DC :: DC : AD,$$
$$x : y :: y : a - x,$$
$$y^2 = x(a - x). \qquad (160.)$$

Or, in a semicircle, a *perpendicular* to the *diameter* terminated by the diameter and the curve *is a geometric mean, or mean proportional, between the two parts into which the perpendicular divides the diameter.*

444.—Circle: Radius from Given Chord and Versed Sine.—Let AB (*Fig.* 295) be a given chord line and CD a versed sine. Extend CD to the opposite side of the circle; it will pass through F, the centre. Join A and C, also E and

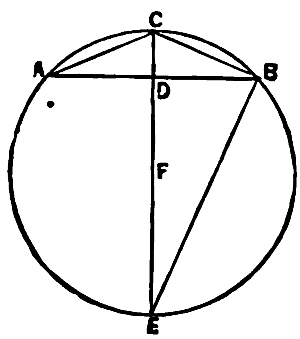


FIG. 295.

B. The line AD, perpendicular to the diameter CE, is a mean proportional between the two parts CD and DE (*Art.* 443); or, putting $a = AD$, $b = CD$, and r equal the radius FE, we have—

$$CD : AD :: AD : DE;$$
$$b : a :: a : 2r - b,$$
$$a^2 = b(2r - b),$$
$$a^2 = 2rb - b^2;$$
$$a^2 + b^2 = 2rb,$$
$$r = \frac{a^2 + b^2}{2b}. \qquad (161.)$$

Or: *The radius of a circle equals the sum of the squares of half the chord and the versed sine, divided by twice the versed sine.*

Another expression for the radius may be obtained; for the two triangles CBD and CEB (*Fig.* 295) are homologous (*Art.* 443) and their corresponding lines in proportion. Putting f for CB, we have—

$$CD : CB :: CB : CE,$$

or—

$$v : f :: f : 2r,$$

or—

$$f^2 = 2rv,$$

and—

$$r = \frac{f^2}{2v}. \qquad (162.)$$

Or: The *radius* of a *circle equals the square of the chord of half the arc divided by twice the versed sine.*

445.—Circle: Segment from Ordinates.—When the curve of a segment of a circle is required for which the radius cannot be used, either by reason of its extreme length, or be-

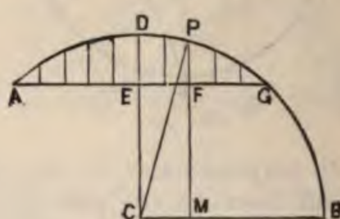


FIG. 296.

cause the centre of the circle is inaccessible, it is desirable to obtain the curve without the use of the radius. This may be done by calculating ordinates, a rule for which will now be developed.

Let DCB (*Fig.* 296) be a right angle, and ADB a circular arc described from C as a centre, with the radius $BC = CD = CP$. Draw PM parallel with DC, and AG parallel with CB. Now, in the segment ADG, we have given AG, its chord, and DE, its versed sine, and it is re-

quired to find an expression by which its ordinates, as PF, may be computed. From *Art.* 416, we have—

$$PM^2 = \overline{CP}^2 - \overline{CM}^2;$$

or, putting for these lines their usual symbols—

$$y^2 = r^2 - x^2,$$

$$y = \sqrt{r^2 - x^2},$$

now we have—

$$EC = DC - DE,$$

$$EC = FM,$$

$$FM = DC - DE,$$

$$FM = r - b.$$

Then we have—

$$PF = PM - FM;$$

or, putting t for PF and substituting for PM and FM their values as above, we have—

$$t = y - (r - b),$$

and for y, substituting its value as above, we have—

$$t = \sqrt{r^2 - x^2} - (r - b). \qquad (163.)$$

Or: The *ordinate* in the *segment* equals the *square root* of the *difference* of the *squares* of the *radius* and the *abscissa minus* the *difference* of the *radius* and the *versed sine.*

For example: let the chord AG (*Fig.* 296) in a given case equal 20 feet, and the versed sine, b, or the rise DE, equal 4 feet; and let the ordinates be located at every 2 feet along the chord line, AG.

In solving this problem we require first to find the radius. This is obtained by means of equation (161.)—

$$r = \frac{a^2 + b^2}{2b}.$$

For a, half the chord, we have 10 feet; for b, the versed sine, we have 4 feet; and, substituting these values, we have—

$$r = \frac{10^2 + 4^2}{2 \times 4} = \frac{116}{8} = 14 \cdot 5$$

The radius equals— 14·5

The versed sine equals— 4·0

$$(r - b) = 10 \cdot 5$$

The square of 14·5, the radius, equals 210·25. Now we have, substituting these values in equation (163.)—

$$t = \sqrt{210 \cdot 25 - x^2} - 10 \cdot 5.$$

The respective values of x, as above required, are 0, 2, 4, 6, 8 and 10 Substituting successively for x one of these values, we shall have, when—

$$x = 0;\ t = \sqrt{210 \cdot 25 - 0^2} - 10 \cdot 5 = 4 \cdot$$

$$x = 2;\ t = \sqrt{210 \cdot 25 - 2^2} - 10 \cdot 5 = 3 \cdot 8614$$

$$x = 4;\ t = \sqrt{210 \cdot 25 - 4^2} - 10 \cdot 5 = 3 \cdot 4374$$

$$x = 6;\ t = \sqrt{210 \cdot 25 - 6^2} - 10 \cdot 5 = 2 \cdot 7004$$

$$x = 8;\ t = \sqrt{210 \cdot 25 - 8^2} - 10 \cdot 5 = 1 \cdot 5934$$

$$x = 10;\ t = \sqrt{210 \cdot 25 - 10^2} - 10 \cdot 5 = 0 \cdot 0$$

Values for t may be taken at points as numerous as desirable for accuracy.

In ordinary cases, however, they need not be nearer than in this example.

After the points are secured, let a flexible piece of wood be bent so as to coincide with at least four of the points at a time, and then draw the curve against the strip.

446.—Circle: Relation of Diameter to Circumference.—In *Art.* 439 it is shown that the area of a polygon equals the radius of the inscribed circle multiplied by half of a side of the polygon and by the number of the sides; or,

$A = r \times \frac{b}{2} n = \frac{r}{2} b n$; or, the area equals half the radius by a side into the number of sides; or, half the radius into the periphery of the polygon. Now, if a polygon have very small sides and many of them, its periphery will approximate the circumference of the circle inscribed within it; indeed when the number of sides becomes infinite, and consequently infinitely small, the periphery and circumference become equal. Consequently, for the area of the circle, we have—

$$A = \frac{r}{2} c, \qquad (164.)$$

where c represents the circumference.

By computing the area of a polygon inscribed within a given circle, and that of one circumscribed about the circle, the area of one will approximate the area of the other in proportion as the number of the sides of the polygon are increased.

For example: if polygons of 4 sides be inscribed within and circumscribed about a circle, the radius of which is 1, the areas will be respectively 2 and 4. If the polygons have 16 sides, the areas are each 3 and a fraction, the fractions being unlike; when they have 128 sides the areas are each 3·14 and with unlike fractions; when the sides are increased to 2048, the areas each equal 3·1415 and unlike fractions, and when the sides reach 32768 in number the areas are equal each to 3·1415926, having like decimals to seven places. The computations have been continued to 127 places (Gregory's "Math. for Practical Men"), but for all possible uses in building operations seven places will be found to be sufficient. From this result we have the diameter in proportion to the circumference as 1 : 3·1415926, or as—

$$1 : 3 \cdot 14159\tfrac{1}{4}.$$
$$1 : 3 \cdot 14159.$$
$$1 : 3 \cdot 1416.$$

Of these proportions, that one may be used which will give

a result most nearly approximating the degree of accuracy required. For many purposes the last proportion will be sufficiently near the truth.

For ordinary purposes the proportion $7 : 22$ is very useful, and is correct for two places of decimals; it fails in the third place.

The proportion $113 : 355$ is correct to six places of decimals.

For the quantity $3 \cdot 1415926$ putting the Greek letter π (called *py*), and $2r = d$ for the diameter, we have—

$$c = \pi d. \qquad (165.)$$

To apply this: in a circle of 50 feet diameter, what is the circumference?

$$c = 3 \cdot 1416 \times 50$$
$$c = 157 \cdot 08 \, ft.$$

If the more accurate value of π be used, we have—

$$c = 3 \cdot 1415926 \times 50,$$
$$c = 157 \cdot 07963.$$

The difference between the two results is $0 \cdot 00037$, which for all ordinary purposes, would be inappreciable.

By the rule of $7 : 22$, we have—

$$c = 50 \times \tfrac{22}{7} = 157 \cdot 1428571,$$

an excess over the more accurate result above, of $0 \cdot 0632271$, which is about $\frac{3}{4}$ of an inch.

By the rule of $113 : 355$, we have—

$$c = 50 \times \tfrac{355}{113} = 157 \cdot 079646.$$

This result gives an excess of only $0 \cdot 000016$; it is sufficiently near for any use required in building.

From these results we have these rules, namely: To obtain the *circumference* of a circle, *multiply its diameter by*

22, *and divide the product by* 7; or, more accurately, *multiply the diameter by* 355 *and divide the product by* 113; or, by multiplication only, *multiply the diameter by* 3·1416; or, *by* 3·14159¼; or, *by* 3·1415926; according to the degree of accuracy required.

And conversely: To obtain the *diameter* from the circumference, *multiply the circumference by* 7 *and divide the product by* 22; or, *multiply by* 113 *and divide by* 355; or, *divide the circumference by* 3·1416; or, *by* 3·14159¼; or, by 3·1415926.

447.—Circle: Length of an Arc.—Considering the circle divided into 360°, the length of an arc of one degree in a circle the diameter of which is unity may be thus found.

The circumference for 360° is 3·14159265;

$$\frac{3 \cdot 14159265}{360} = 0 \cdot 00872664625;$$

which equals an arc of one degree in a circle having unity as its diameter; or, for ordinary use the decimal 0·008727 or 0·0087¼ may be taken; or putting a for the arc and g for the number of degrees, we have—

$$a = 0 \cdot 00872665\, dg. \qquad (166.)$$

Wherefore: To obtain the length of an arc of a circle, *multiply the diameter of the circle by the number of degrees in the arc, and by the decimal* 0·0087¼, or, instead thereof, *by* 0·008727.

448.—Circle: Area.—The area of a circle may be obtained in a manner similar to that for the area of polygons (*Art.* 439), in which $A = B\,n$; $B = r\dfrac{b}{2}$, or—

$$A = \tfrac{1}{2}\, b\, n\, r,$$

where b equals a side of the polygon and n the number of sides; so that $b\,n$ equals the perimeter of the polygon.

Now, if for the perimeter of the polygon there be sub-

stituted the circumference of the circle, we shall have, putting for the circumference $3 \cdot 1416\, d$, or, $\pi\, d$ (*Art.* 446)—

$$A = \tfrac{1}{2}\, \pi\, d\, r,$$

in which r is the radius. Since $2\, r = d$, the diameter, and $r = \frac{d}{2}$, we have—

$$A = \tfrac{1}{2}\, \pi\, d \frac{d}{2},$$

$$A = \tfrac{1}{4}\, \pi\, d^2.$$

And since—

$$\pi = 3 \cdot 14159265,$$

$$\tfrac{1}{4}\, \pi = 0 \cdot 785398 16,$$

or—

$$\tfrac{1}{4}\, \pi = 0 \cdot 7854, \text{ nearly.}$$

Therefore—

$$A = 0 \cdot 7854\, d^2. \qquad (167.)$$

Or: The *area* of a circle *equals the square of the diameter multiplied by* $0 \cdot 7854$.

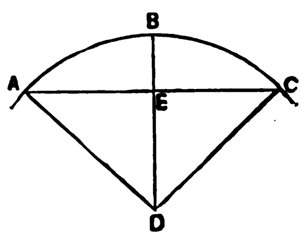


FIG. 297.

As an example, the area of a circle 10 feet in diameter is found thus—

$$10 \times 10 = 100.$$

$$100 \times 0 \cdot 7854 = 78 \cdot 54 \text{ feet.}$$

449.—Circle: Area of a Sector.—The area of $A\, B\, C\, D$ (*Fig.* 297), a sector of a circle, is proportionate to that of the whole circle. For, as the circumference of the whole circle is to its area, so is the arc $A\, B\, C$ to the area of $A\, B\, C\, D$.

The circumference of a circle is (165.) $C = \pi d$. The area of a circle is (167.) $A = \cdot 7854\, d^2$. For the arc $A\ B\ C$ put a, and for the area of $A\ B\ C\ D$ put s. Then we have from the above-named proportion—

$$\pi d : \cdot 7854\, d^2 :: a : s,$$

$$s = \frac{\cdot 7854\, d^2}{\pi d}\, a.$$

The coefficient $0 \cdot 7854$ is $\frac{\pi}{4}$ (*Art.* 448).

Therefore, multiplying the fraction by 4, we have—

$$S = \frac{\pi d^2}{4 \pi d}\, a;$$

or—

$$S = \tfrac{1}{4} d a = \tfrac{1}{2} r a. \qquad (168.)$$

Wherefore: To obtain the *area* of a *sector* of a circle, *multiply a quarter of the diameter by the length of the arc.*

Thus: let $A\ D$ equal 10; also let $A\ B\ C = a$, equal 12. Then the area of $A\ B\ C\ D$ is—

$$S = \tfrac{1}{2} \times 10 \times 12,$$

$$S = 60.$$

The length of the arc may be had by the rule in *Art.* 447.

450.—Circle: Area of a Segment.—In the last article, $A\ B\ C\ D$ (*Fig.* 297) is called the sector of a circle. Of this the portion included within $A\ E\ C\ B$ is a *segment* of a circle. The area of this equals the area of the sector minus the area of the triangle $A\ D\ C$; or, putting M for the area of the segment, S for the area of the sector, and T for the area of the triangle, then—

$$M = S - T.$$

Putting c for $A\ C$ (*Fig.* 297) and h for $D\ E$, then $T = \frac{c}{2}\, h$. In the last article, $s = \frac{1}{2}\, r a$, in which $a =$ the length of the

arc ABC. Substituting this value of s in the above, we have—

$$M = \frac{a}{2} r - \frac{c}{2} h = \frac{ar - ch}{2}; \qquad (169.)$$

Or: When the length of the arc is known, also that of the chord and the perpendicular from the centre of the circle, then the *area* of the *segment equals the difference between the product of half the arc into the radius, aud half the chord into its perpendicular to the centre of the circle.*

But ordinarily the length of the arc and of the chord are unknown. If in this case the number of degrees contained between the two radii, DA, DC, are known, then the area of

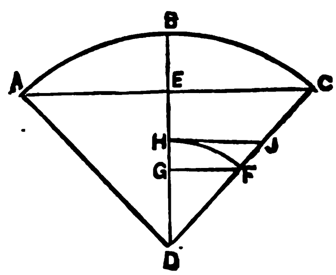


FIG. 298.

the segment may be found by a rule which will now be developed.

In *Fig.* 298 (a repetition of *Fig.* 297) upon D as a centre, and with $DF =$ unity for a radius, describe the arc HF. Then GF is the *sine* of the angle CDB, and DG is the *cosine;* and we have—

$$DF : GF :: DC : EC,$$

or—

$$1 : \sin :: r : \frac{c}{2} = r \sin.$$

Again—

$$DF : DG :: DC : DE,$$

or—

$$1 : \cos :: r : h = r \cos.$$

y equation (166.) we have—

$$a = 0 \cdot 00872665 \, dg,$$

ι which a is the length of the arc; g the number of degrees ɔntained in the arc; and d is the diameter of the circle. ince $d = 2\,r$, therefore—

$$a = 0 \cdot 0174533 \, rg.$$

'utting B for the decimal coefficient, we have—

$$a = Brg.$$

The expression (169.), by substitution of values as above, ecomes—

$$M = \frac{a}{2} r - \frac{c}{2} h,$$

$$M = \frac{Brg}{2} r - r \text{ sin.} \times r \text{ cos.}$$

$$M = \tfrac{1}{2} Bgr^2 - \text{sin. cos.}\, r^2$$

$$M = r^2 (\tfrac{1}{2} Bg - \text{sin. cos.})$$

$$M = r^2 (0 \cdot 00872665\, g - \text{sin. cos.}) \qquad (170.)$$

)r: The *area* of a *segment* of a circle *equals the square of the 'adius into the difference between* 0·00872665 *times the number f degrees contained in the arc of the circle, and the product of he sine and cosine of half the arc.*

When the number of degrees subtended by the arc is ınknown, or tables of sines and cosines are not accessible, hen the area may be obtained by equation (169.), provided he chord and versed sine are known; but before this equaion can be used for this purpose, expressions giving their 'alues in terms of the chord and versed sine must be obained, for a, the arc, r, the radius, and h, the perpendicular o the chord from the centre of the circle.

For the value of the arc we have (from "Penny Cycl.," \rt. *Segment*) as a close approximation—

$$a = \tfrac{1}{3} (8 f - c).$$

By equation (162.) we have—

$$r = \frac{f^2}{2v};$$

Then—

$$h = r - v,$$

or—

$$h = \frac{f^2}{2v} - v.$$

Substituting these values in equation (169.) we have—

$$M = \frac{1}{2}\left[\frac{1}{3}(8f - c)\frac{f^2}{2v} - c\left(\frac{f^2}{2v} - v\right)\right]. \qquad (171.)$$

This rule is the rule (169.) expanded.

The written rule for equation (169.) may be used, substituting for "*half the arc*," *one sixth of the difference between eight times the chord of half the arc and the chord* (or $\frac{1}{6}$ of 8 times $A\,B$, *Fig.* 298, minus $A\,C$, the chord). Also substitute for "*the radius*," *the square of the chord of half the arc divided by twice the versed sine*. Also, for "*its perpendicular to the centre of the circle*," substitute, *the quotient of the square of the chord of half the arc divided by twice the versed sine, minus the versed sine.*

When the arc is small the curve approximates that of a parabola. In this case the equation for the area of the parabola, which is quite simple, may be used. It is this—

$$M = \frac{2}{3} c\,v. \qquad (172).$$

Or, in *segments* of circles where the *versed sine* is *small* in comparison with the chord, *the area equals approximately two thirds of the chord into the versed sine.*

SECTION XII.—THE ELLIPSE.

451.—Ellipse: Definitions.—Let two lines, PF, PF' (*Fig.* 299), be drawn from any point P to any two fixed points FF', and let the point P move in such a manner that the sum of the two lines, PF, PF', shall remain a constant quantity; then the curve $PMKOGADBP$, traced by P, will be an Ellipse; the two fixed points F, F', the Foci; the point C at

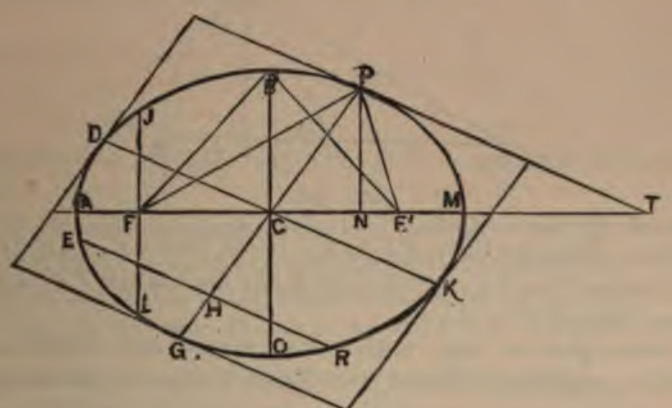


FIG. 299.

the middle of FF', the centre; the line AM drawn through FF' and terminated by the curve, the Major or Transverse Axis; the line BO, drawn through C and at right angles to AM, the Minor or Conjugate Axis; the line GP, drawn through P and C and terminated by the curve, the Diameter to the point P; the line DK drawn through C, parallel with the tangent PT, and terminated by the curve, the diameter Conjugate to PG; the line EHR drawn parallel with DK is a double ordinate to the abscissas GH and HP of the diameter GP ($EH = HR$); the line JL drawn through F at a

right angle to AM and terminated by the curve, the Parameter, or Latus Rectum.

When the point P reaches and coincides with B, the two lines PF and PF' become equal.

The proportion between the major and minor axes depends upon the relative position of F, F', the foci; the nearer these are placed to the extremities of the major axis the smaller will the minor axis be in comparison with the major axis. The nearer F, F' approach C, the centre, the nearer will the minor axis approach the length of the major axis. When F, F' reach and coincide with the centre, the minor axis will equal the major axis, and the ellipse will become a

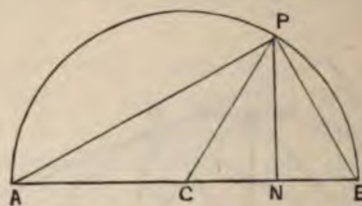


FIG. 300.

circle. Then we have $PF = PF' = BC = AC$. From this we learn $PF + PF' = 2\,AC = AM$; also, when $PF = PF'$, then $PF = BF = AC$.

From this we may, with given major and minor axes, find the position of F and F'. To do this, on B, as a centre, with AC for radius, mark the major axis at F and F'.

452.—Ellipse: Equations to the Curve.—An equation to a curve is an expression containing factors two of which, called co-ordinates, measure the distance to any point in the curve. For example: in a circle it has been shown (*Art.* 443) that PN is a mean proportional to AN and NB. Or, putting $x = AN$, $y = PN$, and $a = AB$, we have—

$$AN : PN :: PN : NB,$$

or—

$$x : y :: y : a - x,$$

or—

$$y^2 = x\,(a - x).$$

This is the equation to the circle having the origin of x and y, the co-ordinates at A, the vertex of the curve. It will be observed that the factors are of such nature in this equation, that it may be employed to measure the distance, rectangularly, to P, wherever in the curve the point P may be located. By this equation the rectangular distance to any and *every* point in the curve may be measured; or, having the curve and one of the lines x or y, the other may be computed.

From this example, the nature and utility of an equation to any curve may be understood. The equation to the ellipse having the origin of co-ordinates at the vertex, is similar to that for the circle. In the form usually given by writers on Conic Sections, it is—

$$y^2 = \frac{b^2}{a^2}(2\,a\,x - x^2), \qquad (173.)$$

in which $a = A\,C$ (*Fig.* 299); $b = B\,C$; x equals $A\,N$, and $y = P\,N$.

If, as before suggested, the foci be drawn towards the centre and finally made to coincide with it, the minor axis would then become equal to the major axis, changing the ellipse into a circle. In this case, the factors a and b in the equation would become equal; and the fraction $\frac{b^2}{a^2}$ would equal $\frac{a^2}{a^2} = 1$, and hence the equation would become—

$$y^2 = 2\,a\,x - x^2,$$

or—

$$y^2 = x\,(2\,a - x);$$

precisely the same as in the equation to the circle above shown. The $2\,a$ of this equation is equivalent to a of the circle; for a in the ellipse represents only half the major axis; while in the equation to the circle a represents the diameter. The relation between the ellipse and the circle is thus shown; indeed, the circle has been said to be an ellipse in its extreme conditions.

453.—Ellipse: Relation of Axis to Abscissas of Axes.—Multiplying equation (173.) by a^2 we have—

$$a^2 y^2 = b^2 (2 a x - x^2),$$

or—

$$a^2 y^2 = b^2 x (2 a - x).$$

These four factors may be put in a proportion, thus—

$$a^2 : b^2 :: x (2 a - x) : y^2,$$

representing—

$$\overline{AC}^2 : \overline{BC}^2 :: AN \times NM : PN^2.$$

Or: The *rectangle* of the two parts into which the ordinate divides the axis major is *in proportion* to the *square of the ordinate*, as the *square of the semi-axis major* is to the *square of the semi-axis minor*.

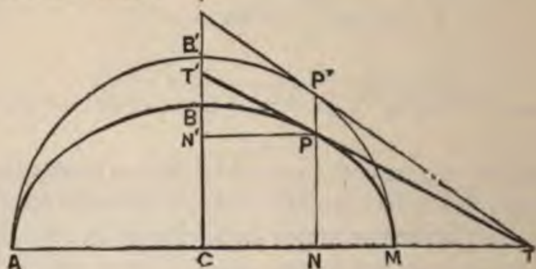


FIG. 301.

It is shown by writers on Conic Sections that this relation is found to subsist, not only with the axes and ordinate, but also between an ordinate to any diameter and the abscissas of that diameter; for example, referring to *Fig.* 299—

$$\overline{CP}^2 : \overline{CD}^2 :: GH \times HP : \overline{EH}^2.$$

If $A B' P' M$ (*Fig.* 301) be a semi-circle, then (*Art.* 443) $\overline{P'N}^2 = AN \times NM$.

Substituting this value of $AN \times NM$ in—

$$\overline{AC}^2 : \overline{BC}^2 :: \overline{P'N}^2 : \overline{PN}^2,$$

we have—

$$AC : BC :: P'N : PN;$$

Or : The *ordinate* in the *circle* is in *proportion* to its *corresponding ordinate* in the *ellipse*, as the *semi-axis major* is to the *semi-axis minor*, or as the *axis major* is to the *axis minor*.

454.—Ellipse: Relation of Parameter and Axes.—The equation to the ellipse when the origin of the co-ordinates is at the centre is, as shown by writers on Conic Sections, thus—

$$a^2 y^2 = a^2 b^2 - b^2 x'^2, \qquad (174.)$$

or—

$$a^2 y^2 = b^2 (a^2 - x'^2).$$

If x' equal CF (*Fig.* 299) then the ordinate will be located at FJ, and—

$$\overline{CF}^2 = \overline{BF}^2 - \overline{BC}^2,$$

$$x'^2 = a^2 - b^2.$$

Then—

$$a^2 - x'^2 = a^2 - (a^2 - b^2),$$

$$= a^2 - a^2 + b^2,$$

$$a^2 - x'^2 = b^2.$$

This is shown also by the figure.

Substituting in the above this value of $a^2 - x'^2$, we have—

$$a^2 y^2 = b^2 b^2 = b^4.$$

From which, taking the square root—

$$a y = b^2,$$

or—

$$a : b :: b : y.$$

Now y, located at FJ, is the semi-parameter; hence we have the semi-minor axis a third proportional to the semi-major axis and the semi-parameter. Or: *The parameter is a third proportional to the two axes of an ellipse.*

455.—Ellipse: Relation of Tangent to the Axes.—Let TT' (*Fig.* 301) be a tangent to P, a point in the ellipse; then, as has been shown by writers on Conic Sections—

$$CN \times CT = \overline{CM}^2,$$

or—

$$CM : CT :: CN : CM.$$

Or: *The semi-major axis is a mean proportional between the abscissa $C N$ and $C T$, the part of the axis intercepted between the centre and the tangent.*

This relation is found also to subsist between the similar parts of the minor axis; for—

$$C N' \times C T' = \overline{C B}^2.$$

This relation affords an easy rule for finding the point T. or T'; for from the above we have—

$$C T = \frac{\overline{C M}^2}{C N};$$

or, putting t for $C T$, we have—

$$t = \frac{a^2}{x'} \qquad (175.)$$

or—

$$t' = \frac{b^2}{y}. \qquad (176.)$$

Since the value of t is not dependent upon y nor upon b, therefore t is constant for all ellipses which may be described upon the same major axis $A M$; and since the circle is an ellipse (*Art.* 452) with equal major and minor axes, therefore rule (175.) is applicable also to a circle, as shown in *Fig.* 301.

The equation (175.) gives the value of $t = C T$. From this deducting $C N = x'$, we have $N T$, the *subtangent*, or—

$$C T - C N = N T,$$

$$t - x' = s;$$

or, substituting for t its value in (175.), we have—

$$s = \frac{a^2}{x'} - x'; \qquad (177.)$$

Or: The *subtangent* to an ellipse *equals the difference between the quotient of the square of the semi-major axis divided by the abscissa, and the abscissa;* the *origin* of the co-ordinates being at C, the *centre*.

456.—Ellipse: Relation of Tangent with the Foci.—Let the two lines from the foci to P (*Fig.* 302), any point in the ellipse, be extended beyond P. With the radius PF' de-

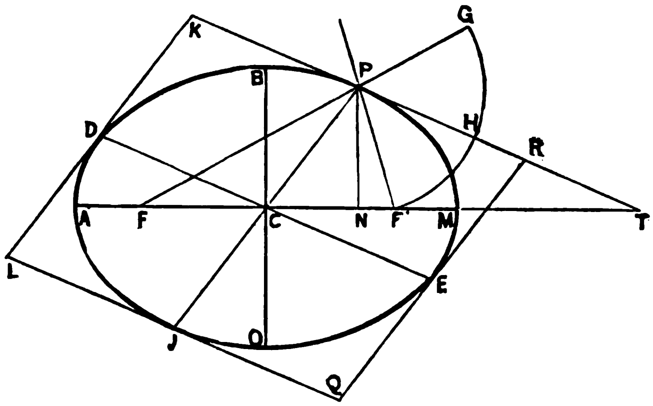


FIG. 302.

scribe from P the arc $F'G$, and bisect it in H. Then the line PT, drawn through H, will be a tangent to the ellipse at P.

This has been shown by writers on Conic Sections. The construction here shown affords a ready method of drawing a tangent. And from the principle here given we learn that a tangent makes equal angles with the lines from the tangential point to the two foci.

For, because $GH = HF'$, we have the angle $F'PH = HPG$. The angles HPG and KPF are opposite, and hence (*Art.* 344) are equal; and since the two triangles $F'PH$ and KPF are each equal to HPG, therefore $F'PH$ and KPF are equal to each other. Or: A *tangent* to an ellipse *makes equal angles with the two lines drawn from the point of tangency to the two foci.*

Experience shows that light shining from one focus is reflected from the ellipse into the other focus. It is for this reason that the two points F and F' are called *foci*, the plural of *focus*, a fireplace.

457.—Ellipse: Relation of Axes to Conjugate Diameters.—Parallel with KT (*Fig.* 302) let DE be drawn through

C, the centre, and $L\,Q$ through J, one end of the diameter from the point P. Parallel with this diameter $P\,J$ draw $L\,K$ and $Q\,R$ through the extremities of the diameter $D\,E$. Then $D\,E$ is a diameter conjugate to the diameter $P\,J$, and $K\,R$, $R\,Q$, $Q\,L$, and $L\,K$ are tangents at the extremities of these conjugate diameters.

Now it is shown by writers on Conic Sections (*Fig.* 302) that—

$$\overline{A\,C}^2 + \overline{B\,C}^2 = \overline{D\,C}^2 + \overline{P\,C}^2,$$

or—

$$a^2 + b^2 = a'^2 + b'^2;$$

Or: The *sum* of the *squares* of the two *axes* equals the *sum* of the *squares* of any two *conjugate diameters*.

From this it is also shown that the area of the parallelogram $K\,C$ equals the rectangle $A\,C \times B\,C$; or, that a *parallelogram* formed by *tangents* at the extremities of *any two conjugate diameters is equal to the rectangle of the axes.*

458.—Ellipse; Area.—Let E equal the area of an ellipse: A the area of a circle, of which the radius a equals the semi-major axis of the ellipse, and let b equal the semi-minor axis. Then it has been shown that—

$$E : A :: b : a,$$

or—

$$E = A\,\frac{b}{a}.$$

The area of a circle (*Art.* 448) is—

$$A = \tfrac{1}{2}\,\pi\,d\,r = \pi\,r^2,$$

and when the radius equals a—

$$A = \pi\,a^2,$$

This value of A, substituted in the above equation, gives—

$$E = \pi\,a^2\,\frac{b}{a},$$

$$E = \pi\,a\,b. \qquad (178.)$$

Or: The *area* of an *ellipse* equals $3 \cdot 14159\frac{1}{4}$ *times the product of the semi-axes; or* $0 \cdot 7854$ *times the product of the axes.*

459.—Ellipse: Practical Suggestions.—In order to describe the curve of an ellipse, it is essential to have the two axes; or, the major axis and the parameter; or, the major axis and the focal distance..

If the two axes are given, then with the semi-major axis for radius, from B (*Fig.* 299) as centre an arc may be made at F and F', the foci; and then the curve may be described by any of the various methods given at *Arts.* 548 to 552.

If the major axis only and the parameter are given, then (*Art.* 454) since—

$$b^2 = a\,y,$$

we have—

$$b = \sqrt{a\,y}. \qquad (179.)$$

Or: The *semi-minor axis* of an *ellipse equals the square root of the product of the semi-major axis into the semi-parameter.* Then, having both of the axes, proceed as before.

If the major axis and the focal distance are given, or the location of the foci; then with the semi-major axis for radius and from the focal points as centres, describe arcs cutting each other at B and O (*Fig.* 299). The intersection of the arcs gives the limit to $B\,O$, the minor axis. With the two axes proceed as before. Points in the curve may be found by computing the length of the ordinates, and then the curve drawn by the side of a flexible rod bent to coincide with the several points.

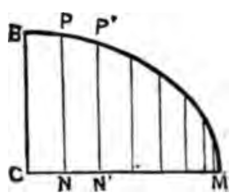


FIG. 303.

For example, let it be required to find points in the curve of an ellipse, the axes of which are 12 and 20 feet; or

the semi-axes 6 and 10 feet, or $6 \times 12 = 72$ inches, and $10 \times 12 = 120$ inches.

Fix the positions of the points $N N'$, etc., along the semi-major axis CM (*Fig.* 303) at any distances apart desirable. It is better to so place them that the ordinates when drawn shall divide the curve BPM into parts approximately equal. If CM be divided into eight parts as shown, these parts measured from C will be well graded if made equal severally to the following decimals multiplied by CM. In this case. $CM = 120$; therefore—

$$\begin{aligned}
CN &= 120 \times 0{\cdot}3 &&= 36{\cdot} = x' \\
CN' &= 120 \times 0{\cdot}475 &&= 57{\cdot} = x' \\
CN'' &= 120 \times 0{\cdot}625 &&= 75{\cdot} = x' \\
\text{Etc.,} &= 120 \times 0{\cdot}75 &&= 90{\cdot} = x' \\
& \quad\ 120 \times 0{\cdot}85 &&= 102{\cdot} = x' \\
& \quad\ 120 \times 0{\cdot}925 &&= 111{\cdot} = x' \\
& \quad\ 120 \times 0{\cdot}975 &&= 117{\cdot} = x' \\
& \quad\ 120 \times 1{\cdot}0 &&= 120{\cdot} = x'.
\end{aligned}$$

The equation of the ellipse having the origin of co-ordinates at the centre (*Art.* 454) is—

$$a^2 y^2 = b^2 (a^2 - x'^2),$$

or, dividing by a^2—

$$y^2 = \frac{b^2}{a^2}(a^2 - x'^2),$$

or—

$$y = \sqrt{\frac{b^2}{a^2}(a^2 - x'^2)},$$

or—

$$y = \frac{b}{a}\sqrt{a^2 - x'^2}; \qquad (180.)$$

in which a and b represent the semi-axes. Substituting for these their values in this case, we have—

$$y = \frac{72}{120}\sqrt{120^2 - x'^2},$$

$$y = 0{\cdot}6\sqrt{14400 - x'^2}.$$

Now, substituting in this equation the several values of x'^2 successively, the values of the corresponding ordinates will be obtained. For example, taking 36, the first value of x', as above, we have—

$$y = 0.6\sqrt{14400 - 36^2}$$
$$y = 68.684;$$
$$y = 0.6\sqrt{14400 - 57^2}$$
$$y = 63.359;$$

and so in like manner compute the others.

The ordinates for this case are as follows, viz.:

When $x' = 0$, $y = 72.0$
" $x' = 36$, $y = 68.684$
" $x' = 57$, $y = 63.359$
" $x' = 75$, $y = 56.205$
" $x' = 90$, $y = 47.624$
" $x' = 102$, $y = 37.928$
" $x' = 111$, $y = 27.358$
" $x' = 117$, $y = 15.999$
" $x' = 120$, $y = 0.0$.

The computation of these ordinates is accomplished easily by the help of a table of square roots and of logarithms.

For example, the work for one ordinate is all comprised within the following, viz.:

$$y = 0.6\sqrt{14400 - 36^2} = 68.684.$$

$$\begin{aligned} 120^2 &= 14400 \\ 36^2 &= \underline{\ 1296} \\ 13104 &= \underline{4.1174039} \\ \text{Half} &= 2.0587020 \\ 0.6 &= \underline{\bar{9}.7781513} \\ 68.684 &= 1.8368533. \end{aligned}$$

The logarithm of $13104 = 4.1174039$. The half of this is the logarithm of the square root of 13104. To the half logarithm add the logarithm of 0.6; the sum is the logarithm of 68.684 found in the table (see *Art.* 427).

SECTION XIII.—THE PARABOLA.

460.—Parabola: Definitions.—The parabola is one of the most interesting of the curves derived from the sections of a cone. The several curves thus produced are as follows: When cut parallel with its base the outline is a *circle*; when the plane passes obliquely through the cone, it is an *ellipse*; when the plane is parallel with the axis, but not in the axis, it is a *hyperbola*; while that which is produced by

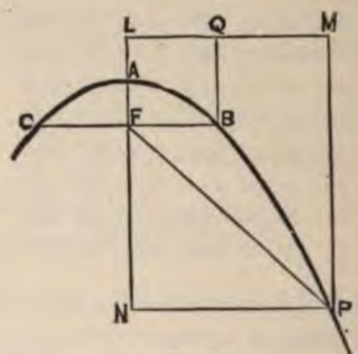


FIG. 304.

a plane cutting it parallel with one side of the cone is a *parabola*.

Let the lines LM and LN (*Fig.* 304) be at right angles; draw CFB parallel with LM; make $LQ = LF$; draw QB parallel with LF; then $FB = BQ$. Now let the line AL move from FL, but remain parallel with it, and as it moves let it gradually increase in length in such manner that the point A shall constantly be equally distant from the line LM and from the point F. Then ABP, the curve described by the point A, will be a semi-parabola. For example, the lines FB and BQ are equal; the lines FP and PM are equal, and so of lines similarly drawn from any point in the curve ABP. Let PN be drawn parallel with LM; then for the

point P, $A\,N$ is the *abscissa* and $N\,P$ its *ordinate* (see *Art.* 452).

The double ordinate $C\,B$ drawn through F, the focus, is the *parameter*. $A\,F$ is the focal distance. A is the vertex of the curve. The line $L\,M$ is the *directrix*.

461.—Parabola : Equation to the Curve.—In *Fig.* 304 $F\,P\,N$ is a right-angled triangle, therefore—

$$\overline{NP}^2 = \overline{FP}^2 - \overline{FN}^2;$$

but— $$FP = MP = LN = A\,N + A\,L;$$

and— $$FN = A\,N - A\,F.$$

Therefore—

$$\overline{NP}^2 = \overline{A\,N + A\,L}^2 - \overline{A\,N - A\,F}^2;$$

or— $$y^2 = (x + \tfrac{1}{2}p)^2 - (x - \tfrac{1}{2}p)^2,$$

p being put for the distance $L\,F = FB$ (see *Art.* 452). As in *Arts.* 412 and 413, we have—

$$(x + \tfrac{1}{2}p)^2 = x^2 + p\,x + \tfrac{1}{4}p^2$$
$$(x - \tfrac{1}{2}p)^2 = x^2 - p\,x + \tfrac{1}{4}p^2$$
$$y^2 = 2\,p\,x \qquad (181.)$$

by subtraction. This is the usual equation to the parabola, in which we have the rule: The *square* of the *ordinate* equals the *rectangle* of the corresponding *abscissa* with the *parameter*.

From (181.) we have—

$$x : y :: y : 2p,$$

or: The *parameter* is a *third proportional* to the *abscissa* and its corresponding ordinate.

462.—Parabola : Tangent.—From M, any point in the directrix, draw a line to F, the focus (*Fig.* 305); bisect $M\,F$ in R, and through R draw $U\,T$ perpendicular to $M\,F$, then the line $T\,U$ will be a tangent to the curve. For, draw $M\,D$

perpendicular to $L V$, and from P, the point of its intersection with the line $T U$, draw a line to F, the focus; then, because $R P$ is a perpendicular from the middle of $M F$, $M P F$ is an isosceles triangle, and therefore the lines $M P$ and $F P$ are equal, or the point P is equidistant from the focus and from the directrix, and therefore is a point in the curve.

To show that the line $T U$ touches the curve but does not pass through it, take U, any point in the line $T U$, other than

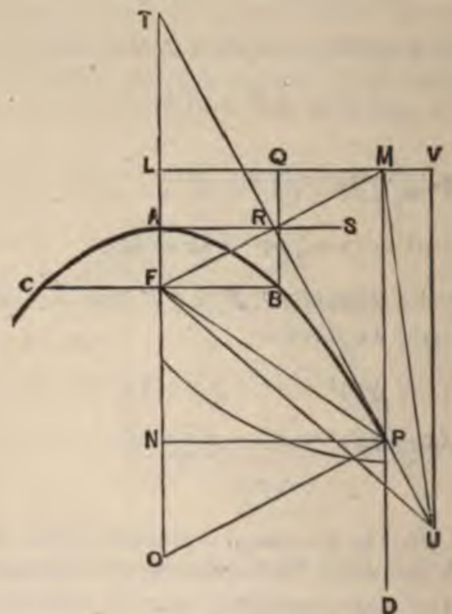


FIG. 305.

the point P; join U to M and to F. Then, since U is a point in the line $T U$, $M U F$, for reasons above given, is an isosceles triangle; from U draw $U V$ perpendicular to $L V$. Now, if the point U be also in the curve, the lines $U V$ and $U F$, by the law of the curve, must be equal; but $U F$, as before shown, is equal to $U M$, a line evidently longer than $U V$; therefore, it is evident that the point U is not in the curve. A similar absurd result will be reached if any other point than the point U in the line $U T$ be assigned, excepting the

point P. Therefore the line TP touches the curve in only one point, P; hence it is a tangent.

Parallel with LV, from A, draw AS, the *vertical tangent*. Now AS bisects MF or intersects it in the point R. For the two right-angled triangles FLM and FAR are homologous; and because $FA = AL$, by construction, therefore $FR = RM$.

Or: The *vertical tangent bisects all lines which can be drawn from the focus to the directrix.*

The lines PF and FT are equal; for the lines MP and NT being parallel, therefore the alternate angles MPT and NTP are equal (*Art.* 345); and because the line PT bisects MF, the base of an isosceles triangle, therefore the angles MPT and FPT are equal. We thus have the two angles NTP and FPT each equal to the angle MPT; therefore the two angles NTP and FPT are equal to each other; hence the triangle PFT is an isosceles triangle, having the points T and P equidistant from F, the focus.

Also because the line MF is perpendicular to PT, therefore the line MF bisects the tangent PT in the point R. And because $TR = RP$, therefore, comparing triangles TRF and TPO, $TF = FO$.

The opposite angles MPT and UPD made by the two intersecting lines UT and MD (*Art.* 344) are equal, and since the angles MPT and FPT are equal, as before shown, therefore the angles FPT and UPD are equal.

It is because these two angles are equal, that, in reflectors, rays of light and heat proceeding from F, the focus, are reflected from the parabolic surface in lines parallel with the axis.

For an equation expressing the value of the tangent, we have—

$$\overline{TP}^2 = \overline{TN}^2 + \overline{NP}^2,$$

$$t^2 = (2x)^2 + y^2,$$

$$t = \sqrt{4x^2 + y^2}. \qquad (182.)$$

Or: The *tangent* to a *parabola equals* the *square root of the sum of four times the square of the abscissa added to the square of the ordinate.*

463.—Parabola: Subtangent.—The line TN (*Fig.* 305), the portion of the axis intercepted between T, the point of intersection of the tangent, and N, the foot of a perpendicular to the axis from P, the point of contact, is the *subtangent.* The subtangent is bisected by the vertex, or $TA = AN$. For, the two triangles TRA and TPN are homologous; and, as shown in the last article, the line MF bisects PT in R; or $TR = RP$.

Therefore, we have—

$$TR : TA :: TP : TN,$$

$$TR \times TN = TA \times TP,$$

but—

$$TR = \tfrac{1}{2}\, TP;$$

therefore—

$$\tfrac{1}{2}\, TP \times TN = TA \times TP,$$

$$\tfrac{1}{2}\, TN = TA.$$

Or: The *subtangent* of a *parabola* is *bisected* by the *vertex*; or is equal to *twice* the *abscissa.*

And because of the similarity of the two triangles TRA and TPN, as above shown, we have—

$$NP = 2\, AR,$$

$$y = 2\, AR.$$

Or: The *ordinate* equals *twice* the *vertical tangent.*

464.—Parabola: Normal and Subnormal.—The line PO (*Fig.* 305) perpendicular to PT, is the *normal* and NO, the part of the axis intercepted between the normal and the ordinate, is the *subnormal.* For the normal, from similar triangles, we have—

$$TN : NP :: TP : PO,$$

$$PO = \frac{NP \times TP}{TN},$$

$$PO = \frac{y\,t}{2\,x}.$$

Or: The *normal* equals the *rectangle* of the *ordinate* and *tangent*, *divided* by *twice* the *abscissa*.

The subnormal equals half the parameter. For (181.)—

$$y^2 = 2\,p\,x,$$

or—

$$NP^2 = 2\,\overline{FB} \cdot \overline{AN}.$$

Dividing by $2\,\overline{AN}$ gives—

$$FB = \frac{\overline{NP}^2}{2\overline{AN}}. \qquad \text{(A.)}$$

In the similar triangles (*Art.* 443) OPN and PTN, we have—

$$NO : NP :: NP : NT,$$

$$NO = \frac{\overline{NP}^2}{\overline{NT}}.$$

As shown in the previous article, $NT = 2\,AN$; therefore—

$$NO = \frac{\overline{NP}^2}{2\overline{AN}}. \qquad \text{(B.)}$$

Comparing equations (A.) and (B.), we have—

$$NO = FB.$$

Or: The *subnormal* of a *parabola* equals *half the parameter*, a constant quantity for the subnormal to all points of the curve.

465.—Parabola: Diameters.— In the parabola BAC (*Fig.* 306), PD, a diameter (a line parallel with the axis) to the point P, is in proportion to $BD \times DC$, the rectangle of the two parts into which the base of the parabola is divided by the diameter.

This may be shown in the following manner:

$$DP = EN = EA - NA. \qquad \text{(A.)}$$

For EA we have, taking the co-ordinates, for the po
C, (181.)—

$$y^2 = 2px,$$

or—

$$\frac{y^2}{2p} = x$$

or—

$$\frac{\overline{EC}^2}{2p} = EA. \qquad \text{(B.}$$

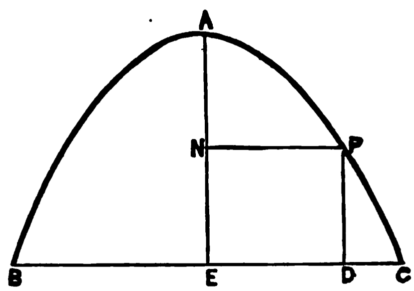


FIG. 306.

For NA we have, taking the co-ordinates to the p
P, (181.)—

$$y^2 = 2px,$$

or—

$$\frac{y^2}{2p} = x,$$

or—

$$\frac{\overline{NP}^2}{2p} = NA. \qquad \text{((}$$

Using these values (B.) and (C.) in (A.), we have—

$$DP = EA - NA,$$

$$DP = \frac{\overline{EC}^2}{2p} - \frac{\overline{NP}^2}{2p} = \frac{\overline{EC}^2 - \overline{NP}^2}{2p}.$$

If l be put for BC and n for DC, then—

$$NP = EC - DC = \tfrac{1}{2}l - n,$$

nd—
$$DP = \frac{(\frac{1}{2}l)^2 - (\frac{1}{2}l - n)^2}{2p},$$

hen (*Art.* 413)—
$$DP = \frac{\frac{1}{4}l^2 - (\frac{1}{4}l^2 - ln + n^2)}{2p},$$

r (*Art.* 415)—
$$DP = \frac{\frac{1}{4}l^2 - \frac{1}{4}l^2 + ln - n^2}{2p},$$

$$DP = \frac{ln - n^2}{2p},$$

$$DP = \frac{n(l - n)}{2p},$$

$$DP = \frac{DC \times BD}{2p}.$$

Now, since $2p$, the parameter, is constant, we have DP, he diameter, in proportion to $DC \times BD$, the two parts of he base.

Putting d for the diameter, we have—

$$d = \frac{n(l - n)}{2p}. \qquad (183.)$$

Or: The *diameter* of a *parabola* equals the *quotient* obtained y *dividing* the *rectangle*—formed by the two parts into which he diameter divides the base—by the *parameter*.

It has been shown by writers on Conic Sections that a liameter, PJ (*Fig.* 307), to any point P in a parabola bisects ll chord lines, SG, DE, etc., drawn parallel with the tangent to the point P; the diameter being parallel with the axis f the parabola.

466.—Parabola: Elements.—From any given parabola, o find the axis, tangent, directrix, parameter and focus, lraw any two parallel lines or chords, SG and DE (*Fig.* 307), and bisect them in H and J; through these points lraw JP; then JP will be a diameter of the parabola—a

line parallel with the axis. Perpendicular to PJ draw the double ordinate PQ and bisect it in N; through N and parallel with PJ draw TO, cutting the curve in A; then TO will be the axis. Make $AT = AN$, join T and P; then TP will be the tangent to the point P; from P draw PO perpendicular to PT; then PO will be the normal, and NO the subnormal.

With NO for radius, from N as a centre, describe the quadrant OR; draw RC parallel with AO, cutting the curve

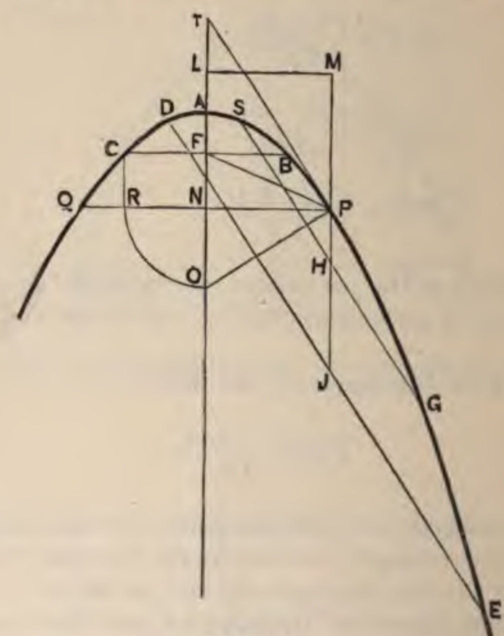


FIG. 307.

in C; from C draw CB perpendicular to AO, cutting AO in F; then F will be the focus and CB the parameter. Make $AL = AF$; draw LM perpendicular to TO; then LM will be the directrix. Extend PJ to meet LM at M; join P and F; then, if the work has been properly performed, FP will equal MP.

467.—Parabola: Described Mechanically.— With NP (*Fig.* 308) a given base, and NA a given height, set perpen-

dicularly to the base, extend NA beyond A, and make AT equal to NA; join T and P; from P perpendicularly to TP draw PO; bisect ON in R; make AL and AF each equal to NR; through L, perpendicular to LO, draw DE, the directrix.

Let the ruler $CDES$ be laid to the line DE, then with JGH, a set-square, the curve may be described in the following manner:

Placing the square against the ruler and with its edge

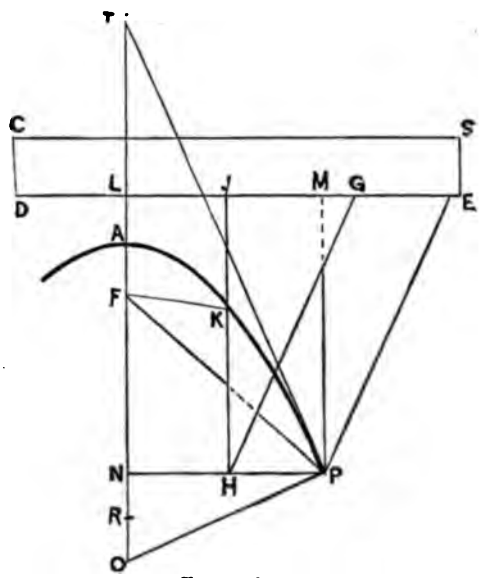


FIG. 308.

JH coincident with the line MP, fasten to it a fine cord on the edge PE, and extend it from P to F, the focus, and secure it to a pin fixed in F. The cord FP will equal the edge MP. To describe the curve set the triangle JGH at MPE, slide it gently along the ruler towards D, keeping the edge JG in contact with the ruler, and, as the square is moved, keep the cord stretched tight, holding for this purpose a pencil, as at K, against the cord. Thus held, as the square is moved the pencil will describe the curve. That this operation will produce the true curve we have but to

consider that at all points the line FK will equal KJ, which is the law of the curve (*Art.* 460).

468.—Parabola: Described from Points.—With given base, NP (*Fig.* 309), and given height, AN, to find the points D, F, M, etc., and describe the curve. Make AT equal to AN (*Art.* 462); join T and P; perpendicular to TP draw

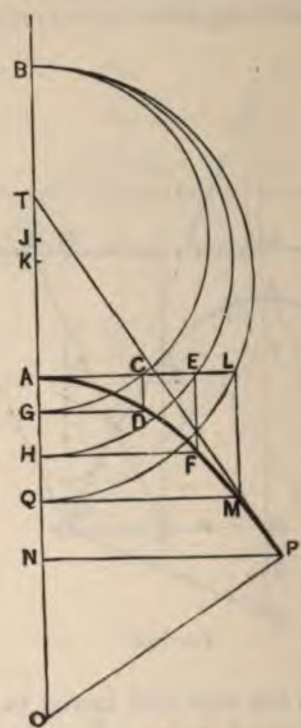


FIG. 309.

PO; make AB equal to twice NO; take G, any point in the axis AO, and bisect BG in J; on J as a centre describe the semi-circle BCG cutting AL, a perpendicular to BO in C; on AC and AG complete the rectangle $ACDG$. Then D is a point in the curve. Take H, another point in the axis; bisect BH in K; on K as a centre describe the semi-circle BEH cutting AL in E; this by EF and HF, gives F, an-

other point in the curve; in like manner procure M, and as many other points as may be desired. This simple and accurate method of obtaining points in the curve depends upon two well-established equations; one, the equation to the parabola, and the other, the equation to the circle. The line $G D$, an ordinate in the parabola, is equal to $A C$, an ordinate in the circle $B C G$; $A G$, the abscissa of the parabola, is also the abscissa of the circle; in which we have (*Art.* 443)—

$$A G : A C :: A C : A B,$$

$$x : y :: y : a - x,$$

$$y^2 = x (a - x).$$

For the parabola, we have (181.)—

$$y^2 = 2 p x.$$

Comparing these two equations, we have—

$$x (a - x) = 2 p x,$$

$$a - x = 2 p,$$

or—

$$B G - A G = 2 p.$$

By construction $A B$ equals $2 N O$, or twice the subnormal; the subnormal (*Art.* 464) equals half the parameter. Hence, twice the subnormal equals the parameter—equals $2 p$. Therefore, the method shown in *Fig.* 309 is correct.

469.—Parabola: Described from Arcs.—Let $N P$ (*Fig.* 310) be the given base and $A N$ the given height of the parabola. Make $A T$ (*Art.* 462) equal $A N$. Join T to P; draw $P O$ perpendicular to $P T$; bisect $N O$ in R; make $A L$ and $A F$ each equal to $N R$; then $L M$, drawn perpendicular to $T O$, will be the directrix. Parallel to $L M$ draw the lines $B D$, $C E$, etc., at discretion. Then with the distance $B L$ for

radius, and on F as a centre, mark the line $B\ D$ with an arc; the intersection of the arc and the line will be a point in the curve (*Art.* 460). Again, with $C\ L$ for radius and on F as a centre, mark the line $C\,E$ with an arc; this gives another point in the curve. In like manner, mark each horizontal

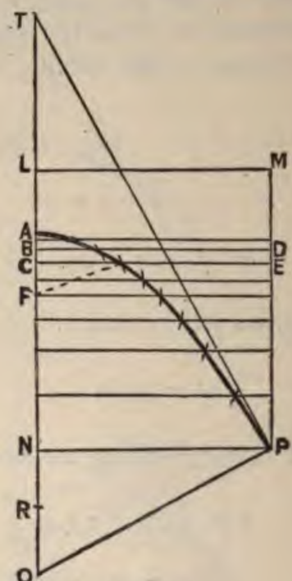


FIG. 310.

line from F as a centre by a radius equal to the perpendicular distance between that line and $L\ M$, the directrix. Then a curve traced through the points of intersection thus obtained will be the required parabola.

470.—Parabola: Described from Ordinates.—With a given base, $N\ P$ (*Fig.* 311), and height, $A\ N$, a parabola may be drawn through points J, H, G, etc., which are the extremities of the ordinates $B\ J$, $C\ H$, $D\ G$, etc.; the lengths of the ordinates being computed from the equation to the curve, (181.)—

$$y^2 = 2\,p\,x.$$

For any given parabola, in base and height, the value of

p may be had by dividing both members of the equation by $2x$; by which we have—

$$p = \frac{y^2}{2x} = \frac{\overline{NP}^2}{2AN}. \qquad \text{(A.)}$$

from which, NP and AN being known, p may be computed.

With the value of p, a constant quantity, determined, the equation is rendered practicable. For, taking the square root of each member of equation (181.), we have—

$$y = \sqrt{2px}. \qquad \text{(B.)}$$

which by computation will produce the value of y, for every assigned value of x, as AB, AC, AD, etc.

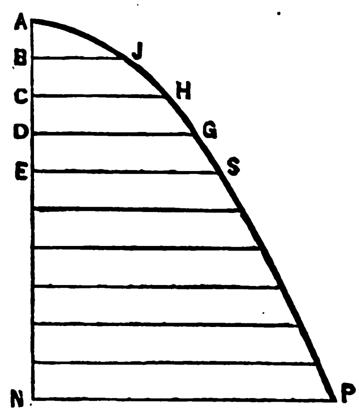


FIG. 311.

As an example: let it be required to compute the ordinates in a parabola in which the base, NP, equals 8 feet, and the height, AN, equals 10 feet. With these values equation (A.) as above becomes—

$$p = \frac{NP^2}{2AN} = \frac{8^2}{2 \times 10} = \frac{64}{20} = 3 \cdot 2;$$

$$2p = 6 \cdot 4.$$

Then, with this value in (B.) as above, we have, for each ordinate—

$$y = \sqrt{6 \cdot 4 x}.$$

In order to assign values to x, let $A N$ be divided into any number of parts at B, C, D, etc., say, for convenience in this example, in ten equal parts; then each part will equal one foot, and we shall have the consecutive values of $x = 1$, 2, 3, 4, etc., to 10, and the corresponding values of y will be as follows. When—

$$x = 1,\ y = \sqrt{6 \cdot 4 \times 1} = \sqrt{6 \cdot 4} = 2 \cdot 5297 = B J,$$

$$x = 2,\ y = \sqrt{6 \cdot 4 \times 2} = \sqrt{12 \cdot 8} = 3 \cdot 5777 = C H,$$

$$x = 3,\ y = \sqrt{6 \cdot 4 \times 3} = \sqrt{19 \cdot 2} = 4 \cdot 3818 = D G,$$

$$x = 4,\ y = \sqrt{6 \cdot 4 \times 4} = \sqrt{25 \cdot 6} = 5 \cdot 0596 = E S,$$

$$x = 5,\ y = \sqrt{6 \cdot 4 \times 5} = \sqrt{32 \cdot} = 5 \cdot 6569 = \text{(etc.)},$$

$$x = 6,\ y = \sqrt{6 \cdot 4 \times 6} = \sqrt{38 \cdot 4} = 6 \cdot 1968 =$$

$$x = 7,\ y = \sqrt{6 \cdot 4 \times 7} = \sqrt{44 \cdot 8} = 6 \cdot 6933 =$$

$$x = 8,\ y = \sqrt{6 \cdot 4 \times 8} = \sqrt{51 \cdot 2} = 7 \cdot 1554 =$$

$$x = 9,\ y = \sqrt{6 \cdot 4 \times 9} = \sqrt{57 \cdot 6} = 7 \cdot 5895 =$$

$$x = 10,\ y = \sqrt{6 \cdot 4 \times 10} = \sqrt{64 \cdot} = 8 \cdot 0 \quad = N P.$$

With these values of y, respectively, set on the corresponding horizontal lines $B J$, $C H$, $D G$, $E S$, etc., points in the curve J, H, G, S, etc., are obtained, through which the curve may be drawn. The decimals above shown are the decimals of a foot; they may be changed to inches and decimals of an inch by multiplying each by 12. For example: $12 \times 0 \cdot 5297 = 6 \cdot 3564$ equals 6 inches and the decimal $0 \cdot 3564$ of an inch, which equals nearly $\frac{3}{8}$ of an inch.

Near the top of the curve, owing to its rapid change in direction and to the approximation of the direction of the curve to a parallel with the direction of the ordinates, it is

desirable to obtain points in the curve more frequent than those obtained by dividing the axis into *equal* parts.

Instead, therefore, of dividing the axis into equal parts, it is better to divide it into parts made gradually smaller toward the apex of the curve—or, to obtain points for this part of the curve as shown in the following article.

471.—Parabola: Described from Diameters.— Let $E\,C$ (*Fig.* 312) be the given base and $A\,E$ the given height, placed perpendicularly to $E\,C$. Divide $E\,C$ in several parts at pleasure, and from the points of division erect perpendiculars to $E\,C$. The problem is to compute the length of these diameters, as $D\,P$, and thereby obtain points in the curve, as at P. For this purpose we have equation (183.), which gives

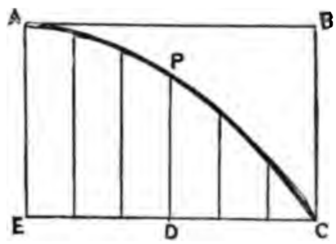


FIG. 312.

the length of the diameters, and in which n equals $D\,C$ (*Fig.* 312), l equals twice $E\,C$, and p equals half the parameter of the curve. The value of p is given in equation (A.), (*Art.* 470), in which y equals $E\,C$ (*Fig.* 312), and x equals $A\,E$. Substituting these symbols in equation (A.), we have—

$$p = \frac{y^2}{2\,x} = \frac{\overline{E\,C}^2}{2 \times A\,E} = \frac{b^2}{2\,h},$$

where $b = E\,C$, the base, and $h = A\,E$, the height. For p, substituting this, its value, in equation (183.), we have—

$$d = \frac{n\,(l-n)}{2\,p} = \frac{n\,(l-n)}{2 \cdot \frac{b^2}{2\,h}}$$

$$d = \frac{h\,n\,(2\,b-n)}{b^2}. \qquad (184.)$$

As an example: let it be required in a parabola in which the base equals 12 feet and the height 8 feet, to compute the length of several diameters, and through their extremities describe the curve. Then h will equal 8, and b 12.

If the base be divided into 6 equal parts, as in *Fig.* 312, each part will equal 2 feet. Then we have—

$$\frac{h}{b^2} = \frac{8}{12^2} = \frac{8}{144} = \frac{1}{18},$$

and—

$$d = \frac{h}{b^2} n (2b - n),$$

$$d = \frac{n(24 - n)}{18}.$$

In this equation, substituting the consecutive values of n, we have, when—

$$n = 0, \quad d = \frac{0 \times 24}{18} = 0$$

$$n = 2, \quad d = \frac{2 \times 22}{18} = 2 \cdot 444$$

$$n = 4, \quad d = \frac{4 \times 20}{18} = 4 \cdot 444$$

$$n = 6, \quad d = \frac{6 \times 18}{18} = 6 \cdot$$

$$n = 8, \quad d = \frac{8 \times 16}{18} = 7 \cdot 111$$

$$n = 10, \quad d = \frac{10 \times 14}{18} = 7 \cdot 777$$

$$n = 12, \quad d = \frac{12 \times 12}{18} = 8 \cdot 0$$

The several diameters, as PD, in *Fig.* 312, may now be made equal respectively to these computed values of d, and the curve traced through their extremities.

472.—Parabola: Area.—From (181.), the equation to the parabola, and by the aid of the calculus, it has been shown that the area of a parabola is equal to two thirds of the circumscribing rectangle. For example: if the height, AE (*Fig.* 312), equals 8 feet, and EC, the base, equals 12 feet, then the area of the part included within the figure $APCEA$ equals $\frac{2}{3}$ of $8 \times 12 = \frac{2}{3} \times 96 = 64$ feet; or, it is equal to $\frac{2}{3}$ of the rectangle $ABCE$.

SECTION XIV.—TRIGONOMETRY.

473.—Right-Angled Triangles: The Sides.—In right-angled triangles, when two sides are given, the third side may be found by the relation of equality which exists of the squares of the sides (*Arts.* 353 and 416). For example,

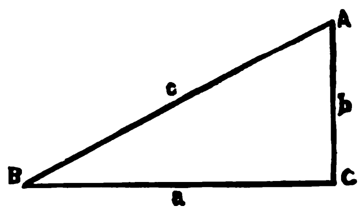


FIG. 313.

if the sides a and b (*Fig.* 313) are given, c, the third side, may be computed from equation (115.)—

$$c^2 = b^2 + a^2.$$

Extracting the square root, we have—

$$c = \sqrt{b^2 + a^2}.$$

When the hypothenuse and one side are given, by transposition of the factors in (115.), we have—

$$a^2 = c^2 - b^2;$$

$$a = \sqrt{c^2 - b^2}; \qquad \text{(A.)}$$

or—

$$b^2 = c^2 - a^2;$$

$$b = \sqrt{c^2 - a^2}. \qquad \text{(B.)}$$

Owing to the factors being involved to the second power in this expression, the labor of computation is greater than that in a more simple method, which will now be shown.

In equation (A.) or (B.) the factors under the *radical* may be simplified. By equation (114.) we have—

$$c^2 - b^2 = (c + b)\,(c - b).$$

Therefore, equation (A.) becomes—

$$a = \sqrt{(c + b)\,(c - b)},$$

a form easy of solution.

For example: let c equal 29·732 and b equal 13·216, then we have—

$$\begin{array}{rr} & 29{\cdot}732 \\ & 13{\cdot}216 \\ \hline \text{The sum} = & 42{\cdot}948 \\ \text{The difference} = & 16{\cdot}516 \end{array}$$

By the use of a table of logarithms (*Art.* 427) the problem may be easily solved; thus—

$$\begin{array}{lr} \text{Log. } 42{\cdot}948 = & 1{\cdot}6329429 \\ 16{\cdot}516 = & 1{\cdot}2179049 \\ \hline \text{To get the square root— } 2\,) & 2{\cdot}8508478 \\ \hline a = 26{\cdot}6332 = & 1{\cdot}4254239 \end{array}$$

This method is applicable to the sides of a triangle, only; for the hypothenuse it will not serve. The length of the hypothenuse as well as that of either side may, however, be obtained by proportion; provided a triangle of known dimensions and with like angles be also given.

For example: in *Fig.* 314, in which the two sides a and b are known, let it be required to find c, the hypothenuse.

Draw the line DE parallel with AC, then the two triangles BDE and BAC are homologous; consequently their

corresponding sides are in proportion (*Art.* 361). Hence, if d equals unity, we have—

$$d : f :: a : c,$$

$$= af,$$

from which, when a and f are known, c is obtained by simple multiplication.

474.—Right-Angled Triangles: Trigonometrical Tables.—To render the simple method last named available, the lengths of d, e and f (*Fig.* 314) have been computed for triangles of all possible angles, and the results arranged in

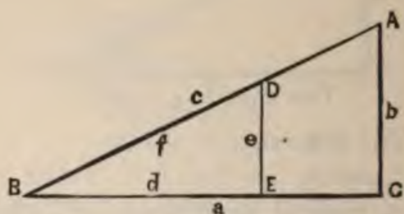


FIG. 314.

tables, termed Trigonometrical Tables. The lines d, e, and f, are known as *sines*, *cosines*, *tangents*, *cotangents*, etc., as shown in *Fig.* 315—where AB is the radius of the circle BCH. Draw a line AF, from A, through any point, C, of the arc BG. From C draw CD perpendicular to AB; from B draw BE perpendicular to AB; and from G draw GF perpendicular to AG.

Then, for the angle FAB, when the radius AC equals unity, CD is the *sine*; AD the *cosine*; DB the *versed sine*; BE the *tangent*; GF the *cotangent*; AE the *secant*; and AF the *cosecant*.

But if the angle be larger than one right angle, yet less than two right angles, as BAH, extend HA to K and EB to K, and from H draw HJ perpendicular to AJ.

Then, for the angle $B A H$, when the radius $A H$ equals unity, $H J$ is the *sine;* $A J$ the *cosine;* $B J$ the *versed sine;* $B K$ the *tangent;* and $A K$ the *secant.*

When the number of degrees contained in a given angle is known, the value of the *sine, cosine, etc.*, corresponding to that angle, may be found in a table of Natural Sines, Co-

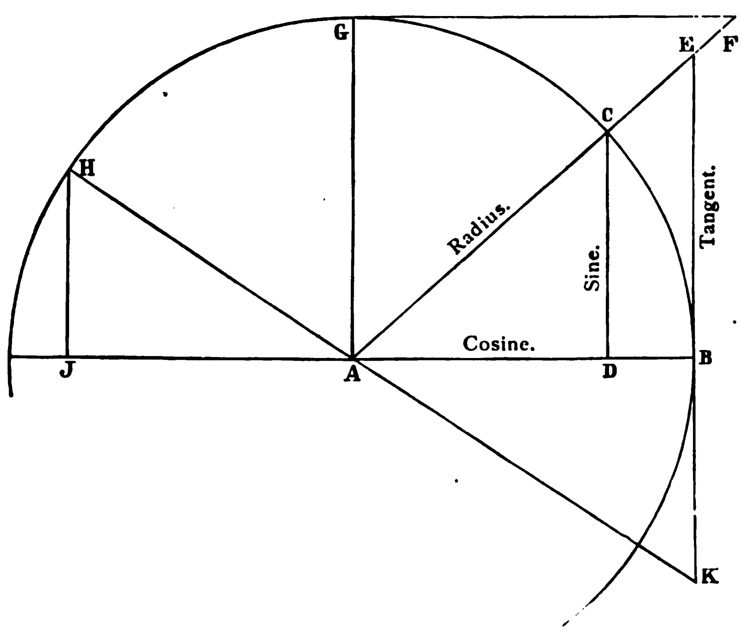


FIG. 315.

sines, etc. Or, the logarithms of the sines, cosines, etc., may be found in logarithmic tables.

In the absence of such a table, and when the degrees contained in the given angle are unknown, the values of the sine, cosine, etc., may be found by computation, as follows:—Let $A B C$ (*Fig.* 316) be the given angle. At any distance from B draw b perpendicular to $B C$. By any scale of equal parts obtain the length of each of the three lines a, b, c. Then for the angle at B we have, by proportion—

$$c : b :: 1 \cdot 0 : \text{sin. } B = \frac{b}{c}.$$

$$c : a :: 1 \cdot 0 : \text{cos. } B = \frac{a}{c}.$$

$$a : b :: 1 \cdot 0 : \text{tan. } B = \frac{b}{a}.$$

$$b : a :: 1 \cdot 0 : \text{cot. } B = \frac{a}{b}.$$

$$a : c :: 1 \cdot 0 : \text{sec. } B = \frac{c}{a}.$$

$$b : c :: 1 \cdot 0 : \text{cosec.} B = \frac{c}{b}.$$

Or, in any right-angled triangle, for the angle contained between the base and hypothenuse—

When	perp.	divided	by hyp.,	the	quotient	equals the	*sine.*
"	base	"	" hyp.,	"	"	"	*cosine.*
"	perp.	"	" base,	"	"	"	*tangent.*
"	base	"	" perp.,	"	"	"	*cotangent.*
"	hyp.	"	" base,	"	"	"	*secant.*
"	hyp.	"	" perp.,	"	"	"	*cosecant.*

To designate the angle to which a trigonometrical term applies, the letter at the intended angle is annexed to the

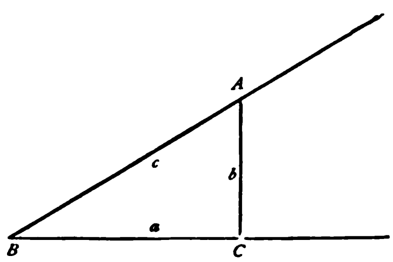


FIG. 316.

name of the trigonometrical term; thus, in the above example, for the sine of $A B C$ we write sin. B: for the cosine, cos. B, etc.

By these proportions the two acute angles of a right-angled triangle may be computed, provided two of the sides are known. For when the perpendicular and hypothenuse are known, the sine and cosecant may be obtained. When the base and hypothenuse are known, the cosine and secant may be computed. And when the base and perpendicular are known, the tangent and cotangent may be computed.

Either one of these, thus obtained, shows by the trigonometrical tables the number of degrees in the angle; and, deducting the angle thus found from 90°, the remainder will be the angle of the other acute angle of the triangle. For

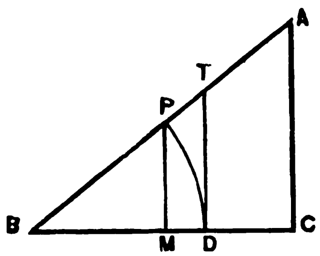


FIG. 317.

example: in a right-angled triangle, of which the base is 8 feet and the perpendicular 6 feet, how many degrees are contained in each of the acute angles?

Having, in this case, the base and perpendicular known, by referring to the above proportions we find that with these two sides we may obtain the tangent; therefore—

$$\text{Tan. } B = \frac{b}{a} = \frac{6}{8} = 0 \cdot 75.$$

Referring to the trigonometrical tables, we find that 0·75 is the tangent of 36° 52′ 12″, nearly; therefore—

The quadrant equals	90· 0· 0
The angle B equals	36·52·12
The angle A equals	53·07·48

475.—Right-Angled Triangles: Trigonometrical Value of Sides.—In the triangle ABC (*Fig.* 317), with $BP = 1$ for radius, and on B as a centre, describe the arc PD, and from its intersection with the lines AB and BC, draw PM and TD perpendicular to the line BC. Then from homologous triangles we have these proportions for the perpendicular—

$$BD : DT :: BC : CA,$$

$$r : \tan. B :: \text{base} : \text{perp.},$$

$$1 : \tan. B :: a : b = a \tan. B. \qquad (185.)$$

Also—

$$BP : PM :: BA : AC,$$

$$r : \sin. B :: \text{hyp.} : \text{perp.},$$

$$1 : \sin. B :: c : b = c \sin. B. \qquad (186.)$$

For the base, we have—

$$BP : BM :: BA : BC,$$

$$r : \cos. B :: \text{hyp.} : \text{base},$$

$$1 : \cos. B :: c : a = c \cos. B. \qquad (187.)$$

Again—

$$TD : BD :: AC : BC,$$

$$\tan. B : r :: \text{perp.} : \text{base},$$

$$\tan. B : 1 :: b : a = \frac{b}{\tan. B}. \qquad (188.)$$

For the hypothenuse, we have—

$$PM : PB :: AC :: AB,$$

$$\sin. B : r :: \text{perp.} : \text{hyp.},$$

$$\sin. B : 1 :: b : c = \frac{b}{\sin. B}. \qquad (189.)$$

Again—

$$BD : BT :: BC : BA,$$

$$r : \sec. B :: \text{base} : \text{hyp.},$$

$$1 : \sec. B :: a : c = a \sec. B = \frac{a}{\cos. B}. \qquad (190.)$$

This substitution of the cos. for the sec. is needed because tables of secants are not always accessible. That it is an equivalent is clear; for we have—

$$BM : BP :: BD : BT,$$

$$\cos. : r :: r : \sec. = \frac{1}{\cos.}.$$

By these equations either side of a right-angled triangle may be computed, provided there are certain parts of the triangle given. As, for example: of the six parts of a triangle (the three sides and the three angles), *three* must be given, and at least one of these must be a side.

As an example: let it be required to find two sides of a right-angled triangle of which the base is 100 feet, and the acute angle at the base is 35 degrees. Here we have given one side and two angles (the base, acute angle, and the right angle) to find the other two sides, the perpendicular and the hypothenuse.

Among the above rules we have, in equation (185.), for the perpendicular—

$$b = a \tan. B.$$

Or: The *perpendicular* equals the *product* of the *base* into the *tangent* of the *acute* angle at the *base*.

Then (*Art.* 427)—

$$\begin{array}{rr} \text{The logarithmic tangent of } B\ (=35^\circ) \text{ is} & \bar{9}\cdot8452268 \\ \text{Log. of } a\ (=100) \text{ is} & 2\cdot0000000 \\ \hline \text{Perpendicular, } b\ (=70\cdot02075) = & 1\cdot8452268 \end{array}$$

And for the hypothenuse, taking equation (190.), we have—

$$c = \frac{a}{\cos. B}.$$

Or: The *hypothenuse* equals the *quotient* of the *base* divided by the *cosine* of the *acute* angle at the *base*.

For this we have—

$$\begin{array}{rr} \text{Log. of } a\ (=100) \text{ is} & 2\cdot0000000 \\ \text{" } \cos. B\ (=35^\circ) \text{ is} & \bar{9}\cdot9133645 \\ \hline \text{Hypothenuse } c\,(=122\cdot0775) = & 2\cdot0866355 \end{array}$$

We thus find that a right-angled triangle, having an angle of 35 degrees at the base, has its three sides, the perpendic-

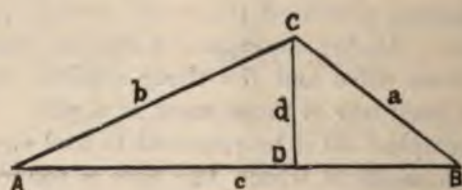


FIG. 318.

ular, base, and hypothenuse, respectively equal to 70·02075, 100, and 122·0775.

N.B.—The angle at A (*Fig.*317) is obtained by deducting the angle at B from 90° (*Art.* 346). Thus, $90 - 35 = 55$; this is the angle at A, in the above case.

If the perpendicular be given, then for the base use equation (188.), and for the hypothenuse use equation (189.). If the hypothenuse be given, then for the base use equation (187.) and for the perpendicular use equation (186.).

476.—Oblique-Angled Triangles: Sines and Sides.—In the oblique-angled triangle $A\,B\,C$ (*Fig.* 318) from C and perpendicular to $A\,B$ draw $C\,D$. This line divides the oblique-angled triangle into two right-angled triangles, the lines and angles of which may be treated by the rules already given; but there is a still more simple method, as will now be shown.

As shown in *Art.* 474: "When the perpendicular is divided by the hypothenuse the quotient equals the sine." Applying this to *Fig.* 318, we have—

$$\text{sin. } A = \frac{d}{b};$$

$$\text{sin. } B = \frac{d}{a}.$$

Let the former be divided by the latter; then—

$$\frac{\text{sin. } A}{\text{sin. } B} = \frac{\frac{d}{b}}{\frac{d}{a}},$$

or, reducing, we have—

$$\frac{\text{sin. } A}{\text{sin. } B} = \frac{a}{b};$$

or, putting the equation in the form of a proportion—

$$\text{sin. } B : \text{sin. } A :: b : a;$$

or; the *sines* are in proportion as the *sides*, respectively *opposite*. Or, as commonly stated, the *sines* are in proportion as the *sides* which *subtend* them.

This is a rule of great utility; by it we obtain the following:

Referring to *Fig.* 318, we have—

$$\text{sin. } B : \text{sin. } A :: b : a = b\,\frac{\text{sin. } A}{\text{sin. } B}. \qquad (191.)$$

$$\sin. C : \sin. A :: c : a = c\frac{\sin. A}{\sin. C}. \qquad (192.)$$

$$\sin. A : \sin. B :: a : b = a\frac{\sin. B}{\sin. A}. \qquad (193.)$$

$$\sin. C : \sin. B :: c : b = c\frac{\sin. B}{\sin. C}. \qquad (194.)$$

$$\sin. A : \sin. C :: a : c = a\frac{\sin. C}{\sin. A}. \qquad (195.)$$

$$\sin. B : \sin. C :: b : c = b\frac{\sin. C}{\sin. B}. \qquad (196.)$$

These expressions give the values of the three sides respectively; two expressions for each, one for each of the two remaining sides; that is to be used which contains the *given* side.

From these expressions we derive the values of the *sines*; thus—

$$\sin. A = \sin. B\frac{a}{b}. \qquad (197.)$$

$$\sin. A = \sin. C\frac{a}{c}. \qquad (198.)$$

$$\sin. B = \sin. A\frac{b}{a}. \qquad (199.)$$

$$\sin. B = \sin. C\frac{b}{c}. \qquad (200.)$$

$$\sin. C = \sin. A\frac{c}{a}. \qquad (201.)$$

$$\sin. C = \sin. B\frac{c}{b}. \qquad (202.)$$

477. — Oblique-Angled Triangles: First Class. — The problems arising in the treatment of oblique-angled triangles have been divided into four classes, one of which, the

first, will here be referred to. The problems of the *first* class are those in which *a side* and *two angles* are given, to find the remaining angle and sides.

As to the required angle, since the three angles of every triangle amount to just two right angles (*Art.* 345), or 180°, the third angle may be found simply by deducting the sum of the two given angles from 180°.

For example: referring to *Fig.* 318, if angle $A = 18°$ and angle $B = 42°$, then their sum is $18 + 42 = 60$, and $180 - 60 = 120° =$ the angle ACB.

To find the two sides: if a be the given side, then to find the side b we have, equation (193.)—

$$b = a \frac{\sin. B}{\sin. A};$$

or, the side b equals the product of the side a into the quotient obtained by a division of the sine of the angle opposite b by the sine of the angle opposite a.

For example: in a triangle (*Fig.* 318) in which the angle $A = 18°$, the angle $B = 42°$ (and, consequently (*Art.* 345) the angle $C = 120°$), and the given side a equals 43 feet; what are the lengths of the sides b and c? Equation (193.) gives—

$$b = a \frac{\sin. B}{\sin. A}.$$

Performing the problem by logarithms (*Art.* 427), we have—

$$\begin{aligned} \text{Log. } a\,(= 43) &= 1 \cdot 6334685 \\ \text{Sin. } B\,(= 42°) &= \underline{\bar{9} \cdot 8255109} \\ & \bar{1} \cdot 4589794 \\ \text{Sin. } A\,(= 18°) &= \underline{\bar{9} \cdot 4899824} \\ \text{Log. } b\,(= 93 \cdot 1102) &= 1 \cdot 9689970. \end{aligned}$$

Thus the side b equals $93 \cdot 1102$ feet, or 93 feet 1 inch and nearly one third of an inch.

For the side c, we have, equation (195.)—

$$c = a \frac{\sin. C}{\sin. A};$$

or—

$$\begin{array}{rl}
\text{Log. } a\,(= 43) = & \bar{1}\cdot 6334685 \\
\text{Sin. } C\,(= 120^\circ) = & \bar{9}\cdot 9375306 \\
\hline
& \bar{1}\cdot 5709991 \\
\text{Sin. } A\,(= 18^\circ) = & \bar{9}\cdot 4899824 \\
\hline
\text{Log. } c\,(= 120\cdot 508) = & 2\cdot 0810167
\end{array}$$

or, the base c equals 120 feet 6 inches and one tenth of an inch, nearly. But if instead of a the side b be given, then for a use equation (191.), and for c use equation (196.).

And, lastly, if c be the given side, then for a use equation (192.), and for b use equation (194.).

478.—Oblique-Angled Triangles: Second Class.—The problems which comprise the *second* class are those in which *two sides* and an *angle opposite* to one of them are given, to find the two remaining angles and the third side.

The only requirement really needed here is to find a second angle; for, with this second angle found, the problem is reduced to one of the first class; and the third side may then be found under rules given in *Art.* 477.

To find a second angle, use one of the equations (197.) to (202.).

For example: in the triangle $A\,B\,C$ (*Fig.* 318), let $a\,(= 43)$ and $b\,(= 93\cdot 11)$ be the two given sides, and A, the angle opposite a, be the given angle ($= 18^\circ$). Then to find the angle B, we have equation (199.)—(selecting that which in the right hand member contains the given angle and sides)—

$$\sin. B = \sin. A \frac{b}{a}$$

$$= \sin. A \frac{93\cdot 11}{43}.$$

By logarithms (*Art.* 427), we have—

Log. sin. A (= 18°)	=	$\bar{9}\cdot4899824$
" 93·11	=	$1\cdot9689970$
		$1\cdot4589794$
" 43	=	$1\cdot63334685$
" sin. B (= 42°)	=	$\bar{9}\cdot8255109$

By reference to the log. tables, the last line of figures, as above, is found to be the sine of 42°; therefore, the required angle B is 42°. Then $180° - (18° + 42°) = 120°$ = the angle C.

With these angles, or with any two of them, the third side c may be found by rules given in *Art.* 477.

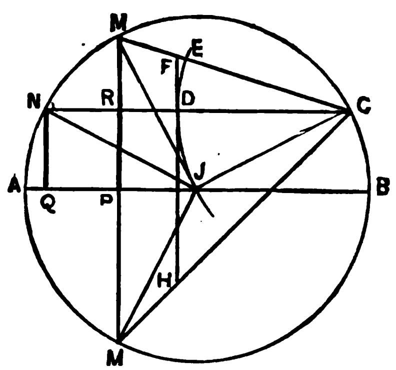


FIG. 319.

479.—Oblique-Angled Triangles: Sum and Difference of Two Angles.—Preliminary to a consideration of problems in the third class of triangles, it is requisite to show the relation between the *sum* and *difference* of two angles.

In *Fig.* 319, let the angle AJM and the angle AJN be the two given angles; and let AJM be called angle A, and AJN, angle B. Now the sum and difference of the angles may be ascertained by the use of the sum and difference of the *sines* of the angles, and by the sum and difference of the *tangents*. In the diagram, in which the radius AJ equals

unity, we have MP, the sine of angle A $(= A\,J\,M)$, and $N\,Q = R\,P$, the sine of angle $B (= A\,J\,N)$. Then—

$$MP - RP = MR$$

equals the *difference* of the *sines* of the angles; and since $PM' = PM$—

$$PM' + RP = RM',$$

equals the *sum* of the sines of the angles.

With the radius JC describe the arc JDE, and tangent to this arc draw FH parallel with MM', or perpendicular to AB.

Then FD is the tangent of the angle MCN, and DH is the tangent of the angle NCM'.

Now since an angle at the circumference is equal to half the angle at the centre standing on the same arc (*Art.* 355), therefore the measure of the angle MCN is the half of MN, equals—

$$\tfrac{1}{2}(AM - AN) = \tfrac{1}{2}(A - B).$$

Similarly, we have—

$$\tfrac{1}{2}(AM' + AN) = \tfrac{1}{2}(A + B),$$

for the angle NCM'.

Therefore we have for the tangent of the angle MCN—

$$FD = \tan.\ \tfrac{1}{2}(A - B),$$

and, for the tangent of the angle NCM'—

$$DH = \tan.\ \tfrac{1}{2}(A + B).$$

And, because FCD and MCR are homologous triangles, as, also, DCH and RCM', therefore—

$$M'R : MR :: DH : DF,$$

$$\sin. A + \sin. B : \sin. A - \sin. B : : \tan. \tfrac{1}{2}(A + B) : \tan. \tfrac{1}{2}(A - B),$$

from which we have—

$$\frac{\sin. A - \sin. B}{\sin. A + \sin. B} = \frac{\tan. \frac{1}{2}(A - B)}{\tan. \frac{1}{2}(A + B)}. \tag{D.}$$

To obtain a proper substitute for the first member of this expression we have, equation (195.)—

$$c = a\frac{\sin. C}{\sin. A},$$

or—

$$c \sin. A = a \sin. C. \tag{M.}$$

We also have, equation (196.)—

$$c = b\frac{\sin. C}{\sin. B},$$

or—

$$c \sin. B = b \sin. C. \tag{N.}$$

These two equations, (M.) and (N.), added, give—

$$c \sin. A + c \sin. B = a \sin. C + b \sin. C.$$

or—

$$c(\sin. A + \sin. B) = \sin. C(a + b). \tag{P.}$$

But, if equation (N.) be subtracted from equation (M.), we have—

$$c \sin. A - c \sin. B = a \sin. C - b \sin. C,$$

or—

$$c(\sin. A - \sin. B) = A \sin. C(a - b). \tag{R.}$$

If equation (R.) be divided by equation (P.), we have—

$$\frac{c(\sin. A - \sin. B)}{c(\sin. A + \sin. B)} = \frac{\sin. C(a - b)}{\sin. C(a + b)},$$

which reduces to—

$$\frac{\sin. A - \sin. B}{\sin. A + \sin. B} = \frac{a - b}{a + b}.$$

The first member of this equation is identical with the first member of the above equation (D.), and therefore its equal, the second member, may be substituted for it; thus—

$$\frac{a - b}{a + b} = \frac{\tan. \frac{1}{2}(A - B)}{\tan. \frac{1}{2}(A + B)}.$$

From which we have—

$$\tan. \tfrac{1}{2}(A - B) = \tan. \tfrac{1}{2}(A + B)\frac{a - b}{a + b}. \qquad (203.)$$

We have (*Art.* 431) the proposition, that if half the difference of two quantities be subtracted from half their sum, the remainder will equal the smaller quantity. For example: if A represent the larger quantity and B the smaller, then—

$$\tfrac{1}{2}(A + B) - \tfrac{1}{2}(A - B) = B; \qquad (204.)$$

and, again, we also have (*Art.* 431)—

$$\tfrac{1}{2}(A + B) + \tfrac{1}{2}(A - B) = A. \qquad (205.)$$

480.—Oblique-Angled Triangles: Third Class.—The *third* class of problems comprises all those cases in which two sides of a triangle and their included angle are given, to find the other side and angles.

In this case, as in the problems of the second class, the only requirement here is to find a second angle; for then the problem becomes one belonging to the first class. But the finding of the second angle, in problems of the third class, is attended with more computation than it is in problems of the second class. The process is as follows: Having one angle of a triangle, the *sum* of the two remaining

ngles is obtained by subtracting the given angle from 80° — the sum of the three angles.

Then with equation (203.) the *difference* of the two angles ; obtained. And then, having the *sum* and *difference* of the wo angles, either may be found by one of the equations 204.) and (205.).

For example: let *Fig.* 320 represent the triangle in which $a\,(= 36$ feet) and $b\,(= 27$ feet) are the given sides; and

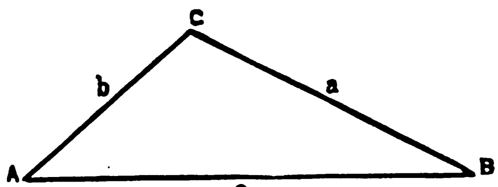


FIG. 320.

$C\,(= 105°)$ the angle included between the given sides, a and . The sum of the two angles A and B, therefore, will be—

$$(A + B) = 180 - 105 = 75°,$$

nd the half of the sum of A and B is $\frac{75}{2} = 37°\ 30'$.

The sum of the given sides is $36 + 27 = 63$, and their difference is $36 - 27 = 9$.

Then from equation (203.) we have—

$$\tan.\ \tfrac{1}{2}(A - B) = \tan.\ 37°\ 30\tfrac{9}{63}'.$$

Solving this by logs. (*Art.* 427), we have—

$$\begin{array}{rl}
\text{Log. tan. } 37°\ 30' & = \bar{9}\cdot8849805 \\
9 & = \underline{0\cdot9542425} \\
 & \ 0\cdot8392230 \\
63 & = \underline{1\cdot7993405} \\
\tan.\ \tfrac{1}{2}(A - B)(= 6°\ 15'\ 20\cdot5'') & = \bar{9}\cdot0398825
\end{array}$$

Thus half the difference of A and B is $6°\ 15'\ 20\cdot5''$, nearly.

By equation (204.)—

$$\begin{array}{r} 37^\circ\ 30' \\ 6^\circ\ 15'\ 20\cdot5'' \\ \hline \text{The difference, } 31^\circ\ 14'\ 39\cdot5'' = B, \end{array}$$

and by equation (205.)—

$$\begin{array}{rll} & 37.30 & \\ & 6.15.20\cdot5 & \\ \cline{2-2} \text{The sum,} & 43.45.20\cdot5 & = A \\ \text{From above,} & 31.14.39\cdot5 & = B \\ \text{The given angle,} & 105.\ 0.\ 0 & = C \\ \cline{2-3} \text{The three angles,} & 180.\ 0.\ 0 & \end{array}$$

Thus, by adding together the three angles, the work is tested and proved.

Having the three angles, the third side may now be found by the rule for problems of the first class.

481.— Oblique-Angled Triangles: Fourth Class.—The fourth class comprises those problems in which the three sides of the triangle are given, to find the three angles.

The method by which the problems of the fourth class are solved is to divide the triangle into two right-angled triangles; then, by the use of equation (129.), to find one side of one of these triangles, and then with this side to find one of the angles, then by rules for the second class problems, obtain the second and third angles.

Thus, from equation (129.), we have—

$$g = \frac{c^2 - (a + b)(a - b)}{2c}.$$

By the relation of sines to sides (*Art.* 476), we have (*Fig.* 321)—

$$b : g :: \sin. E : \sin. F.$$

But the angle E is a right angle, of which the sine is unity, therefore—

$$b : g :: 1 : \sin. F = \frac{g}{b}.$$

Substituting for g its value as above, we have—

$$\sin. F = \frac{c^2 - (a + b)(a - b)}{2bc}. \qquad (206.)$$

To illustrate: let a, b, c (*Fig.* 321) be the three given sides

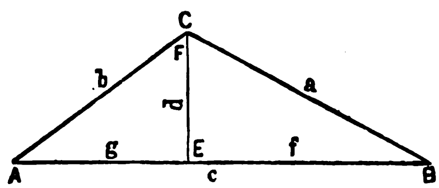


Fig. 321.

of the triangle ABC, respectively equal to 12, 8 and 16 feet. With these, equation (206.) becomes—

$$\sin. F = \frac{16^2 - (12 + 8)(12 - 8)}{2 \times 8 \times 16},$$

$$\sin. F = \frac{256 - (20 \times 4)}{256},$$

$$\sin. F = \frac{176}{256}.$$

Solving this by logarithms (*Art.* 427), we have—

$$\begin{aligned} \text{Log. } 176 &= 2 \cdot 2455127 \\ \text{"} \quad 256 &= \underline{2 \cdot 4082400} \\ \text{Log. sin. } 43^\circ\ 26' &= \bar{9} \cdot 8372727 \end{aligned}$$

or, the angle at F equals $43^\circ\ 26'$, nearly. Of the triangle

ACE (*Fig.* 321), E is a right angle, therefore the sum of F and A, the two remaining angles, equals 90° (*Art.* 346).

Hence, for the angle at A, we have—

$$A = 90° - 43° \ 26' = 46° \ 34'.$$

We now have two sides a and b and A, an angle opposite to one of them, to find B, a second angle. For this, equation (199.) is appropriate. Thus—

$$\sin. B = \sin. A \frac{b}{a}.$$

This may be solved as shown in *Art.* 478.

And, when the second angle is obtained, the third angle is found by subtracting the sum of the first and second angles from 180°.

But to test the accuracy of the work, it is well to *compute* the angle C from the angle A, and the sides a and c. For this, equation (201.) will be appropriate.

482.—Trigonometric Formulæ: Right-Angled Triangles.—For facility of reference the formulæ of previous

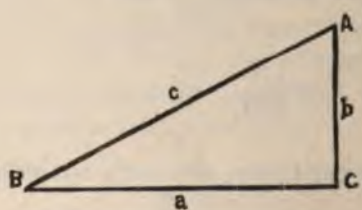


FIG. 322.

articles are here presented in tabular form. The symbols referred to are those of *Fig.* 322.

RIGHT-ANGLED TRIANGLES.

GIVEN.	REQUIRED.	FORMULÆ.
a, b,	c,	$c = \sqrt{a^2 + b^2}$.
a, c,	b,	$b = \sqrt{(c+a)(c-a)}$.
b, c,	a,	$a = \sqrt{(c+b)(c-b)}$.
A,	B,	$B = 90^\circ - A$.
B,	A,	$A = 90^\circ - B$.
B, a,	b,	$b = a \tan. B$.
	c,	$c = \frac{a}{\cos. B}$.
B, b,	a,	$a = \frac{b}{\tan. B}$.
	c,	$c = \frac{b}{\sin. B}$.
B, c,	a,	$a = c \cos. B$.
	b,	$b = c \sin. B$.

483.—Trigonometrical Formulæ: First Class, Oblique.

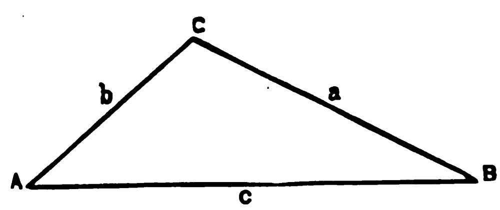


FIG. 323.

—The symbols of the formulæ of the following table indicate quantities represented in *Fig.* 323 by like symbols.

OBLIQUE-ANGLED TRIANGLES: FIRST CLASS.

GIVEN.	REQUIRED.	FORMULÆ.
$A, B,$	$C,$	$C = 180 - \overline{A + B}.$
$A, C,$	$B,$	$B = 180 - \overline{A + C}.$
$B, C,$	$A,$	$A = 180 - \overline{B + C}.$
$A, B, b,$	$a,$	$a = b\dfrac{\sin. A}{\sin. B}.$
$A, C, c,$	$a,$	$a = c\dfrac{\sin. A}{\sin. C}.$
$A, B, a,$	$b,$	$b = a\dfrac{\sin. B}{\sin. A}.$
$B, C, c,$	$b,$	$b = c\dfrac{\sin. B}{\sin. C}.$
$A, C, a,$	$c,$	$c = a\dfrac{\sin. C}{\sin. A}.$
$B, C, b,$	$c,$	$c = b\dfrac{\sin. C}{\sin. B}.$

484.—Trigonometrical Formulæ: Second Class, Oblique. —The symbols in the formulæ of the following table refer to quantities represented in *Fig.* 323, by like symbols.

Oblique-Angled Triangles: Second Class.

Given.	Required.	Formulæ.
$B, a, b,$	$A,$	$\sin. A = \sin. B \frac{a}{b}.$
$C, a, c,$	$A,$	$\sin. A = \sin. C \frac{a}{c}.$
$A, a, b,$	$B,$	$\sin. B = \sin. A \frac{b}{a}.$
$C, b, c,$	$B,$	$\sin. B = \sin. C \frac{b}{c}.$
$A, a, c,$	$C,$	$\sin. C = \sin. A \frac{c}{a}.$
$B, b, c,$	$C,$	$\sin. C = \sin. B \frac{c}{b}.$
$B, C,$	$A,$	$A = 180 - \overline{B + C}.$
$A, C,$	$B,$	$B = 180 - \overline{A + C}.$
$A, B,$	$C,$	$C = 180 - \overline{A + B}.$
For—	$a,$ $b,$ $c,$	See Formulæ, First Class.

485.—Trigonometrical Formulæ: Third Class, Oblique.—The symbols in the formulæ of the following table refer to quantities shown by like symbols in *Fig.* 323.

OBLIQUE-ANGLED TRIANGLES: THIRD CLASS.

GIVEN.	REQUIRED.	FORMULÆ.
	$A+B$,	$A+B=180-C$.
C, a, b,	$A-B$,	$\tan. \frac{1}{2}(A-B)=\tan. \frac{1}{2}(A+B)\frac{a-b}{a+b}$.
	A,	$A=\frac{1}{2}(A+B)+\frac{1}{2}(A-B)$.
	B,	$B=\frac{1}{2}(A+B)-\frac{1}{2}(A-B)$.
	$C+B$,	$C+B=180-A$.
A, b, c,	$C-B$,	$\tan. \frac{1}{2}(C-B)=\tan. \frac{1}{2}(C+B)\frac{c-b}{c+b}$.
	C,	$C=\frac{1}{2}(C+B)+\frac{1}{2}(C-B)$.
	B,	$B=\frac{1}{2}(C+B)-\frac{1}{2}(C-B)$.
	$C+A$.	$C+A=180-B$.
B, a, c,	$C-A$,	$\tan. \frac{1}{2}(C-A)=\tan. \frac{1}{2}(C+A)\frac{c-a}{c+a}$.
	C,	$C=\frac{1}{2}(C+A)+\frac{1}{2}(C-A)$.
	A,	$A=\frac{1}{2}(C+A)+\frac{1}{2}(C-A)$.

For the remaining side consult formulæ for the *first* class.

486.—Trigonometrical Formulæ: Fourth Class, Oblique.—The symbols in the formulæ of the following table refer to quantities shown by like symbols in *Fig.* 321.

OBLIQUE-ANGLED TRIANGLES: FOURTH CLASS.

Given a, b, c, to find A, B, C.
$\sin. F = \frac{c^2 - (a + b)(a - b)}{2bc}$. $A = 90 - F$. $\sin. B = \sin. A \frac{b}{a}$. $\sin. C = \sin. A \frac{c}{a}$. $C = 180 - (A + B)$.

SECTION XV.—DRAWING.

487.—General Remarks.—A knowledge of the properties and principles of *lines* can best be acquired by practice. Although the various diagrams throughout this work may be understood by inspection, yet they will be impressed upon the mind with much greater force, if they are actually drawn out with pencil and paper by the student. Science is acquired by study—art by practice; he, therefore, who would have anything more than a theoretical (which must of necessity be a superficial) knowledge of carpentry and geometry, will provide himself with the articles here specified, and perform all the operations described in the foregoing and following pages. Many of the problems may appear, at the first reading, somewhat confused and intricate; but by making one line at a time, according to the explanations, the student will not only succeed in copying the figures correctly, but by ordinary attention will learn the principles upon which they are based, and thus be able to make them available in any unexpected case to which they may apply.

488.—Articles Required.—The following articles are necessary for drawing, viz.: a drawing-board, paper, drawing-pins or mouth-glue, a sponge, a T-square, a set-square, two straight-edges, or flat rulers, a lead pencil, a piece of india-rubber, a cake of india-ink, a set of drawing-instruments, and a scale of equal parts.

489.—The Drawing-Board.—The size of the *drawing-board* must be regulated according to the size of the drawings which are to be made upon it. Yet for ordinary practice, in learning to draw, a board about fifteen by twenty inches, and one inch thick, will be found large enough, and

more convenient than a larger one. This board should be well seasoned, perfectly square at the corners, and without clamps on the ends. A board is better without clamps, because the little service they are supposed to render by preventing the board from warping is overbalanced by the consideration that the shrinking of the panel leaves the ends of the clamps projecting beyond the edge of the board, and thus interfering with the proper working of the stock of the T-square. When the stuff is well-seasoned, the warping of the board will be but trifling; and by exposing the rounding side to the fire, or to the sun, it may be brought back to its proper shape.

490.—Drawing-Paper.—For mere line drawings, it is unnecessary to use the *best* drawing-paper; and since, where much is used, the expense will be considerable, it is desirable for economy to procure a paper of as low a price as will be suitable for the purpose. The best paper is made in England and water-marked "Whatman." This is a hand-made paper. There is also a machine-made paper at about half-price, and the manilla paper, of various tints of russet color, is still less in price. These papers are of the various sizes needed, and are quite sufficient for ordinary drawings.

491.—To Secure the Paper to the Board.—A *drawing-pin* is a small brass button, having a steel pin projecting from the underside. By having one of these at each corner, the paper can be fixed to the board; but this can be done in a better manner with *mouth-glue*. The pins will prevent the paper from changing its position on the board; but, more than this, the glue keeps the paper perfectly tight and smooth, thus making it so much the more pleasant to work on.

To attach the paper with mouth-glue, lay it with the bottom side up, on the board; and with a straight-edge and penknife cut off the rough and uneven edge. With a sponge moderately wet rub all the surface of the paper, except a strip around the edge about half an inch wide. As soon as the glistening of the water disappears turn the sheet

over and place it upon the board just where you wish it glued. Commence upon one of the longest sides, and proceed thus: lay a flat ruler upon the paper, parallel to the edge, and within a quarter of an inch of it. With a knife, or anything similar, turn up the edge of the paper against the edge of the ruler, and put one end of the cake of mouth-glue between your lips to dampen it. Then holding it upright, rub it against and along the entire edge of the paper that is turned up against the ruler, bearing moderately against the edge of the ruler, which must be held firmly with the left hand. Moisten the glue as often as it becomes dry, until a sufficiency of it is rubbed on the edge of the paper. Take away the ruler, restore the turned-up edge to the level of the board, and lay upon it a strip of pretty stiff paper. By rubbing upon this, not very hard but pretty rapidly, with the thumb-nail of the right hand, so as to cause a gentle friction and heat to be imparted to the glue that is on the edge of the paper, you will make it adhere to the board. The other edges in succession must be treated in the same manner.

Some short distances along one or more of the edges may afterward be found loose; if so, the glue must again be applied, and the paper rubbed until it adheres. The board must then be laid away in a warm or dry place; and in a short time the surface of the paper will be drawn out, perfectly tight and smooth, and ready for use. The paper dries best when the board is laid level. When the drawing is finished lay a straight-edge upon the paper and cut it from the board, leaving the glued strip still attached. This may afterward be taken off by wetting it freely with the sponge, which will soak the glue and loosen the paper. Do this as soon as the drawing is taken off, in order that the board may be dry when it is wanted for use again. Care must be taken that, in applying the glue, the edge of the paper does not become damper than the rest; if it should, the paper must be laid aside to dry (to use at another time) and another sheet be used in its place.

Sometimes, especially when the drawing-board is new, the paper will not stick very readily; but by persevering

this difficulty may be overcome. In the place of the mouth-glue a strong solution of gum-arabic may be used, and on some accounts is to be preferred ; for the edges of the paper need not be kept dry, and it adheres more readily. Dissolve the gum in a sufficiency of warm water to make it of the consistency of linseed-oil. It must be applied to the paper with a brush, when the edge is turned up against the ruler, as was described for the mouth-glue. If two drawing-boards are used, one may be in use while the other is laid away to dry ; and as they may be cheaply made, it is advisable to have two. The drawing-board having a frame around it, commonly called a panel board, may afford rather more facility in attaching the paper when this is of the size to

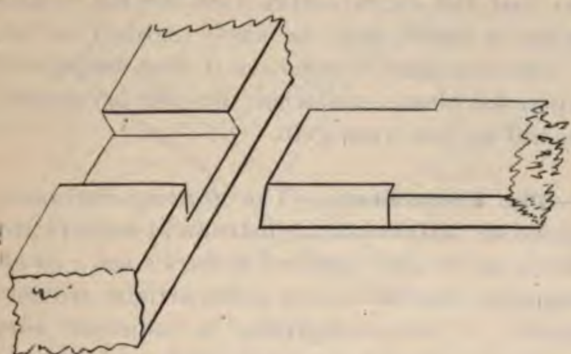


FIG. 324.

suit ; yet it has objections which overbalance that consideration.

492.—The T-Square.—A *T-square* of mahogany, at once simple in its construction and affording all necessary service, may be thus made: let the stock or handle be seven inches long, two and a quarter inches wide, and three eighths of an inch thick ; the blade, twenty inches long (exclusive of the stock), two inches wide, and one eighth of an inch thick. In joining the blade to the stock, a very firm and simple joint may be made by dovetailing it—as shown at *Fig.* 324.

493.—The Set-Square.—The *set-square* is in the form of a right-angled triangle ; and is commonly made of mahogany,

one eighth of an inch in thickness. The size that is most convenient for general use is six inches and three inches respectively for the sides which contain the right angle, although a particular length for the sides is by no means necessary. Care should be taken to have the square corner exactly true. This, as also the T-square and rulers, should have a hole bored through them, by which to hang them upon a nail when not in use.

494.—The Rulers.—One of the *rulers* may be about twenty inches long, and the other six inches. The *pencil* ought to be hard enough to retain a fine point, and yet not so hard as to leave ineffaceable marks. It should be used lightly, so that the extra marks that are not needed when the drawing is inked, may be easily rubbed off with the rubber. The best kind of *india-ink* is that which will easily rub off upon the plate; and, when the cake is rubbed against the teeth, will be free from grit.

495.—The Instruments.—The *drawing-instruments* may be purchased of mathematical instrument makers at various prices; from one to one hundred dollars a set. In choosing a set, remember that the lowest price articles are not always the cheapest. A set, comprising a. sufficient number of instruments for ordinary use, well made and fitted in a mahogany box, may be purchased of the mathematical instrument makers in New York for four or five dollars. But for permanent use those which come at ten or twelve dollars will be found to be better.

496.—The Scale of Equal Parts.—The best *scale of equal parts* for carpenters' use, is one that has one eighth, three sixteenths, one fourth, three eighths, one half, five eighths, three fourths, and seven eighths of an inch, and one inch, severally divided into *twelfths*, instead of being divided, as they usually are, into tenths. By this, if it be required to proportion a drawing so that every foot of the object represented will upon the paper measure one fourth of an inch, use that part of the scale which is divided into one

fourths of an inch, taking for every foot one of those divisions, and for every inch one of the subdivisions into twelfths; and proceed in like manner in proportioning a drawing to any of the other divisions of the scale. An instrument in the form of a semi-circle, called a *protractor*, and used for laying down and measuring angles, is of much service to surveyors, and occasionally to carpenters.

497.—The Use of the Set-Square.—In drawing parallel lines, when they are to be parallel to either side of the board, use the T-square; but when it is required to draw lines parallel to a line which is drawn in a direction oblique

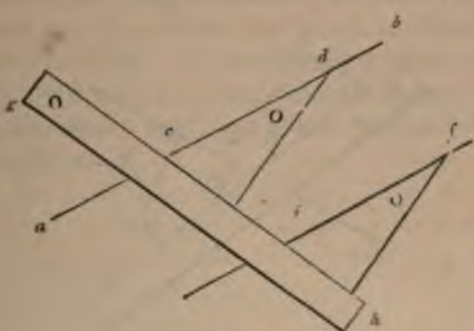


FIG. 325.

to either side of the board, the set-square must be used. Let *a b* (*Fig.* 325) be a line, parallel to which it is desired to draw one or more lines. Place any edge, as *c d*, of the set-square even with said line; then place the ruler *g h* against one of the other sides, as *c e*, and hold it firmly; slide the set-square along the edge of the ruler as far as it is desired, as at *f*; and a line drawn by the edge *i f* will be parallel to *a b*.

To draw a line, as *k l* (*Fig.* 326), perpendicular to another, as *a b*, set the shortest edge of the set-square at the line *a b*; place the ruler against the longest side (the hypothenuse of the right-angled triangle); hold the ruler firmly, and slide the set-square along until the side *e d* touches the point *k*; then the line *l k*, drawn by it, will be perpendicular to *a b*.

In like manner, the drawing of other problems may be facilitated, as will be discovered in using the instruments.

498.—Directions for Drawing.—In drawing a problem, proceed, with the pencil sharpened to a point, to lay down the several lines until the whole figure is completed, observing to let the lines cross each other at the several angles, instead of merely meeting. By this, the length of every line will be clearly defined. With a drop or two of water, rub one end of the cake of ink upon a plate or saucer, until a sufficiency adheres to it. Be careful to dry the cake of

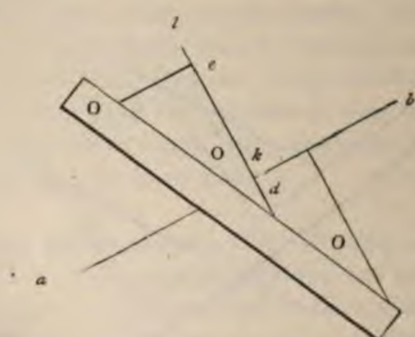


FIG 326.

ink; because if it is left wet it will crack and crumble in pieces. With an inferior camel's-hair pencil add a little water to the ink that was rubbed on the plate, and mix it well. It should be diluted sufficiently to flow freely from the pen, and yet be thick enough to make a *black* line. With the hair pencil place a little of the ink between the nibs of the drawing-pen, and screw the nibs together until the pen makes a fine line. Beginning with the curved lines, proceed to ink *all* the lines of the figure, being careful now to make every line of its requisite length. If they are a trifle too short or too long the drawing will have a ragged appearance; and this is opposed to that neatness and accuracy which is indispensable to a good drawing. When the ink is dry efface the pencil-marks with the india-rubber. If the

pencil is used lightly they will all rub off, leaving those lines only that were inked.

In problems all auxiliary lines are drawn light; while the lines given and those sought, in order to be distinguished at a glance, are made much heavier. The heavy lines are made so by passing over them a second time, having the nibs of the pen separated far enough to make the lines as heavy as desired. If the heavy lines are made before the drawing is cleaned with the rubber they will not appear so black and neat, because the india-rubber takes away part of the ink. If the drawing is a ground-plan or elevation of a house, the shade-lines, as they are termed, should not be put in until the drawing is shaded; as there is danger of the heavy lines spreading when the brush, in shading or coloring, passes over them. If the lines are inked with common writing-ink they will, however fine they may be made, be subject to the same evil; for which reason india-ink is the only kind to be used.

SECTION XVI.—PRACTICAL GEOMETRY.

499.—Definitions.—*Geometry* treats of the properties of magnitudes.

A *point* has neither length, breadth, nor thickness.

A *line* has length only.

Superficies has length and breadth only.

A *plane* is a surface, perfectly straight and even in every direction; as the face of a panel when not warped nor winding.

A *solid* has length, breadth, and thickness.

A *right*, or *straight*, line is the shortest that can be drawn between two points.

Parallel lines are equidistant throughout their length.

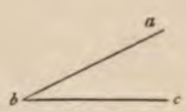


FIG. 327.

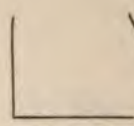


FIG. 328.

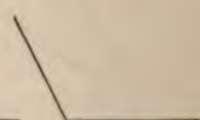


FIG. 329.

An *angle* is the inclination of two lines towards one another (*Fig.* 327).

A *right angle* has one line perpendicular to the other (*Fig.* 328).

An *oblique angle* is either greater or less than a right angle (*Figs.* 327 and 329).

An *acute angle* is less than a right angle (*Fig.* 327).

An *obtuse angle* is greater than a right angle (*Fig.* 329).

When an angle is denoted by three letters, the middle one, in the order they stand, denotes the angular point, and the other two the sides containing the angle; thus, let *a, b, c* (*Fig.* 327) be the angle, then *b* will be the angular point, and *a b* and *b c* will be the two sides containing that angle.

A *triangle* is a superficies having three sides and angles (*Figs.* 330, 331, 332, and 333).

An *equilateral triangle* has its three sides equal (*Fig.* 330).

An *isosceles triangle* has only two sides equal (*Fig.* 331).

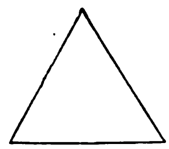


FIG. 330.

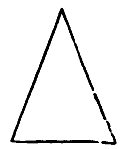


FIG 331.

A *scalene triangle* has all its sides unequal (*Fig.* 332).

A *right-angled triangle* has one right angle (*Fig.* 333).

An *acute-angled triangle* has all its angles acute (*Figs.* 330 and 331).

FIG. 332.

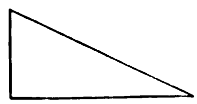


FIG. 333.

An *obtuse-angled triangle* has one obtuse angle (*Fig.* 332).

A *quadrangle* has four sides and four angles (*Figs.* 334 to 339).

A *parallelogram* is a quadrangle having its opposite sides parallel (*Figs.* 334 to 337).

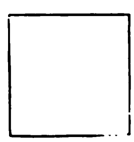


FIG. 334.

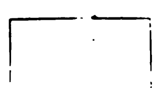


FIG. 335.

A *rectangle* is a parallelogram, its angles being right angles (*Figs.* 334 and 335).

A *square* is a rectangle having equal sides (*Fig.* 334).

A *rhombus* is an equilateral parallelogram having oblique angles (*Fig.* 336).

A *rhomboid* is a parallelogram having oblique angles (*Fig.* 337).

A *trapezoid* is a quadrangle having only two of its sides parallel (*Fig.* 338).

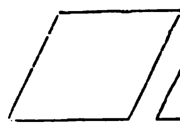


FIG. 336.

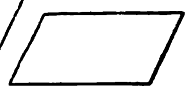


FIG. 337.

A *trapezium* is a quadrangle which has no two of its sides parallel (*Fig.* 339).

A *polygon* is a figure bounded by right lines.

A *regular polygon* has its sides and angles equal.

An *irregular polygon* has its sides and angles unequal.

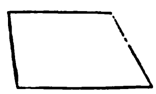


FIG. 338.

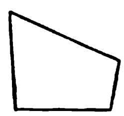


FIG. 339.

A *trigon* is a polygon of three sides (*Figs.* 330 to 333); a *tetragon* has four sides (*Figs.* 334 to 339); a *pentagon* has five (*Fig.* 340); a *hexagon* six (*Fig.* 341); a *heptagon* seven (*Fig.* 342); an *octagon* eight (*Fig.* 343); a *nonagon* nine; a *decagon* ten; an *undecagon* eleven; and a *dodecagon* twelve sides.

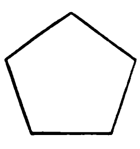


FIG. 340.

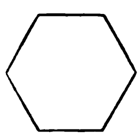


FIG. 341.

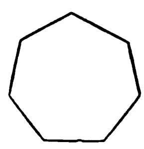


FIG. 342.

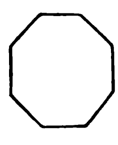


FIG. 343.

A *circle* is a figure bounded by a curved line, called the *circumference*, which is everywhere equidistant from a certain point within, called its *centre*.

The circumference is also called the *periphery*, and sometimes the *circle*.

The *radius* of a circle is a right line drawn from the centre to any point in the circumference (*a b*, *Fig.* 334).

All the *radii* of a circle are equal.

The *diameter* is a right line passing through the centre, and terminating at two opposite points in the circumference. Hence it is twice the length of the radius (*c d*, *Fig.* 344.)

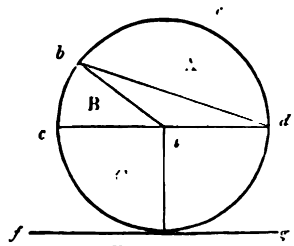


FIG. 344.

An *arc* of a circle is a part of the circumference (*c b*, or *b e d*, *Fig.* 344).

A *chord* is a right line joining the extremities of an arc (*b d*, *Fig.* 344).

A *segment* is any part of a circle bounded by an arc and its chord (*A*, *Fig.* 344).

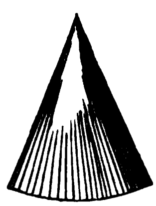


FIG. 345.

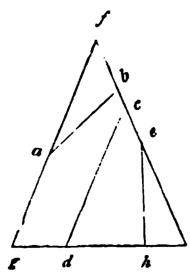


FIG. 346.

A *sector* is any part of a circle bounded by an arc and two radii, drawn to its extremities (*B*, *Fig.* 344).

A *quadrant*, or quarter of a circle, is a sector having a quarter of the circumference for its arc (*C*, *Fig.* 344).

A *tangent* is a right line which, in passing a curve, touches, withont cutting it (*f g*, *Fig.* 344).

A *cone* is a solid figure standing upon a circular base diminishing in straight lines to a point at the top, called its vertex (*Fig.* 345).

The *axis* of a cone is a right line passing through it, from the vertex to the centre of the circle at the base.

An *ellipsis* is described if a cone be cut by a plane, not parallel to its base, passing quite through the curved surface (*a b*, *Fig.* 346).

A *parabola* is described if a cone be cut by a plane, parallel to a plane touching the curved surface (*c d*, *Fig.* 346—*c d* being parallel to *f g*).

An *hyperbola* is described if a cone be cut by a plane,

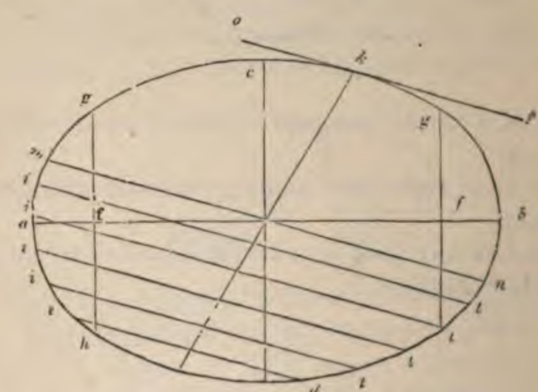


FIG. 347.

parallel to any plane within the cone that passes through its vertex (*e h*, *Fig.* 346).

Foci are the points at which the pins are placed in describing an ellipse (see *Art.* 548, and *f*, *f*, *Fig.* 347).

The *transverse axis* is the longest diameter of the ellipsis (*a b*, *Fig.* 347).

The *conjugate axis* is the shortest diameter of the ellipsis; and is, therefore, at right angles to the transverse axis (*c d*, *Fig.* 347).

The *parameter* is a right line passing through the focus of an ellipsis, at right angles to the transverse axis, and terminated by the curve (*g h* and *g t*, *Fig.* 347).

A *diameter of an ellipsis* is any right line passing through the centre, and terminated by the curve (*k l*, or *m n*, *Fig.* 347).

A diameter is *conjugate* to another when it is parallel to a tangent drawn at the extremity of that other—thus, the diameter *m n* (*Fig.* 347) being parallel to the tangent *o p*, is therefore conjugate to the diameter *k l*.

A *double ordinate* is any right line, crossing a diameter of an ellipsis, and drawn parallel to a tangent at the extremity of that diameter (*i t*, *Fig.* 347).

A *cylinder* is a solid generated by the revolution of a right-angled parallelogram, or rectangle, about one of its

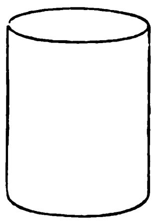


FIG. 348.

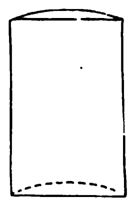


FIG. 349.

sides; and consequently the ends of the cylinder are equal circles (*Fig.* 348).

The *axis* of a cylinder is a right line passing through it from the centres of the two circles which form the ends.

A *segment* of a cylinder is comprehended under three planes, and the curved surface of the cylinder. Two of these are segments of circles; the other plane is a parallelogram, called by way of distinction, the *plane of the segment*. The circular segments are called the ends of the cylinder (*Fig.* 349).

PROBLEMS.

RIGHT LINES AND ANGLES.

500.—To Bisect a Line.—Upon the ends of the line *a b* (*Fig.* 350) as centres, with any distance for radius greater than half *a b*, describe arcs cutting each other in *c* and *d*;

draw the line cd, and the point e, where it cuts ab, will be the middle of the line ab.

In practice, a line is generally divided with the compasses, or dividers; but this problem is useful where it is

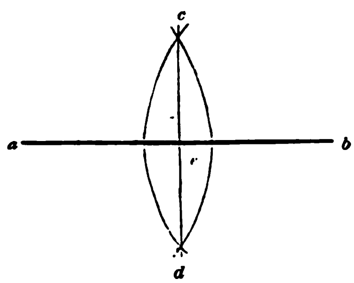


FIG. 350.

desired to draw, at the middle of another line, one at right angles to it. (See *Art.* 514.)

501.—To Erect a Perpendicular.—From the point a (*Fig.* 351) set off any distance, as ab, and the same distance from a to c; upon c, as a centre, with any distance for radius greater than ca, describe an arc at d; upon b, with the same

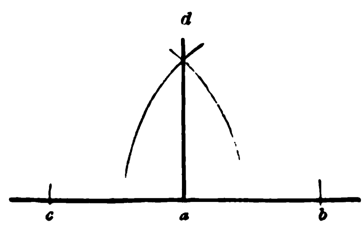


FIG. 351.

radius, describe another at d; join d and a, and the line da will be the perpendicular required.

This, and the three following problems, are more easily performed by the use of the set-square (see *Art.* 493). Yet they are useful when the operation is so large that a set-square cannot be used.

502.—To let Fall a Perpendicular.—Let a (*Fig.* 352) be the point above the line $b\,c$ from which the perpendicular is required to fall. Upon a, with any radius greater than $a\,d$, describe an arc, cutting $b\,c$ at e and f; upon the points e and f, with any radius greater than $e\,d$, describe arcs, cutting

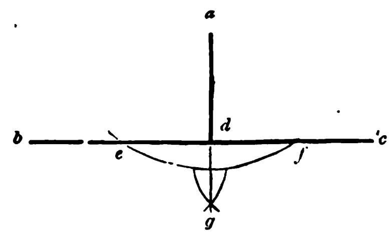


FIG. 352.

each other at g; join a and g, and the line $a\,d$ will be the perpendicular required.

503.—To Erect a Perpendicular at the End of a Line.—Let a (*Fig.* 353), at the end of the line $c\,a$, be the point at which the perpendicular is to be erected. Take any point, as b, above the line $c\,a$, and with the radius $b\,a$ describe the arc $d\,a\,e$; through d and b draw the line $d\,e$; join e and a, then $e\,a$ will be the perpendicular required.

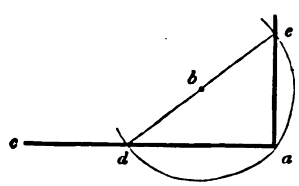


FIG. 353.

The principle here made use of is a very important one, and is applied in many other cases (see *Art.* 510, 3d, and *Art.* 513. For proof of its correctness, see *Art.* 352).

A second method. Let b (*Fig.* 354), at the end of the line $a\,b$, be the point at which it is required to erect a perpendicular. Upon b, with any radius less than $b\,a$, describe the arc $c\,e\,d$; upon c, with the same radius, describe the small arc at e,

and upon c, another at d; upon c and d, with the same or any other radius greater than half $c\,d$, describe arcs intersecting at f; join f and b, and the line $f\,b$ will be the perpendicular required. This method of erecting a perpendicular, and that of the following article, depend for accuracy upon the

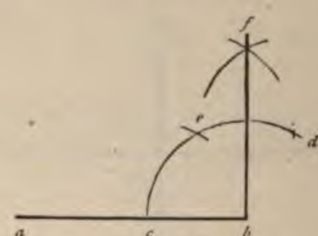


FIG. 354.

fact that the side of a hexagon is equal to the radius of the circumscribing circle.

A third method. Let b (*Fig.* 355) be the given point at which it is required to erect a perpendicular. Upon b, with any radius less than $b\,a$, describe the quadrant $d\,e\,f$; upon d, with the same radius, describe an arc at e, and upon e another at c; through d and e draw $d\,c$, cutting the arc in c; join c and b, then $c\,b$ will be the perpendicular required.

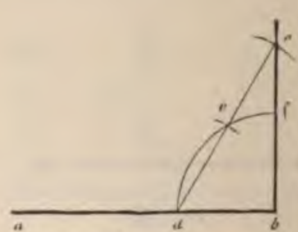


FIG. 355.

This problem can be solved by the *six, eight and ten* rule, as it is called, which is founded upon the same principle as the problems at *Arts.* 536, 537, and is applied as follows: let $a\,d$ (*Fig.* 353) equal eight, and $a\,e$, six; then, if $d\,e$ equals ten, the angle $e\,a\,d$ is a right angle. Because the square of six and that of eight, added together, equal the square of

ten, thus: $6 \times 6 = 36$, and $8 \times 8 = 64$; $36 + 64 = 100$, and $10 \times 10 = 100$. Any sizes, taken in the same proportion, as six, eight and ten, will produce the same effect; as 3, 4 and 5, or 12, 16 and 20. (See *Art.* 536.)

By the process shown at *Fig.* 353, the end of a board may be squared without a carpenters'-square. All that is necessary is a pair of compasses and a ruler. Let *c a* be the edge of the board, and *a* the point at which it is required to be squared. Take the point *b* as near as possible at an angle of forty-five degrees, or on a *mitre*-line from *a*, and at about the middle of the board. This is not necessary to the working of the problem, nor does it affect its accuracy, but the result is more easily obtained. Stretch the compasses from *b* to *a*, and then bring the leg at *a* around to *d*; draw a line from *d*, through *b*, out indefinitely; take the distance *d b* and place it from *b* to *e*; join *e* and *a*; then *e a* will be at right angles to *c a*. In squaring the foundation of a building, or laying out a garden, a rod and chalk-line may be used instead of compasses and ruler.

504.—To let Fall a Perpendicular near the End of a Line.—Let *e* (*Fig.* 353) be the point above the line *c a*, from which the perpendicular is required to fall. From *e* draw any line, as *e d*, obliquely to the line *c a*; bisect *e d* at *b*; upon *b*, with the radius *b e*, describe the arc *e a d*; join *e* and *a*; then *e a* will be the perpendicular required.

505.—To Make an Angle (as *e d f*, *Fig.* 356) Equal to a Given Angle (as *b a c*).—From the angular point *a*, with any

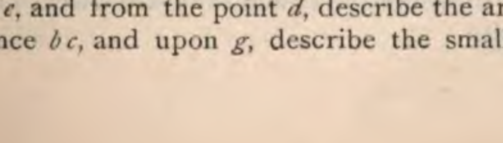


FIG. 356.

radius, describe the arc *b c*; and with the same radius, on the line *d e*, and from the point *d*, describe the arc *f g*; take the distance *b c*, and upon *g*, describe the small arc at *f*;

join f and d; and the angle $e\,d\,f$ will be equal to the angle $b\,a\,c$.

If the given line upon which the angle is to be made is situated parallel to the similar line of the given angle, this may be performed more readily with the set-square. (See *Art.* 497.)

506.—To Bisect an Angle.—Let $a\,b\,c$ (*Fig.* 357) be the angle to be bisected. Upon b, with any radius, describe the

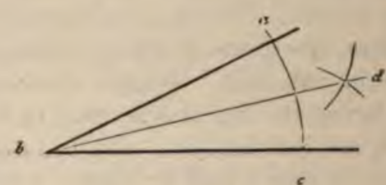


FIG. 357.

arc $a\,c$; upon a and c, with a radius greater than half $a\,c$, describe arcs cutting each other at d; join b and d; and $b\,d$ will bisect the angle $a\,b\,c$, as was required.

This problem is frequently made use of in solving other problems; it should therefore be well impressed upon the memory.

507.—To Trisect a Right Angle.—Upon a (*Fig.* 358), with any radius, describe the arc $b\,c$; upon b and c, with the

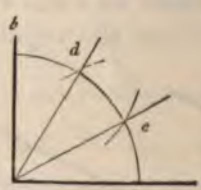


FIG. 358.

same radius, describe arcs cutting the arc $b\,c$ at d and e; from d and e draw lines to a, and they will trisect the angle, as was required.

The truth of this is made evident by the following operation: divide a circle into quadrants; also, take the radius in the dividers, and space off the circumference. This will divide the circumference into just six parts. A semi-circumference, therefore, is equal to three, and a quadrant to one and a half of those parts. The radius, therefore, is equal to two thirds of a quadrant; and this is equal to a right angle.

508.—Through a Given Point, to Draw a Line Parallel to a Given Line.—Let *a* (*Fig.* 359) be the given point, and

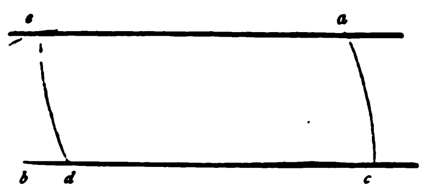


FIG. 359.

bc the given line. Upon any point, as *d*, in the line *bc*, with the radius *da*, describe the arc *ac*; upon *a*, with the same radius, describe the arc *dc*; make *dc* equal to *ac*; through *e* and *a* draw the line *ea*, which will be the line required.

This is upon the same principle as *Art.* 505.

509.—To Divide a Given Line into any Number of Equal Parts.—Let *ab* (*Fig.* 360) be the given line, and 5 the number of parts. Draw *ac* at any angle to *ab*; on *ac*, from

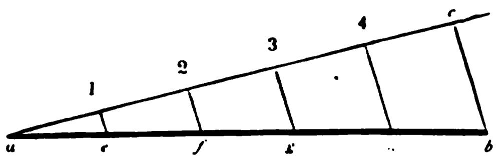


FIG. 360.

a, set off five equal parts of any length, as at 1, 2, 3, 4 and *c*; join *c* and *b*; through the points 1, 2, 3, and 4, draw 1 *e*, 2 *f*, 3 *g* and 4 *h*, parallel to *cb*; which will divide the line *ab*, as was required.

The lines *a b* and *a c* are divided in the same proportion. (See *Art.* 542.)

THE CIRCLE.

510.—To Find the Centre of a Circle.—Draw any chord, as *a b* (*Fig.* 361), and bisect it with the perpendicular *c d*; bi-

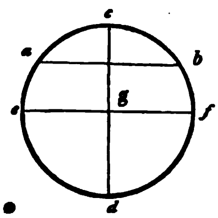


FIG. 361.

sect *c d* with the line *e f*, as at *g*; then *g* is the centre, as was required.

A second method. Upon any two points in the circumference nearly opposite, as *a* and *b* (*Fig.* 362), describe arcs cutting each other at *c* and *d*; take any other two points, as *e* and *f*, and describe arcs intersecting, as at *g* and *h*; join *g* and *h* and *c* and *d*; the intersection *o* is the centre.

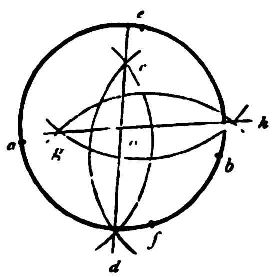


FIG. 362.

This is upon the same principle as *Art.* 514.

A third method. Draw any chord, as *a b* (*Fig.* 363), and from the point *a* draw *a c* at right angles to *a b*; join *c* and *b*; bisect *c b* at *d*—which will be the centre of the circle.

If a circle be not too large for the purpose, its centre may very readily be ascertained by the help of a carpenters'-square, thus: apply the corner of the square to any point in the circumference, as at *a*; by the edges of the square (which the lines *a b* and *a c* represent) draw lines cutting the

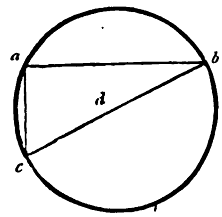


FIG. 363.

circle, as at *b* and *c*; join *b* and *c*; then if *b c* is bisected, as at *d*, the point *d* will be the centre. (See *Art.* 352.)

511.—At a Given Point in a Circle to Draw a Tangent thereto.—Let *a* (*Fig.* 364) be the given point, and *b* the centre of the circle.

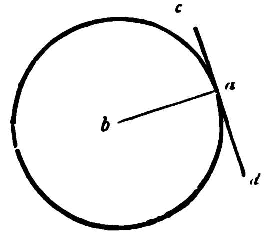


FIG. 364.

tre of the circle. Join *a* and *b*; through the point *a*, and at right angles to *a b*, draw *c d*; then *c d* is the tangent required.

512.—The Same, without making use of the Centre of the Circle.—Let *a* (*Fig.* 365) be the given point. From *a* set off any distance to *b*, and the same from *b* to *c*; join *a* and *c*; upon *a*, with *a b* for radius, describe the arc *d b e*; make *d b* equal to *b e*; through *a* and *d* draw a line; this will be the tangent required.

The correctness of this method depends upon the fact that the angle formed by a chord and tangent is equal to any inscribed angle in the opposite segment of the circle (*Art.* 358); *a b* being the chord, and *b c a* the angle in the opposite segment of the circle. Now, the angles *d a b* and *b c a* are equal, because the angles *d a b* and *b a c* are, by construction,

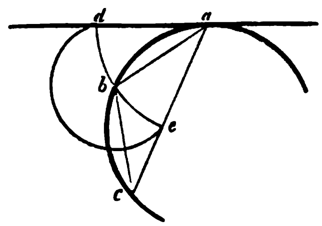


Fig. 365.

equal; and the angles *b a c* and *b c a* are equal, because the triangle *a b c* is an isosceles triangle, having its two sides *a b* and *b c*, by construction equal; therefore the angles *d a b* and *b c a* are equal.

513.—A Circle and a Tangent Given, to Find the Point of Contact.—From any point, as *a* (*Fig.* 366), in the tangent

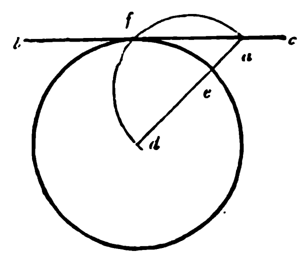


Fig. 366.

b c, draw a line to the centre *d*; bisect *a d* at *e*; upon *e*, with the radius *e a*, describe the arc *a f d*; *f* is the point of contact required.

If *f* and *d* were joined, the line would form right angles with the tangent *b c*. (See *Art.* 352.)

514.—Through any Three Points not in a Straight Line, to Draw a Circle.—Let *a*, *b* and *c* (*Fig.* 367) be the three given points. Upon *a* and *b*, with any radius greater than half *a b*, describe arcs intersecting at *d* and *e*; upon *b* and *c*, with any radius greater than half *b c*, describe arcs intersecting at *f* and *g*; through *d* and *e* draw a right line, also

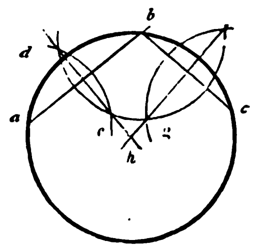


FIG. 367.

another through *f* and *g*; upon the intersection *h*, with the radius *h a*, describe the circle *a b c*, and it will be the one required.

515.—Three Points not in a Straight Line being Given, to Find a Fourth that shall, with the Three, Lie in the Circumference of a Circle.—Let *a b c* (*Fig.* 368) be the given points. Connect them with right lines, forming the triangle

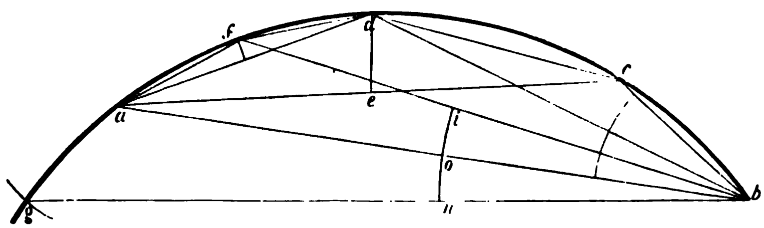


FIG. 368.

a c b; bisect the angle *c b a* (*Art.* 506) with the line *b d*; also bisect *c a* in *e*, and erect *e d* perpendicular to *a c*, cutting *b d* in *d*; then *d* is the *fourth* point required.

A fifth point may be found, as at *f*, by assuming *a*, *d* and *b*, as the three given points, and proceeding as before. So,

also, any number of points may be found simply by using any three already found. This problem will be serviceable in obtaining short pieces of very flat sweeps. (See *Art.* 240.)

The proof of the correctness of this method is found in the fact that equal chords subtend equal angles (*Art.* 357). Join d and c; then since $a\,e$ and $e\,c$ are, by construction, equal, therefore the chords $a\,d$ and $d\,c$ are equal; hence the angles they subtend, $d\,b\,a$ and $d\,b\,c$, are equal. So, likewise, chords drawn from a to f, and from f to d, are equal, and subtend the equal angles $d\,b\,f$ and $f\,b\,a$. Additional points *beyond* a or b may be obtained on the same principle. To obtain a point beyond a, on b, as a centre, describe with any radius the arc $i\,o\,n$; make $o\,n$ equal to $o\,i$; through b and n draw $b\,g$; on a as centre and with $a\,f$ for radius, describe the arc, cutting $g\,b$ at g, then g is the point sought.

516.—To Describe a Segment of a Circle by a Set-Triangle.—Let $a\,b$ (*Fig.* 369) be the chord, and $c\,d$ the height

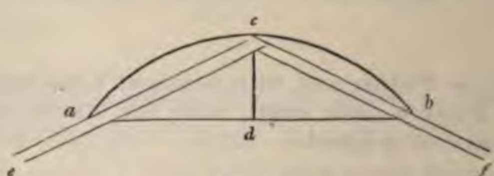


FIG. 369.

of the segment. Secure two straight-edges, or rulers, in the position $c\,e$ and $c\,f$, by nailing them together at c, and affixing a brace from e to f; put in pins at a and b; move the angular point c in the direction $a\,c\,b$; keeping the edges of the triangle hard against the pins a and b; a pencil held at c will describe the arc $a\,c\,b$.

A curve described by this process is accurately *circular*, and is not a mere approximation to a circular arc, as some may suppose. This method produces a circular curve, because all inscribed angles on one side of a chord-line are equal (*Art.* 356). To obtain the radius from a chord and its versed sine, see *Art.* 444.

If the angle formed by the rulers at c be a right angle,

the segment described will be a semi-circle. This problem is useful in describing centres for brick arches, when they are required to be rather flat. Also, for the head hanging-stile of a window-frame, where a brick arch, instead of a stone lintel, is to be placed over it.

517.—To Find the Radius of an Arc of a Circle when the Chord and Versed Sine are Given.—The radius is equal to the sum of the squares of half the chord and of the versed sine, divided by twice the versed sine. This is expressed, algebraically, thus: $r = \frac{(\frac{c}{2})^2 + v^2}{2v}$, where r is the radius, c the chord, and v the versed sine (*Art.* 444).

Example.—In a given arc of a circle a chord of 12 feet has the rise at the middle, or the versed sine, equal to 2 feet, what is the radius?

Half the chord equals 6, the square of 6 is,	$6 \times 6 = 36$
The square of the versed sine is,	$2 \times 2 = 4$
Their sum equals,	40

Twice the versed sine equals 4, and 40 divided by 4 equals 10. Therefore the radius, in this case, is 10 feet. This result is shown in less space and more neatly by using the above algebraical formula. For the letters substituting their value, the formula $r = \frac{(\frac{c}{2})^2 + v^2}{2v}$ becomes $r = \frac{(\frac{12}{2})^2 + 2^2}{2 \times 2}$, and performing the arithmetical operations here indicated equals—

$$\frac{6^2 + 2^2}{4} = \frac{36 + 4}{4} = \frac{40}{4} = 10.$$

518.—To Find the Versed Sine of an Arc of a Circle when the Radius and Chord are Given.—The versed sine is equal to the radius, less the square root of the difference of the squares of the radius and half chord; expressed algebraically thus: $v = r - \sqrt{r^2 - (\frac{c}{2})^2}$, where r is the radius, v the versed sine, and c the chord. (Equation (161.) reduced.)

Example.—In an arc of a circle whose radius is 75 feet, what is the versed sine to a chord of 120 feet? By the table in the Appendix it will be seen that—

The square of the radius, 75, equals . .	5625
The square of half the chord, 60, equals .	3600
The difference is	2025
The square root of this is	45
This deducted from the radius . . .	75
The remainder is the versed sine, =	30

This is expressed by the formula, thus—

$$v = 75 - \sqrt{75^2 - (\tfrac{120}{2})^2} = 75 - \sqrt{5625 - 3600} = 75 - 45 = 30.$$

519.—To Describe the Segment of a Circle by Intersection of Lines.—Let $a\,b$ (*Fig.* 370) be the chord, and $c\,d$ the

FIG. 370.

height of the segment. Through c draw $e\,f$ parallel to $a\,b$; draw $b\,f$ at right angles to $c\,b$; make $c\,e$ equal to $c\,f$; draw $a\,g$ and $b\,h$ at right angles to $a\,b$; divide $c\,e$, $c\,f$, $d\,a$, $d\,b$, $a\,g$, and $b\,h$, each into a like number of equal parts, as four; draw the lines 1 1, 2 2, etc., and from the points o, o, and o, draw lines to c; at the intersection of these lines trace the curve, $a\,c\,b$, which will be the segment required.

In very large work, or in laying out ornamental gardens, etc., this will be found useful; and where the centre of the proposed arc of a circle is inaccessible it will be invaluable. (To trace the curve, see note at *Art.* 550.)

The lines $e\,a$, $c\,d$, and $f\,b$, would, were they extended, meet in a point, and that point would be in the opposite side of the circumference of the circle of which $a\,c\,b$ is a

segment. The lines 1 1, 2 2, 3 3, would likewise, if extended, meet in the same point. The line *c d*, if extended to the opposite side of the circle, would become a diameter. The line *f b* forms, by construction, a right angle with *b c*, and hence the extension of *f b* would also form a right angle with *b c*, on the opposite side of *b c*; and this right angle would be the inscribed angle in the semi-circle; and since this is required to be a *right* angle (*Art.* 352), therefore the construction thus far is correct, and it will be found likewise that at each point in the curve formed by the intersection of the radiating lines, these intersecting lines are at right angles.

520. — Ordinates. — Points in the circumference of a circle may be obtained arithmetically, and positively accurate, by the calculation of *ordinates*, or the parallel lines o 1,

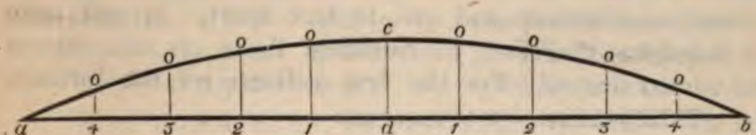


FIG. 371.

o 2, o 3, o 4 (*Fig.* 371). These ordinates are drawn at right angles to the chord-line *a b*, and they may be drawn at any distance apart, either equally distant or unequally, and there may be as many of them as is desirable; the more there are the more points in the curve will be obtained. If they are located in pairs, equally distant from the versed sine *c d*, calculation need be made only for those on one side of *c d*, as those on the opposite side will be of equal lengths, respectively; for example: o 1, on the left-hand side of *c d*, is equal to o 1 on the right-hand side, o 2 on the right equals o 2 on the left, and in like manner for the others.

The length of any ordinate is equal to the square root of the difference of the squares of the radius and abscissa, less the difference between the radius and versed sine (*Art.* 445). The abscissa being the distance from the foot of the versed sine to the foot of the ordinate. Algebraically,

$t = \sqrt{r^2 - x^2} - (r - b)$, where t is put to represent the ordinate; x, the abscissa; b, the versed sine; and r, the radius.

Example.—An arc of a circle has its chord $a\,b$ (*Fig.* 371) 100 feet long, and its versed sine $c\,d$, 5 feet. It is required to ascertain the length of ordinates for a sufficient number of points through which to describe the curve. To this end it is requisite, first, to ascertain the radius. This is readily done in accordance with *Art.* 517. For $\frac{\left(\frac{c}{2}\right)^2 + v^2}{2\,v}$ becomes $\frac{50^2 + 5^2}{2 \times 5} = 252 \cdot 5 =$ radius. Having the radius, the curve might at once be described without the ordinate points, but for the impracticability that usually occurs, in large, flat segments of the circle, of getting a location for the centre, the centre usually being inaccessible. The ordinates are, therefore, to be calculated. In *Fig.* 371 the ordinates are located equidistant, and are 10 feet apart. It will only be requisite, therefore, to calculate those on one side of the versed sine $c\,d$. For the first ordinate o 1, the formula $t = \sqrt{r^2 - x^2} - (r - b)$ becomes—

$$t = \sqrt{252 \cdot 5^2 - 10^2} - (252 \cdot 5 - 5).$$
$$= \sqrt{63756 \cdot 25 - 100} - 247 \cdot 5.$$
$$= \quad 252 \cdot 3019 - 247 \cdot 5.$$
$$= \quad 4 \cdot 8019 = \text{the first ordinate, o 1.}$$

For the second—

$$t = \sqrt{252 \cdot 5^2 - 20^2} - (252 \cdot 5 - 5).$$
$$= \quad 251 \cdot 7066 - 247 \cdot 5.$$
$$= \quad 4 \cdot 2066 = \text{the second ordinate, o 2.}$$

For the third—

$$t = \sqrt{252 \cdot 5^2 - 30^2} - 247 \cdot 5.$$
$$= \quad 250 \cdot 7115 - 247 \cdot 5.$$
$$= \quad 3 \cdot 2115 = \text{the third ordinate, o 3.}$$

For the fourth—

$$t = \sqrt{252\cdot5^2 - 40^2} - 247\cdot5.$$

$$= \quad 249\cdot3115 - 247\cdot5.$$

$$= \quad 1\cdot8115 = \text{the fourth ordinate, } o\,4.$$

The results here obtained are in feet and decimals of a foot. To reduce these to feet, inches, and eighths of an inch, proceed as at Reduction of Decimals in the Appendix. If the two-feet rule, used by carpenters and others, were decimally divided, there would be no necessity of this reduction, and it is to be hoped that the rule will yet be thus divided, as such a reform would much lessen the labor of computations, and insure more accurate measurements.

Versed sine $c\,d$ =	ft. 5·0	=	ft. 5·0 inches.
Ordinates o 1 =	4·8019	=	4·9⅝ inches, nearly.
" o 2 =	4·2066	=	4·2½ inches, nearly.
" o 3 =	3·2115	=	3·2½ inches, nearly.
" o 4 =	1·8115	=	1·9¾ inches, nearly.

521.—In a Given Angle, to Describe a Tanged Curve. —Let $a\,b\,c$ (*Fig.* 372) be the given angle, and 1 in the line $a\,b$,

FIG. 372.

and 5 in the line $b\,c$, the termination of the curve. Divide 1 b and b 5 into a like number of equal parts, as at 1, 2, 3, 4, and 5; join 1 and 1, 2 and 2, 3 and 3, etc.; and a regular curve will be formed that will be tangical to the line $a\,b$, at the point 1, and to $b\,c$ at 5.

This is of much use in stair-building, in easing the angles formed between the wall-string and the base of the hall, also between the front string and level facia, and in many other instances. The curve is not circular, but of the form of the parabola (*Fig.* 418); yet in large angles the difference

is not perceptible. This problem can be applied to describing the curve for door-heads, window-heads, etc., to rather better advantage than *Art.* 516. For instance, let *a b* (*Fig.* 373) be the width of the opening, and *c d* the height of the

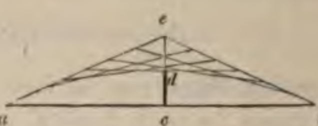


FIG. 373.

arc. Extend *c d*, and make *d e* equal to *c d*; join *a* and *e*, also *e* and *b*; and proceed as directed above.

522.—To Describe a Circle within any Given Triangle, so that the Sides of the Triangle shall be Tangical.—Let *a b c* (*Fig.* 374) be the given triangle. Bisect the angles

FIG. 374.

a and *b* according to *Art.* 506; upon *d*, the point of intersection of the bisecting lines, with the radius *d e*, describe the required circle.

523.—About a Given Circle, to Describe an Equilateral Triangle.—Let *a d b c* (*Fig.* 375) be the given circle. Draw the diameter *c d*; upon *d*, with the radius of the given circle, describe the arc *a e b*; join *a* and *b*; draw *f g* at right angles to *d c*; make *f c* and *c g* each equal to *a b*; from *f*, through *a*, draw *f h*, also from *g*, through *b*, draw *g h*; then *f g h* will be the triangle required.

524.—To Find a Right Line nearly Equal to the Circumference of a Circle.—Let *a b c d* (*Fig.* 376) be the given

. Draw the diameter $a c$; on this erect an equilateral ;le $a e c$ according to *Art.* 525; draw $g f$ parallel to $a c$; d $e c$ to f, also $e a$ to g; then $g f$ will be nearly the

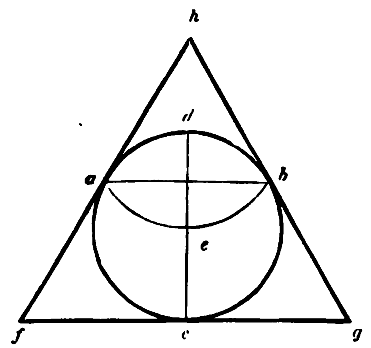


FIG. 375.

ı of the semi-circle $a d c$; and twice $g f$ will nearly the circumference of the circle $a b c d$, as was required. nes drawn from e, through any points in the circle, as ıd o, to p, p and p, will divide $g f$ in the same way as :mi-circle $a d c$ is divided. So, any portion of a circle ɔe transferred to a straight line. This is a very useful

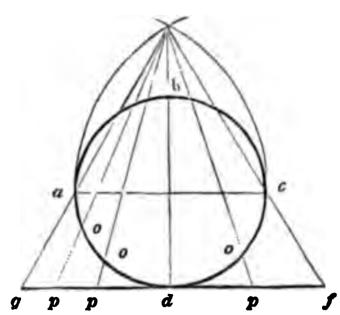


FIG. 376.

əm, and should be well studied, as it is frequently :o solve problems on stairs, domes, etc.

ıother method. Let $a b f c$ (*Fig.* 377) be the given circle. the diameter $a c$; from d, the centre, and at right an-

gles to *a c*, draw *d b*; join *b* and *c*; bisect *b c* at *e*; from *d*, through *e*, draw *d f*; then *e f* added to three times the diameter, will equal the circumference of the circle sufficiently near for many uses. The result is a trifle too large. If the

FIG. 377.

circumference found by this rule be divided by 648·22 the quotient will be the excess. Deduct this excess, and the remainder will be the true circumference. This problem is rather more curious than useful, as it is less labor to perform the operation arithmetically, simply multiplying the given diameter by 3·1416, or, where a greater degree of accuracy is needed, by 3·1415926. (See *Art.* 446.)

POLYGONS, ETC.

525.—Upon a Given Line to Construct an Equilateral Triangle.—Let *a b* (*Fig.* 378) be the given line. Upon *a* and

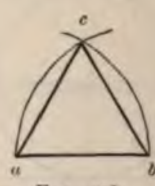


FIG. 378.

b, with *a b* for radius, describe arcs, intersecting at *c*; join *a* and *c*, also *c* and *b*; then *a c b* will be the triangle required.

526.—To Describe an Equilateral Rectangle, or Square.—Let *a b* (*Fig.* 379) be the length of a side of the proposed

quare. Upon *a* and *b*, with *ab* for radius, describe the arcs *d* and *bc*; bisect the arc *ac* in *f*; upon *e*, with *ef* for radius, describe the arc *cfd*; join *a* and *c*, *c* and *d*, *d* and *b*; hen *acdb* will be the square required.

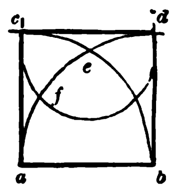


FIG. 379.

527.—Within a Given Circle, to Inscribe an Equilateral Triangle, Hexagon or Dodecagon.—Let *abcd* (*Fig.* 380) be he given circle. Draw the diameter *bd*; upon *b*, with the adius of the given circle, describe the arc *aec*; join *a* and *c*, lso *a* and *d*, and *c* and *d*—and the triangle is completed. For the hexagon: from *a*, also from *c*, through *e*, draw the ines *af* and *cg*; join *a* and *b*, *b* and *c*, *c* and *f*, etc., and the

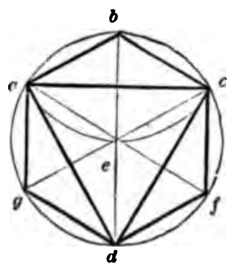


FIG. 380.

iexagon is completed. The dodecagon may be formed by bisecting the sides of the hexagon.

Each side of a regular hexagon is exactly equal to the adius of the circle that circumscribes the figure. For the adius is equal to a chord of an arc of 60 degrees; and, as every circle is supposed to be divided into 360 degrees, there s just 6 times 60, or 6 arcs of 60 degrees, in the whole circumference. A line drawn from each angle of the hexagon

to the centre (as in the figure) divides it into six equal, equilateral triangles.

528.—Within a Square to Inscribe an Octagon.—Let *a b c d* (*Fig.* 381) be the given square. Draw the diagonals

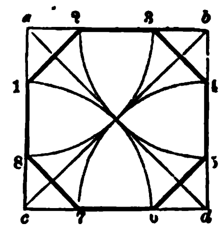


FIG 381.

a d and *b c*; upon *a*, *b*, *c*, and *d*, with *a e* for radius, describe arcs cutting the sides of the square at 1, 2, 3, 4, 5, 6, 7, and 8: join 1 and 2, 3 and 4, 5 and 6, etc., and the figure is completed.

In order to eight-square a hand-rail, or any piece that is

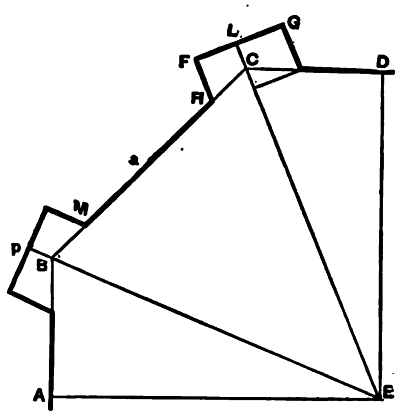


FIG. 382.

to be afterwards rounded, draw the diagonals *a d* and *b c* upon the end of it, after it has been squared-up. Set a gauge to the distance *a e* and run it upon the whole length of the stuff, from each corner both ways. This will show

how much is to be chamfered off, in order to make the piece octagonal (*Art.* 354).

529.—To Find the Side of a Buttressed Octagon.—Let $A\,B\,C\,D\,E$ (*Fig.* 382) represent one quarter of an octagon structure, having a buttress $H\,F\,G\,J$ at each angle. The distance $M\,H$, between the buttresses, being given, as also $F\,G$, the width of a buttress; to find $H\,C$ or $C\,J$, in order to obtain $B\,C$, the side of the octagon. Let $B\,C$, a side of the octagon, be represented by b; or $D\,C$ by $\frac{1}{2}b$. Let $M\,H = a$; or $J\,D = \frac{1}{2}a$; and $J\,C = x$.

Then we have—

$$J\,D + J\,C = C\,D,$$

$$\tfrac{1}{2}a + x = \tfrac{1}{2}b,$$

$$a + 2x = b.$$

For $F\,G$ put p; or $L\,G = K\,J = \frac{1}{2}p$.

Now $E\,D$ is the radius of an inscribed circle and, as per equation (140.), equals $r = (\sqrt{2} + 1)\dfrac{b}{2}$.

Also, $E\,C$ is the radius of a circumscribed circle, and, as per equation (141.), equals $R = \sqrt{2\sqrt{2} + 4}\,\dfrac{b}{2}$.

The two triangles, $C\,J\,K$ and $C\,E\,D$, are homologous; for the angles at C are common and the angles at K and D are right angles. Having thus two angles of one equal respectively to the two angles of the other, therefore (*Art.* 345) the remaining angles must be equal. Hence, the sides of the triangles are proportionate, or—

$$E\,D : E\,C :: J\,K : C\,J$$

$$r : R :: \tfrac{1}{2}p : x = \tfrac{1}{2}p\,\frac{R}{r}.$$

The value of the side, as above, is—

$$b = a + 2x = a + p\,\frac{R}{r}.$$

And taking the value of R and r, as above, we have—

$$\frac{R}{r} = \frac{\sqrt{2\sqrt{2}+4}^{\frac{b}{2}}}{(\sqrt{2}+1)^{\frac{b}{2}}} = \frac{\sqrt{2\sqrt{2}+4}}{\sqrt{2}+1}.$$

Substituting this for $\frac{R}{r}$, we have—

$$b = a + p\frac{\sqrt{2\sqrt{2}+4}}{\sqrt{2}+1}.$$

The numerical coefficient of p reduces to $1 \cdot 0823923$ or $1 \cdot 0824$, nearly.

Therefore we have—

$$b = a + 1 \cdot 0824\, p. \qquad (207.)$$

Or: The *side* of a buttressed octagon equals the *distance between the buttresses plus* $1 \cdot 0824$ times the width of the *face* of the *buttress*.

For example: let there be an octagon building, which measures between the buttresses, as at MH, 18 feet, and the face of the buttresses, as FG, equals 3 feet; what, in such a building, is the length of a side BC? For this, using equation (207.), we have—

$$\begin{aligned} b &= 18 + 1 \cdot 0824 \times 3 \\ &= 18 + 3 \cdot 2472 \\ &= 21 \cdot 2472. \end{aligned}$$

Or: The side of the octagon BC equals 21 feet and nearly 3 inches.

530.—Within a Given Circle to Inscribe any Regular Polygon.—Let abc 2 (*Figs.* 383, 384, and 385) be given circles. Draw the diameter ac; upon this erect an equilateral triangle aec, according to *Art.* 525; divide ac into as many equal parts as the polygon is to have sides, as at 1, 2, 3, 4, etc.; from e, through each even number, as 2, 4, 6, etc., draw lines

tting the circle in the points 2, 4, etc.; from these points d at right angles to *a c* draw lines to the opposite part the circle; this will give the remaining points for the lygon, as *b*, *f*, etc.

In forming a hexagon, the sides of the triangle erected

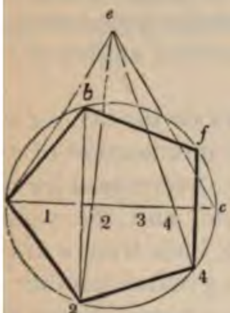


FIG. 383.

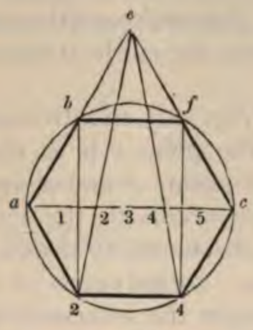


FIG. 384.

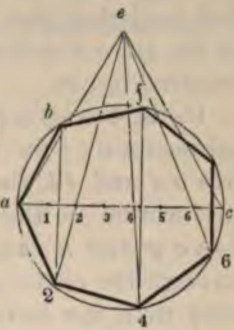


FIG. 385.

on *a c* (as at *Fig.* 384) mark the points *b* and *f*. This ethod of locating the angles of a polygon is an approxima- n sufficiently near for many purposes; it is based upon e like principle with the method of obtaining a right line *arly* equal to a circle (*Art.* 524). The method shown at *t.* 531 is accurate.

FIG. 386.

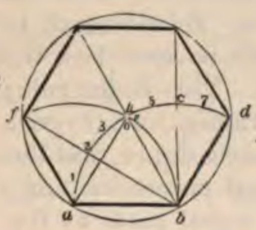


FIG. 387.

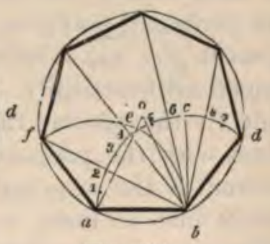


FIG. 388

531.—Upon a Given Line to Describe any Regular lygon.—Let *a b* (*Figs.* 386, 387, and 388) be given lines, ual to a side of the required figure. From *b* draw *b c* at ght angles to *a b*; upon *a* and *b*, with *a b* for radius, describe e arcs *a c d* and *f e b*; divide *a c* into as many equal parts

as the polygon is to have sides. and extend those divisions from *c* towards *d*; from the second point of division, counting from *c* towards *a*, as 3 (*Fig.* 386), 4 (*Fig.* 387), and 5 (*Fig.* 388), draw a line to *b*; take the distance from said point of division to *a*, and set it from *b* to *e*; join *e* and *a*; upon the intersection *o* with the radius *oa*, describe the circle *afdb*; then radiating lines, drawn from *b* through the even numbers on the arc *ad*, will cut the circle at the several angles of the required figure.

In the hexagon (*Fig.* 387), the divisions on the arc *ad* are not necessary; for the point *o* is at the intersection of the arcs *ad* and *fb*, the points *f* and *d* are determined by the intersection of those arcs with the circle, and the points above *g* and *h* can be found by drawing lines from *a* and *b* through the centre *o*. In polygons of a greater number of sides than the hexagon the intersection *o* comes above the arcs; in such case, therefore, the lines *ae* and *b* 5 (*Fig.* 388) have to be extended before they will intersect. This method of describing polygons is founded on correct principles, and is therefore accurate. In the circle equal arcs subtend equal angles (*Arts.* 357 and 515). Although this method is accurate, yet polygons may be described as accurately and more simply in the following manner. It will be observed that much of the process in this method is for the purpose of ascertaining the centre of a circle that will circumscribe the proposed polygon. By reference to the Table of Polygons in *Art.* 442 it will be seen how this centre may be obtained arithmetically. This is the rule: multiply the given side by the tabular radius for polygons of a like number of sides with the proposed figure, and the product will be the radius of the required circumscribing circle. Divide this circle into as many equal parts as the polygon is to have sides, connect the points of division by straight lines, and the figure is complete. For example: It is desired to describe a polygon of 7 sides, and 20 inches a side. The tabular radius is 1·15238. This multiplied by 20, the product, 23·0476 is the required radius in inches. The Rules for the Reduction of Decimals, in the Appendix, show how to change decimals to the fractions of a foot or an inch. From

this, 23·0476 is equal to $23\frac{1}{16}$ inches, nearly. It is not needed to take all the decimals in the table, three or four of them will give a result sufficiently near for all ordinary practice.

532.—To Construct a Triangle whose Sides shall be severally Equal to Three Given Lines.—Let *a*, *b* and *c* (*Fig.* 389) be the given lines. Draw the line *d e* and make it equal

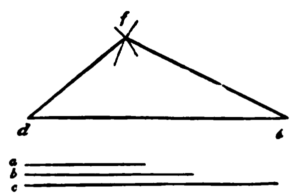


FIG. 389.

c; upon *e*, with *b* for radius, describe an arc at *f*; upon *d*, with *a* for radius, describe an arc intersecting the other at *f*; join *d* and *f*, also *f* and *e*; then *d f e* will be the triangle required.

533.—To Construct a Figure Equal to a Given, Right-lined Figure.—Let *a b c d* (*Fig.* 390) be the given figure. Make *e f* (*Fig.* 391) equal to *c d*; upon *f*, with *d a* for radius,

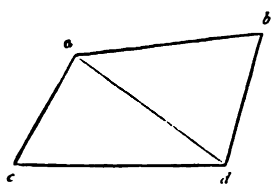


FIG. 390.

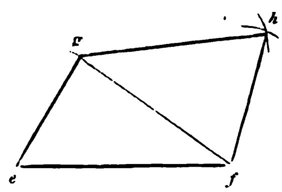


FIG. 391.

describe an arc at *g*; upon *e*, with *c a* for radius, describe an arc intersecting the other at *g*; join *g* and *e*; upon *f* and *g*, with *d b* and *a b* for radius, describe arcs intersecting at *h*; join *g* and *h*, also *h* and *f*; then *Fig.* 391 will every way equal *Fig.* 390.

So, right-lined figures of any number of sides may be copied, by first dividing them into triangles, and then pro-

ceeding as above. The shape of the floor of any room, or of any piece of land, etc., may be accurately laid out by this problem, at a scale upon paper; and the contents in square feet be ascertained by the next.

534.—To Make a Parallelogram equal to a Given Triangle.—Let *a b c* (*Fig.* 392) be the given triangle. From *a* draw *a d* at right angles to *b c*; bisect *a d* in *e*; through *e*

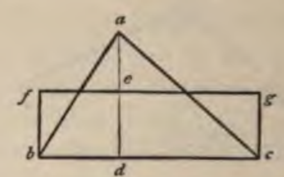


FIG. 392.

draw *f g* parallel to *b c*; from *b* and *c* draw *b f* and *c g* parallel to *d e*; then *b f g c* will be a parallelogram containing a surface exactly equal to that of the triangle *a b c*.

Unless the parallelogram is required to be a rectangle, the lines *b f* and *c g* need not be drawn parallel to *d e*. If a rhomboid is desired they may be drawn at an oblique angle, provided they be parallel to one another. To ascertain the area of a triangle, multiply the base *b c* by half the perpendicular height *d a*. In doing this it matters not which side is taken for base.

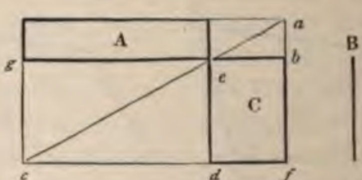


FIG. 393.

535.—A Parallelogram being Given, to Construct Another Equal to it, and Having a Side Equal to a Given Line.—Let *A* (*Fig.* 393) be the given parallelogram, and *B* the given line. Produce the sides of the parallelogram, as at

a, *b*, *c*, and *d*; make *e d* equal to *B*; through *d* draw *c f* parallel to *g b*; through *e* draw the diagonal *c a*; from *a* draw *a f* parallel to *e d*; then *C* will be equal to *A*. (See *Art.* 340.)

536.—To Make a Square Equal to two or more Given Squares.—Let *A* and *B* (*Fig.* 394) be two given squares.

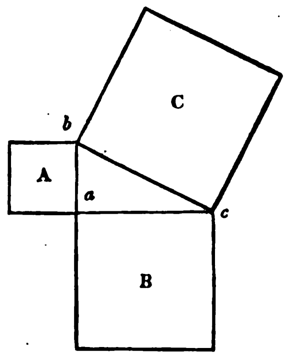


FIG. 394.

Place them so as to form a right angle, as at *a*; join *b* and *c*; then the square *C*, formed upon the line *b c*, will be equal in extent to the squares *A* and *B* added together. Again: if *a b* (*Fig.* 395) be equal to the side of a given square, *c a*, placed at right angles to *a b*, be the side of another given square,

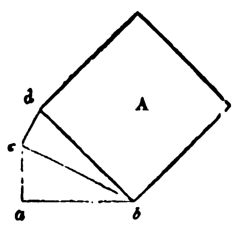


FIG. 395.

and *c d*, placed at right angles to *c b*, be the side of a third given square, then the square *A*, formed upon the line *d b*, will be equal to the three given squares. (See *Art.* 353.)

The usefulness and importance of this problem are proverbial. To ascertain the length of braces and of rafters in

framing, the length of stair-strings, etc., are some of the purposes to which it may be applied in carpentry. (See note to *Art.* 503.) If the lengths of any two sides of a right-angled triangle are known, that of the third can be ascertained. Because the square of the hypothenuse is equal to the united squares of the two sides that contain the right angle.

(1.)—The two sides containing the right angle being known, to find the hypothenuse.

Rule.—Square each given side, add the squares together, and from the product extract the square root; this will be the answer.

For instance, suppose it were required to find the length of a rafter for a house, 34 feet wide—the ridge of the roof to be 9 feet high, above the level of the wall-plates. Then 17 feet, half of the span, is one, and 9 feet, the height, is the other of the sides that contain the right angle. Proceed as directed by the rule:

```
  17                   9
  17                   9
 ---                 ---
 119                  81 = square of 9.
 17                  289 = square of 17.
 ---                 ---
 289 = square of 17. 370 Product.
```

```
1 ) 370 ( 19·235 + = square root of 370; equal 19 feet 2⅞ in.,
1   1                  nearly; which would be the required
    ---                length of the rafter.
29 ) 270
 9   261
     ------
382 ) ·900
  2    764
     ------
3843 ) 13600
   3   11529
     --------
38465 ) ·207100
         192325
         ------
```

(By reference to the table of square roots in the Appendix, the root of almost any number may be found ready calculated; also, to change the decimals of a foot to inches and parts, see Rules for the Reduction of Decimals in the Appendix.)

Again: suppose it be required, in a frame building, to find the length of a brace having a run of three feet each way from the point of the right angle. The length of the sides containing the right angle will be each 3 feet; then, as before—

$$
\begin{array}{rl}
 & 3 \\
 & \underline{3} \\
 & 9 = \text{square of one side.} \\
3 \text{ times } 3 = & \underline{9} = \text{square of the other side.} \\
 & 18 \text{ Product: the square root of which is } 4 \cdot 2426+ \text{ ft.,}
\end{array}
$$

or 4 feet 2 inches and $\frac{7}{8}$ full.

(2.)—The hypothenuse and one side being known, to find the other side.

Rule.—Subtract the square of the given side from the square of the hypothenuse, and the square root of the product will be the answer.

Suppose it were required to ascertain the greatest perpendicular height a roof of a given span may have, when pieces of timber of a given length are to be used as rafters. Let the span be 20 feet, and the rafters of 3 x 4 hemlock joist. These come about 13 feet long. The known hypothenuse, then, is 13 feet, and the known side, 10 feet—that being half the span of the building.

$$
\begin{array}{rl}
 & 13 \\
 & \underline{13} \\
 & 39 \\
 & \underline{13} \\
 & 169 = \text{square of hypothenuse.} \\
10 \text{ times } 10 = & \underline{100} = \text{square of the given side.} \\
 & 69 \text{ Product: the square root of which is}
\end{array}
$$

8·3066+ feet, or 8 feet 3 inches and $\frac{5}{8}$ full. This will be the greatest perpendicular height, as required. Again: suppose that in a story of 8 feet, from floor to floor, a step-ladder is required, the strings of which are to be of plank 12 feet long, and it is desirable to know the greatest run such a length of string will afford. In this case, the two given sides are—hypothenuse 12, perpendicular 8 feet.

12 times 12 = 144 = square of hypothenuse,
8 times 8 = 64 = square of perpendicular.

80 Product: the square root of which is 8·9442+ feet, or 8 feet 11 inches and $\frac{5}{16}$—the answer, as required.

Many other cases might be adduced to show the utility of this problem. A practical and ready method of ascertaining the length of braces, rafters, etc., when not of a great length, is to apply a rule across the carpenters'-square. Suppose, for the length of a rafter, the base be 12 feet and the height 7. Apply the rule diagonally on the square, so that it touches 12 inches from the corner on one side, and 7 inches from the corner on the other. The number of inches on the rule which are intercepted by the sides of the square, $13\frac{7}{8}$, nearly, will be the length of the rafter in feet; viz., 13 feet and $\frac{7}{8}$ of a foot. If the dimensions are large, as 30 feet and 20, take the half of each on the sides of the square, viz., 15 and 10 inches; then the length in inches across will be one half the number of feet the rafter is long. This method is just as accurate as the preceding; but when the length of a very long rafter is sought, it requires great care and precision to ascertain the fractions. For the least variation on the square, or in the length taken on the rule, would make perhaps several inches difference in the length of the rafter. For shorter dimensions, however, the result will be true enough.

537.—To Make a Circle Equal to two Given Circles.—Let A and B (*Fig.* 396) be the given circles. In the right-angled triangle $a\,b\,c$ make $a\,b$ equal to the diameter of the

circle B, and cb equal to the diameter of the circle A; then the hypothenuse ac will be the diameter of a circle C, which will be equal in area to the two circles A and B, added together.

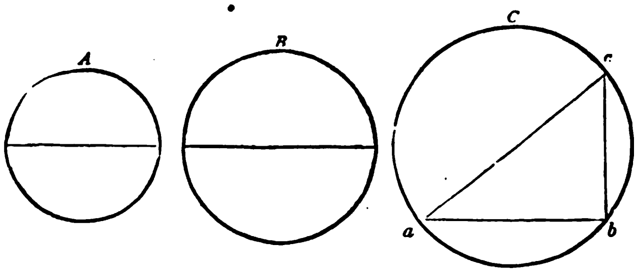


FIG. 396.

Any polygonal figure, as A (*Fig.* 397), formed on the hypothenuse of a right-angled triangle, will be equal to two similar figures,* as B and C, formed on the two legs of the triangle.

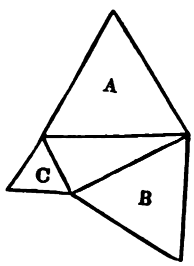


FIG. 397.

538.—To Construct a Square Equal to a Given Rectangle.—Let A (*Fig.* 398) be the given rectangle. Extend the side ab and make bc equal to be; bisect ac in f, and upon f, with the radius fa, describe the semi-circle agc; extend eb till it cuts the curve in g; then a square $bghd$, formed on the line bg, will be equal in area to the rectangle A.

* Similar figures are such as have their several angles respectively equal, and their sides respectively proportionate.

Another method. Let *A* (*Fig.* 399) be the given rectangle. Extend the side *a b* and make *a d* equal to *a c*; bisect *a d* in *e*; upon *e*, with the radius *e a*, describe the semi-circle *a f d*; extend *g b* till it cuts the curve in *f*; join *a* and *f*; then

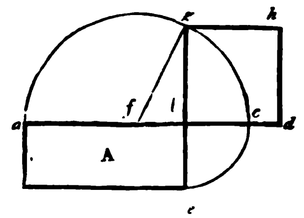


FIG. 398.

the square *B*, formed on the line *a f*, will be equal in area to the rectangle *A*. (See *Arts.* 352 and 353.)

539.—To Form a Square Equal to a Given Triangle.— Let *a b* (*Fig.* 398) equal the base of the given triangle, and *b c* equal half its perpendicular height (see *Fig.* 392); then proceed as directed at *Art.* 538.

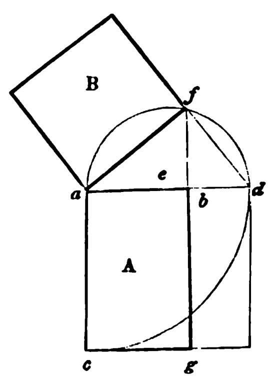


FIG. 399.

540.—Two Right Lines being Given, to Find a Third Proportional Thereto.—Let *A* and *B* (*Fig.* 400) be the given lines. Make *a b* equal to *A*; from *a* draw *a c* at any angle

with *a b*; make *a c* and *a d* each equal to *B*; join *c* and *b*; **from *d*** draw ***d e*** parallel to *c b*; then *a e* will be the third proportional required. That is, *a e* bears the same proportion to *B* as *B* does to *A*.

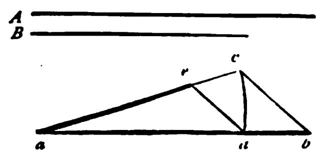


FIG. 400.

541.—Three Right Lines being Given, to Find a Fourth Proportional Thereto.—Let *A*, *B*, and *C* (*Fig.* 401) be the given lines. Make *a b* equal to *A*; from *a* draw *a c* at any

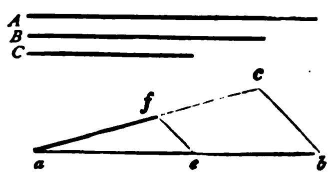


FIG. 401.

angle with *a b*; make *a c* equal to *B* and *a e* equal to *C*; join *c* and *b*; from *e* draw *e f* parallel to *c b*; then *a f* will be the fourth proportional required. That is, *a f* bears the same proportion to *C* as *B* does to *A*.

To apply this problem, suppose the two axes of a given ellipsis and the longer axis of a proposed ellipsis are given. Then, by this problem, the length of the shorter axis to the proposed ellipsis can be found; so that it will bear the same proportion to the longer axis as the shorter of the given ellipsis does to its longer. (See also *Art.* 559.)

542.—A Line with Certain Divisions being Given, to Divide Another, Longer or Shorter, Given Line in the Same Proportion.—Let *A* (*Fig.* 402) be the line to be divided, and *B* the line with its divisions. Make *a b* equal to *B* with all its divisions, as at 1, 2, 3, etc.; from *a* draw *a c* at any angle with *a b*; make *a c* equal to *A*; join *c* and *b*; from

the points 1, 2, 3, etc., draw lines parallel to cb; then these will divide the line ac in the same proportion as B is divided —as was required.

This problem will be found useful in proportioning the

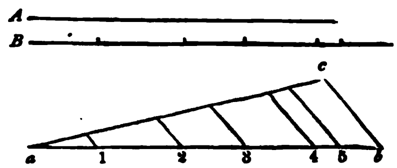


FIG. 402.

members of a proposed cornice, in the same proportion as those of a given cornice of another size. (See *Art.* 321.) So of a pilaster, architrave, etc.

543.—Between Two Given Right Lines, to Find a Mean Proportional.—Let A and B (*Fig.* 403) be the given lines. On the line ac make ab equal to A and bc equal to B; bisect ac in e; upon e, with ea for radius, describe the semi-circle adc; at b erect bd at right angles to ac; then

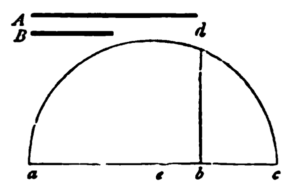


FIG. 403.

bd will be the mean proportional between A and B. That is, ab is to bd as bd is to bc. This is usually stated thus: $ab : bd :: bd : bc$, and since the product of the means equals the product of the extremes, therefore, $ab \times bc = \overline{bd}^2$. This is shown geometrically at *Art.* 538.

CONIC SECTIONS.

544.—Definitions.—If a cone, standing upon a base that is at right angles with its axis, be cut by a plane, per-

pendicular to its base and passing through its axis, the section will be an isosceles triangle (as *a b c*, *Fig.* 404); and the base will be a semi-circle. If a cone be cut by a plane in the direction *e f* the section will be an *ellipsis;* if in the direction *m l*, the section will be a *parabola;* and if in the direction *r o*, an *hyperbola*. (See *Art.* 499.) If the cutting planes be at right angles with the plane *a b c*, then—

545.—To Find the Axes of the Ellipsis: bisect *e f* (*Fig.* 404) in *g*; through *g* draw *h i* parallel to *a b*; bisect *h i* in *j*;

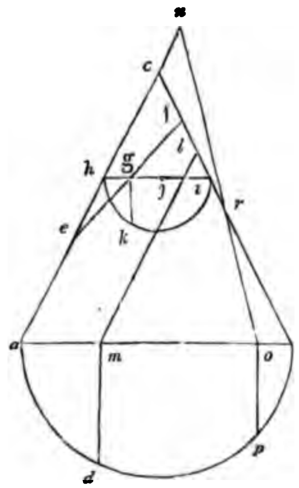


FIG. 404.

upon *j*, with *j h* for radius, describe the semi-circle *h k i*; from *g* draw *g k* at right angles to *h i*; then twice *g k* will be the conjugate axis and *e f* the transverse.

546.—To Find the Axis and Base of the Parabola.—Let *m l* (*Fig.* 404), parallel to *a c*, be the direction of the cutting plane. From *m* draw *m d* at right angles to *a b*; then *l m* will be the axis and height, and *m d* an ordinate and half the base, as at *Figs.* 417, 418.

547.—To Find the Height, Base, and Transverse Axis of an Hyperbola.—Let *o r* (*Fig.* 404) be the direction of the

cutting plane. Extend *o r* and *a c* till they meet at *n*; from *o* draw *o p* at right angles to *a b*; then *r o* will be the height, *n r* the transverse axis, and *o p* half the base; as at *Fig.* 419.

548.—The Axes being Given, to Find the Foci, and to Describe an Ellipsis with a String.—Let *a b* (*Fig.* 405) and *c d* be the given axes. Upon *c*, with *a e* or *b e* for radius, describe the arc *f f*; then *f* and *f*, the points at which the arc cuts the transverse axis, will be the *foci*. At *f* and *f* place two pins, and another at *c*; tie a string about the three pins, so as to form the triangle *f f c*; remove the pin from *c* and place a pencil in its stead; keeping the string taut,

FIG. 405.

move the pencil in the direction *c g a*; it will then describe the required ellipsis. The lines *f g* and *g f* show the position of the string when the pencil arrives at *g*.

This method, when performed correctly, is perfectly accurate; but the string is liable to stretch, and is, therefore, not so good to use as the trammel. In making an ellipse by a string or twine, that kind should be used which has the least tendency to elasticity. For this reason, a cotton cord, such as chalk-lines are commonly made of, is not proper for the purpose; a linen or flaxen cord is much better.

549.—The Axes being Given, to Describe an Ellipsis with a Trammel.—Let *a b* and *c d* (*Fig.* 406) be the given axes. Place the trammel so that a line passing through the centre of the grooves would coincide with the axes; make

the distance from the pencil *e* to the nut *f* equal to half *c d*; also, from the pencil *e* to the nut *g* equal to half *a b*; letting the pins under the nuts slide in the grooves, move the trammel *e g* in the direction *c b d*; then the pencil at *e* will describe the required ellipse.

A trammel may be constructed thus: take two straight strips of board, and make a groove on their face, in the centre of their width; join them together, in the middle of their length, at right angles to one another; as is seen at *Fig.* 406. A rod is then to be prepared, having two movable nuts made of wood, with a mortise through them of the size of the rod, and pins under them large enough to fill the grooves. Make a hole at one end of the rod, in which to

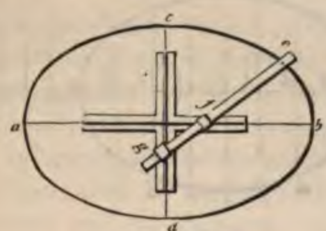


FIG. 406.

place a pencil. In the absence of a regular trammel a temporary one may be made, which, for any short job, will answer every purpose. Fasten two straight-edges at right angles to one another. Lay them so as to coincide with the axes of the proposed ellipse, having the angular point at the centre. Then, in a rod having a hole for the pencil at one end, place two brad-awls at the distances described at *Art.* 549. While the pencil is moved in the direction of the curve, keep the brad-awls hard against the straight-edges, as directed for using the trammel-rod, and one quarter of the ellipse will be drawn. Then, by shifting the straight-edges, the other three quarters in succession may be drawn. If the required ellipse be not too large, a carpenters'-square may be made use of, in place of the straight-edges.

An improved method of constructing the trammel is as

follows: make the sides of the grooves bevelling from the face of the stuff, or dove-tailing instead of square. Prepare two slips of wood, each about two inches long, which shall be of a shape to just fill the groove when slipped in at the end. These, instead of pins, are to be attached one to each of the movable nuts with a screw, loose enough for the nut to move freely about the screw as an axis. The advantage of this contrivance is, in preventing the nuts from slipping out of their places during the operation of describing the curve.

550.—To Describe an Ellipsis by Ordinates.—Let *ab* and *cd* (*Fig.* 407) be given axes. With *ce* or *ed* for radius

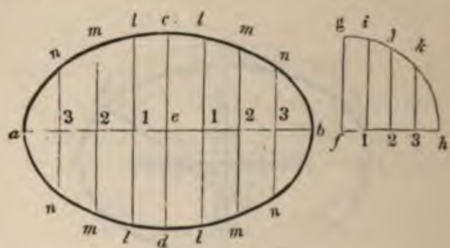


FIG. 407.

describe the quadrant *fgh*; divide *fh*, *ae*, and *eb*, each into a like number of equal parts, as at 1, 2, and 3; through these points draw ordinates parallel to *cd* and *fg*; take the distance 1 *i* and place it at 1 *l*, transfer 2 *j* to 2 *m*, and 3 *k* to 3 *n*; through the points *a*, *n*, *m*, *l*, and *c*, trace a curve, and the ellipsis will be completed.

The greater the number of divisions on *a*, *e*, etc., in this and the following problem, the more points in the curve can be found, and the more accurate the curve can be traced. If pins are placed in the points *n*, *m*, *l*, etc., and a thin slip of wood bent around by them, the curve can be made quite correct. This method is mostly used in tracing face-moulds for stair hand-railing.

551.—To Describe an Ellipsis by Intersection of Lines.—Let *ab* and *cd* (*Fig.* 408) be given axes. Through *c*, draw

fg parallel to *ab*; from *a* and *b* draw *af* and *bg* at right angles to *ab*; divide *fa*, *gb*, *ae*, and *eb*, each into a like number of equal parts, as at 1, 2, 3, and *o*, *o*, *o*; from 1, 2, and 3, draw lines to *c*; through *o*, *o*, and *o*, draw lines from *d*,

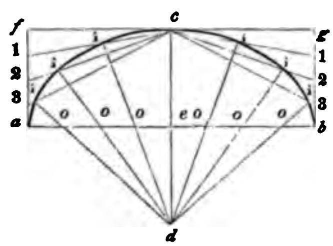


FIG. 408.

intersecting those drawn to *c*; then a curve, traced through the points *i*, *i*, *i*, will be that of an ellipsis.

Where neither trammel nor string is at hand, this, perhaps, is the most ready method of drawing an ellipsis. The divisions should be small, where accuracy is desirable. By this method an ellipsis may be traced without the axes, provided that a diameter and its conjugate be given. Thus, *ab*

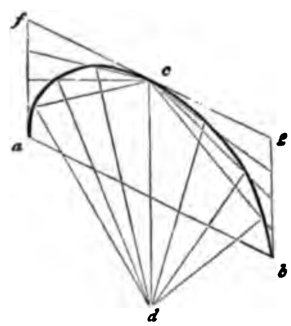


FIG. 409.

and *cd* (*Fig.* 409) are conjugate diameters: *fg* is drawn parallel to *ab*, instead of being at right angles to *cd*; also, *fa* and *gb* are drawn parallel to *cd*, instead of being at right angles to *ab*.

552.—To Describe an Ellipsis by Intersecting Arcs.— Let $a\,b$ and $c\,d$ (*Fig.* 410) be given axes. Between one of the foci, f and f, and the centre e, mark any number of points, at random, as 1, 2, and 3; upon f and f, with b 1 for radius, describe arcs at g, g, g, and g; upon f and f, with a 1 for

FIG. 410.

radius, describe arcs intersecting the others at g, g, g, and g; then these points of intersection will be in the curve of the ellipsis. The other points, h and i, are found in like manner, viz.: h is found by taking b 2 for one radius, and a 2 for the other; i is found by taking b 3 for one radius, and a 3 for the

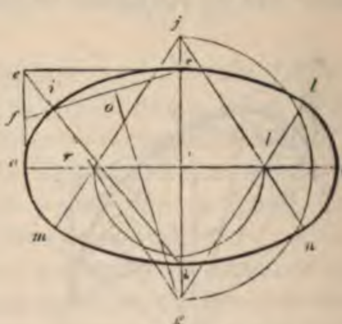


FIG. 411.

other, always using the foci for centres. Then by tracing a curve through the points c, g, h, i, b, etc., the ellipse will be completed.

This problem is founded upon the same principle as that of the string. This is obvious, when we reflect that the length of the string is equal to the transverse axis, added to

the distance between the foci. See *Fig.* 405, in which *cf* equals *ae*, the half of the transverse axis.

553.—To Describe a Figure Nearly in the Shape of an Ellipsis, by a Pair of Compasses.—Let *ab* and *cd* (*Fig.* 411) be given axes. From *c* draw *ce* parallel to *ab*; from *a* draw *ae* parallel to *cd*; join *e* and *d*; bisect *ea* in *f*; join *f* and *c*, intersecting *ed* in *i*; bisect *ic* in *o*; from *o* draw *og* at right angles to *ic*, meeting *cd* extended to *g*; join *i* and *g*, cutting the transverse axis in *r*; make *hj* equal to *hg*, and *hk* equal to *hr*; from *j*, through *r* and *k*, draw *jm* and *jn*; also, from *g*, through *k*, draw *gl*; upon *g* and *j*, with *gc* for radius, describe the arcs *il* and *mn*; upon *r* and *k*, with *ra* for

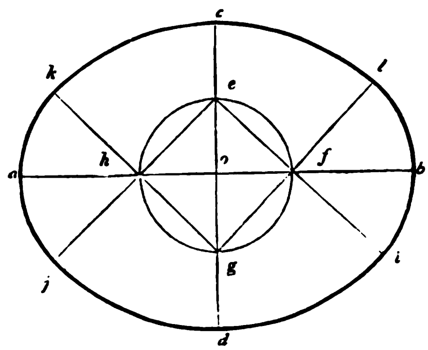


FIG. 412.

radius, describe the arcs *mi* and *ln*; this will complete the figure.

When the axes are proportioned to one another, as at 2 to 3, the extremities, *c* and *d*, of the shortest axis, will be the centres for describing the arcs *il* and *mn*; and the intersection of *cd* with the transverse axis will be the centre for describing the arc *m*, *i*, etc. As the elliptic curve is continually changing its course from that of a circle, a true ellipsis cannot be described with a pair of compasses. The above, therefore, is only an approximation.

554.—To Draw an Oval in the Proportion Seven by Nine.—Let *cd* (*Fig.* 412) be the given conjugate axis. Bisect

c d in *o*, and through *o* draw *a b* at right angles to *c d*; bisect *c o* in *e*; upon *o*, with *o e* for radius, describe the circle *e f g h*; from *e*, through *h* and *f*, draw *e j* and *e i*; also, from *g*, through *h* and *f*, draw *g k* and *g l*; upon *g*, with *g c* for radius, describe the arc *k l*; upon *e*, with *e d* for radius, describe the arc *j i*; upon *h* and *f*, with *h k* for radius, describe the arcs *j k* and *l i*; this will complete the figure.

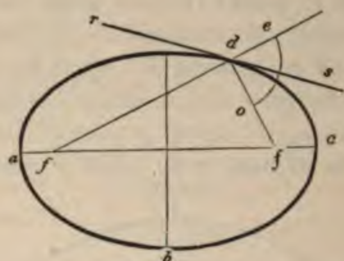


FIG. 413.

This is an approximation to an ellipsis; and perhaps no method can be found by which a well-shaped oval can be drawn with greater facility. By a little variation in the process, ovals of different proportions may be obtained. If quarter of the transverse axis is taken for the radius of the circle *e f g h*, one will be drawn in the proportion five by seven.

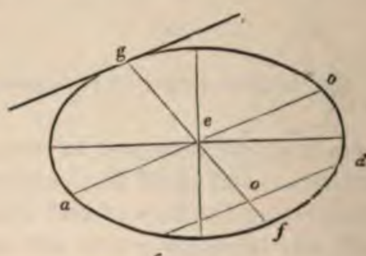


FIG. 414.

555.—To Draw a Tangent to an Ellipsis.—Let *a b c d* (*Fig.* 413) be the given ellipsis, and *d* the point of contact. Find the foci (*Art.* 548) *f* and *f*, and from them, through *d*, draw *f e* and *f d*; bisect the angle (*Art.* 506) *e d o* with the line *s r*; then *s r* will be the tangent required.

556.—An Ellipsis with a Tangent Given, to Detect the Point of Contact.—Let *a g b f* (*Fig.* 414) be the given ellipsis and tangent. Through the centre *e* draw *a b* parallel to the tangent; anywhere between *e* and *f* draw *c d* parallel to *a b*; bisect *c d* in *o*; through *o* and *e* draw *f g*; then *g* will be the point of contact required.

557.—A Diameter of an Ellipsis Given, to Find its Conjugate.—Let *a b* (*Fig.* 414) be the given diameter. Find the line *f g* by the last problem; then *f g* will be the diameter required.

558.—Any Diameter and its Conjugate being Given, to Ascertain the Two Axes, and thence to Describe the Ellipsis.—Let *a b* and *c d* (*Fig.* 415) be the given diameters, conjugate

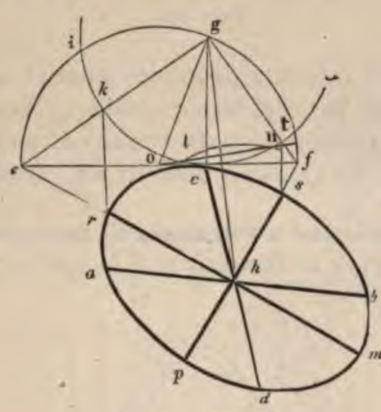


FIG. 415.

to one another. Through *c* draw *c f* parallel to *a b*; from *c* draw *c g* at right angles to *e f*; make *c g* equal to *a h* or *h b*; join *g* and *h*; upon *g*, with *g c* for radius, describe the arc *i k c j*; upon *h*, with the same radius, describe the arc *l n*; through the intersections *l* and *n* draw *n o*, cutting the tangent *e f* in *o*; upon *o*, with *o g* for radius, describe the semicircle *e i g f*; join *e* and *g*, also *g* and *f*, cutting the arc *i c j* in *k* and *t*; from *e*, through *h*, draw *e m*, also from *f*, through *h*, draw *f p*; from *k* and *t* draw *k r* and *t s* parallel to *g h*

cutting *e m* in *r*, and *f p* in *s*; make *h m* equal to *h r*, and *h p* equal to *h s*; then *r m* and *s p* will be the axes required, by which the ellipsis may be drawn in the usual way.

559.—To Describe an Ellipsis, whose Axes shall be Proportionate to the Axes of a Larger or Smaller Given One.—Let *a c b d* (*Fig.* 416) be the given ellipsis and axes, and

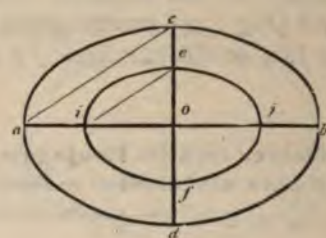


FIG. 416.

i j the transverse axis of a proposed smaller one. Join *a* and *c*; from *i* draw *i e* parallel to *a c*; make *o f* equal to *o e*; then *e f* will be the conjugate axis required, and will bear the same proportion to *i j* as *c d* does to *a b*. (See *Art.* 541.)

560.—To Describe a Parabola by Intersection of Lines.—Let *m l* (*Fig.* 417) be the axis and height (see *Fig.* 404) and

FIG. 417.

d d a double ordinate and base of the proposed parabola. Through *l* draw *a a* parallel to *d d*; through *d* and *d* draw *d a* and *d a* parallel to *m l*; divide *a d* and *d m*, each into a like number of equal parts; from each point of division in

dm draw the lines 1 1, 2 2, etc., parallel to *m l*; from each point of division in *d a* draw lines to *l*; then a curve traced through the points of intersection *o*, *o*, and *o*, will be that of a parabola.

Another method. Let *m l* (*Fig.* 418) be the axis and height, and *dd* the base. Extend *m l* and make *la* equal to *m l*; join *a* and *d*, and *a* and *d*; divide *a d* and *a d*, each into a like number of equal parts, as at 1, 2, 3, etc.; join 1 and 1, 2 and 2, etc., and the parabola will be completed. (See *Arts.* 460 to 472.)

561.—To Describe an Hyperbola by Intersection of Lines.—Let *r o* (*Fig.* 419) be the height, *p p* the base, and *n r* the transverse axis. (See *Fig.* 404.) Through *r* draw *a a*

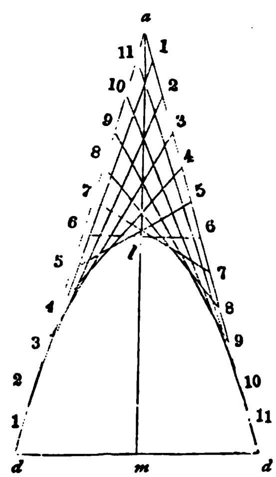


FIG. 418.

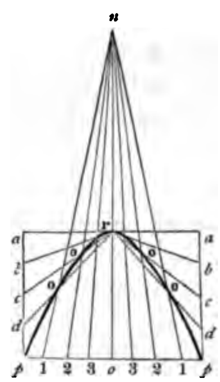


FIG. 419.

parallel to *p p*; from *p* draw *a p* parallel to *r o*; divide *a p* and *p o*, each into a like number of equal parts; from each of the points of division in the base, draw lines to *n*; from each of the points of division in *a p*, draw lines to *r*; then a curve traced through the points of intersection *o*, *o*, etc., will be that of an hyperbola.

The parabola and hyperbola afford handsome curves for various mouldings. (See *Figs.* 191 to 205; 222 to 224; 241 and 242; also note to *Art.* 318.)

SECTION XVII.—SHADOWS.

562.—The Art of Drawing consists in representing solids upon a plane surface, so that a curious and nice adjustment of lines is made to present the same appearance to the eye as does the human figure, a tree, or a house. It is by the effects of light, in its reflection, shade, and shadow, that the presence of an object is made known to us; so upon paper it is necessary, in order that the delineation may appear real, to represent fully all the shades and shadows that would be seen upon the object itself. In this section I propose to illustrate, by a few plain examples, the simple elementary principles upon which shading, in architectural subjects, is based. The necessary knowledge of drawing, preliminary to this subject, is treated of in Section XV., from *Arts.* 487 to 498.

563.—The Inclination of the Line of Shadow.—This is always, in architectural drawing, 45 degrees, both on the elevation and on the plan; and the sun is supposed to be behind the spectator, and over his left shoulder. This can be illustrated by reference to *Fig.* 420, in which *A* represents a horizontal plane, and *B* and *C* two vertical planes placed at right angles to each other. *A* represents the plan, *C* the elevation, and *B* a vertical projection from the elevation. In finding the shadow of the plane *B*, the line *a b* is drawn at an angle of 45 degrees with the horizon, and the line *c b* at the same angle with the vertical plane *B*. The plane *B* being a rectangle, this makes the true direction of the sun's rays to be in a course parallel to *d b*, which direction has been proved to be at an angle of 35 degrees and 16 minutes with the horizon. It is convenient, in shading, to have a set-square with the two sides that contain the

CRYPT OF ST. EUTROPE.

right angle of equal length; this will make the two acute angles each 45 degrees, and will give the requisite bevel when worked upon the edge of the T-square. One reason why this angle is chosen in preference to another is that when shadows are properly made upon the drawing by it, the depth of every recess is more readily known, since the breadth of shadow and the depth of the recess will be equal.

To distinguish between the terms *shade* and *shadow*, it will be understood that all such parts of a body as are not exposed to the direct action of the sun's rays are in *shade;*

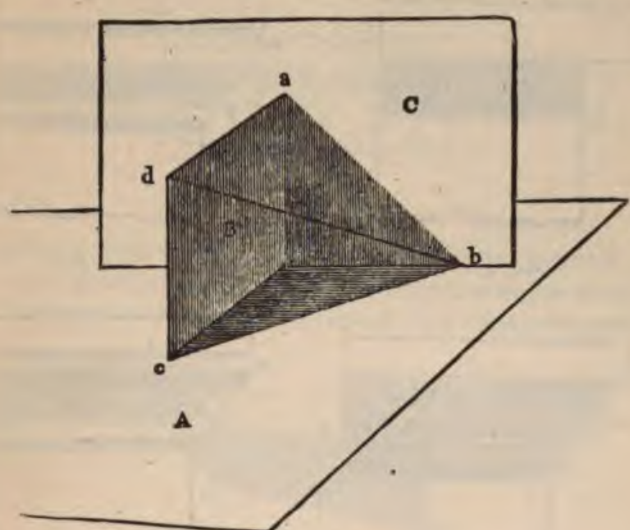


FIG. 420.

while those parts which are deprived of light by the interposition of other bodies are in *shadow*.

564.—To Find the Line of Shadow on Mouldings and other Horizontally Straight Projections.—*Figs.* 421, 422, 423, and 424 represent various mouldings in elevation, returned at the left, in the usual manner of mitering around a projection. A mere inspection of the figures is sufficient to see how the line of shadow is obtained, bearing in mind that the ray *a b* is drawn from the projections at an angle of 45

degrees. When there is no return at the end, it is necessary to draw a section, at any place in the length of the mouldings, and find the line of shadow from that.

565.—To Find the Line of Shadow Cast by a Shelf.—In *Fig.* 425, A is the plan and B is the elevation of a shelf attached to a wall. From a and c draw $a\,b$ and $c\,d$, according to the angle previously directed; from b erect a perpendicular intersecting $c\,d$ at d; from d draw $d\,e$ parallel to

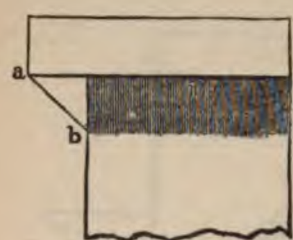


FIG. 421.

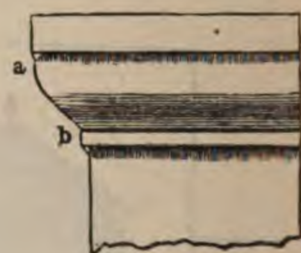


FIG. 422.

FIG. 423.

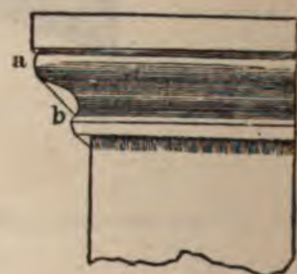


FIG. 424.

the shelf; then the lines $c\,d$ and $d\,e$ will define the shadow cast by the shelf. There is another method of finding the shadow, without the plan A. Extend the lower line of the shelf to f, and make $c\,f$ equal to the projection of the shelf from the wall; from f draw $f\,g$ at the customary angle, and from c drop the vertical line $c\,g$ intersecting $f\,g$ at g; from g draw $g\,e$ parallel to the shelf, and from c draw $c\,d$ at the usual angle; then the lines $c\,d$ and $d\,e$ will determine the extent of the shadow as before.

566.—To Find the Shadow Cast by a Shelf which is ler at one End than at the Other.—In *Fig.* 426, *A* is the ı, and *B* the elevation. Find the point *d*, as in the pre-

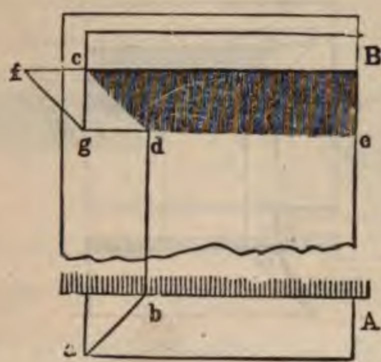


FIG. 425.

us example, and from any other point in the front of the lf, as *a*, erect the perpendicular *a e*; from *a* and *e* draw *a b* *e c*, at the proper angle, and from *b* erect the perpendicu-

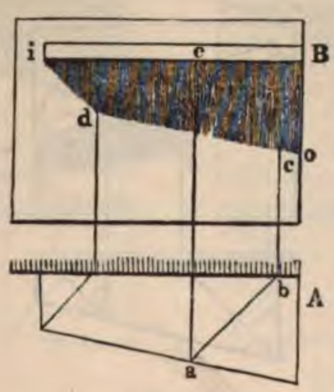


FIG. 426.

b c, intersecting *e c* in *c*; from *d*, through *c*, draw *d o*; n the lines *i d* and *d o* will give the limit of the shadow t by the shelf.

567.—To Find the Shadow of a Shelf having one End Acute or Obtuse Angled.—*Fig.* 427 shows the plan and elevation of an acute-angled shelf. Find the line *e g* as before;

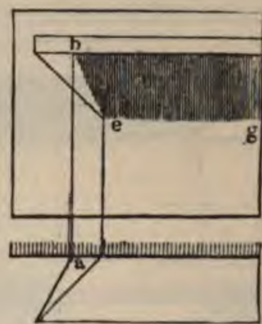


Fig. 427.

from *a* erect the perpendicular *a b*; join *b* and *e*; then *b e* and *e g* will define the boundary of shadow.

568.—To Find the Shadow Cast by an Inclined Shelf.—In *Fig.* 428 the plan and elevation of such a shelf are shown, having also one end wider than the other. Proceed as directed for finding the shadows of *Fig.* 426, and find the points

Fig 428.

d and *c*; then *a d* and *d c* will be the shadow required. If the shelf had been parallel in width on the plan, then the line *d c* would have been parallel with the shelf *a b*.

569.—To Find the Shadow Cast by a Shelf Inclined in ts Vertical Section either Upward or Downward.—From (*Figs.* 429 and 430) draw *a b* at the usual angle, and from *b* raw *b c* parallel with the shelf; obtain the point *e* by draw-

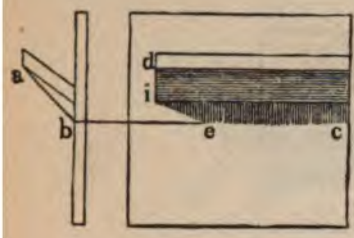


FIG. 429.

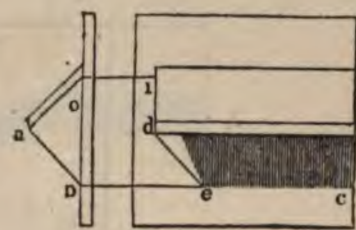


FIG. 430.

ng a line from *d* at the usual angle. In *Fig.* 429 join *e* and ; then *i e* and *e c* will define the shadow. In *Fig.* 430, from draw *o i* parallel with the shelf; join *i* and *e*; then *i e* and *e* will be the shadow required.

The projections in these several examples are bounded

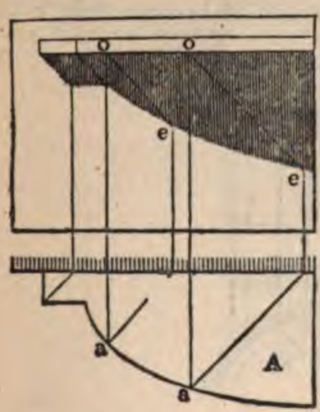


FIG. 431.

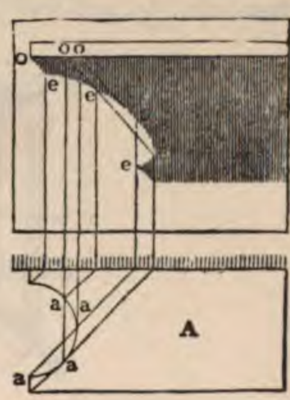


FIG. 432.

y straight lines; but the shadows of curved lines may be und in the same manner, by projecting shadows from sev-al points in the curved line, and tracing the curve of adow through these points. (*Figs.* 431 and 432.)

570.—To Find the Shadow of a Shelf having its Front Edge, or End, Curved on the Plan.—In *Figs.* 431 and 432 *A* and *A* show an example of each kind. From several points, as *a, a*, in the plan, and from the corresponding points *o, o* in the elevation, draw rays and perpendiculars intersect-

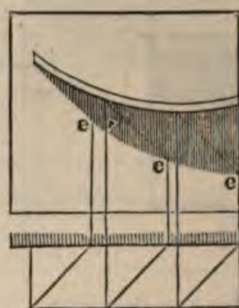


FIG. 433.

ing at *e, e*, etc.; through these points of intersection trace the curve, and it will define the shadow.

571.—To Find the Shadow of a Shelf Curved in the Elevation.—In *Fig.* 433 find the points of intersection, *e, e* and

FIG. 434.

e, as in the last examples, and a curve traced through them will define the shadow.

The preceding examples show how to find shadows when cast upon a *vertical plane;* shadows thrown upon *curved surfaces* are ascertained in a similar manner. (*Fig.* 434.)

572.—To Find the Shadow Cast upon a Cylindrical Wall by a Projection of any Kind.—By an inspection of *Fig.* 434, it will be seen that the only difference between this and the last examples is that the rays in the plan die against the circle *a b*, instead of a straight line.

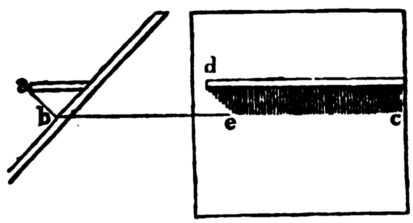


FIG. 435.

573.—To Find the Shadow Cast by a Shelf upon an Inclined Wall.—Cast the ray *a b* (*Fig.* 435) from the end of the shelf to the face of the wall, and from *b* draw *b c* parallel to the shelf; cast the ray *d e* from the end of the shelf; then the lines *d e* and *e c* will define the shadow.

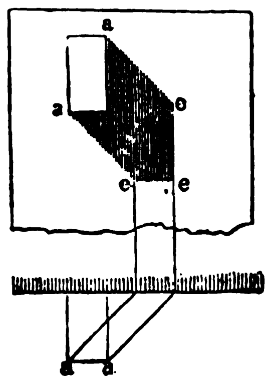


FIG. 436.

These examples might be multiplied, but enough has been given to illustrate the general principle by which shadows in all instances are found. Let us attend now to the application of this principle to such familiar objects as are likely to occur in practice.

574.—To Find the Shadow of a Projecting Horizontal Beam.—From the points *a, a*, etc. (*Fig.* 436), cast rays upon the wall; the intersections *c, c, c* of those rays with the perpendiculars drawn from the plan will define the shadow. If the beam be inclined, either on the plan or elevation, at any angle other than a right angle, the difference in the manner

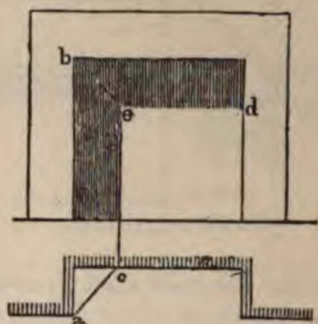


FIG. 437.

of proceeding can be seen by reference to the preceding examples of inclined shelves, etc.

575.—To Find the Shadow in a Recess.—From the point *a* (*Fig.* 437) in the plan, and *b* in the elevation, draw the rays *a c* and *b e*; from *c* erect the perpendicular *c e*, and from *e*

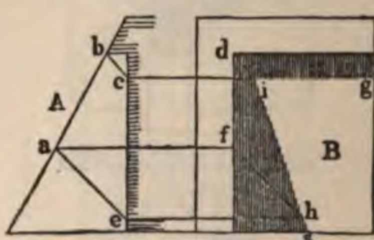


FIG. 438.

draw the horizontal line *e d*; then the lines *c e* and *e d* will show the extent of the shadow. This applies only where the back of the recess is parallel with the face of the wall.

576.—To Find the Shadow in a Recess, when the Face of the Wall is Inclined, and the Back of the Recess is Vertical.—In *Fig.* 438, *A* shows the section and *B* the eleva-

tion of a recess of this kind. From *b*, and from any other point in the line *b a*, as *a*, draw the rays *b c* and *a e*; from *c*, *a*, and *e* draw the horizontal lines *c g*, *a f*, and *e h*; from *d*

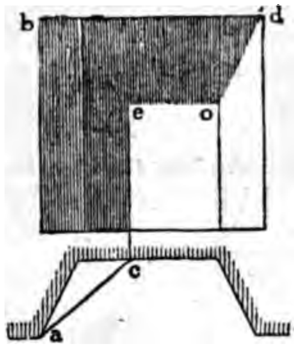


FIG. 439.

and *f* cast the rays *d i* and *f h*; from *i*, through *h*, draw *i s*; then *s i* and *i g* will define the shadow.

577.—To Find the Shadow in a Fireplace.—From *a* and *b* (*Fig.* 439) cast the rays *a c* and *b e*, and from *c* erect the

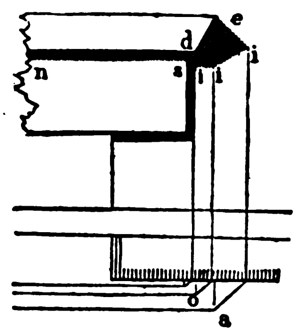


FIG. 440.

perpendicular *c e*; from *e* draw the horizontal line *e o*, and join *o* and *d*; then *c e*, *e o*, and *o d* will give the extent of the shadow.

578.—To Find the Shadow of a Moulded Window-Lintel.—Cast rays from the projections *a*, *o*, etc., in the plan (*Fig.* 440), and *d*, *e*, etc., in the elevation, and draw the usual perpendiculars intersecting the rays at *i*, *i*, and *i*; these intersections connected, and horizontal lines drawn from them, will define the shadow. The shadow on the face of the lintel is found by casting a ray back from *i* to *s*, and drawing the horizontal line *s n*.

579.—To Find the Shadow Cast by the Nosing of a Step.—From *a* (*Fig.* 441) and its corresponding point *c*, cast the

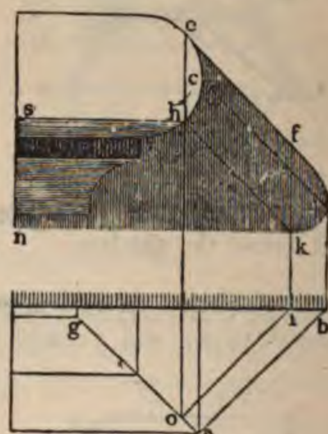


FIG. 441.

rays *a b* and *c d*, and from *b* erect the perpendicular *b d*; tangical to the curve at *e* cast the ray *e f*, and from *e* drop the perpendicular *e o*, meeting the mitre-line *a g* in *o*; cast a ray from *o* to *i*, and from *i* erect the perpendicular *i f*; from *h* draw the ray *h k*; from *f* to *d* and from *d* to *k* trace the curve as shown in the figure; from *k* and *h* draw the horizontal lines *k n* and *h s*; then the limit of the shadow will be completed.

580.—To Find the Shadow Thrown by a Pedestal upon Steps.—From *a* (*Fig.* 442) in the plan, and from *c* in the elevation, draw the rays *a b* and *c e*; then *a o* will show the ex-

tent of the shadow on the first riser, as at *A*; *fg* will determine the shadow on the second riser, as at *B*; *cd* gives the amount of shadow on the first tread, as at *C*, and *hi* that on the second tread, as at *D*; which completes the shadow of

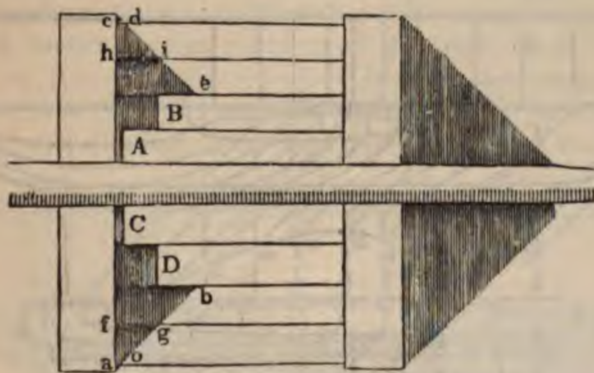


Fig. 442.

the left-hand pedestal, both on the plan and elevation. A mere inspection of the figure will be sufficient to show how the shadow of the right-hand pedestal is obtained.

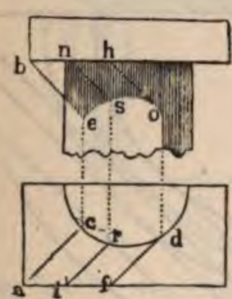


Fig. 443.

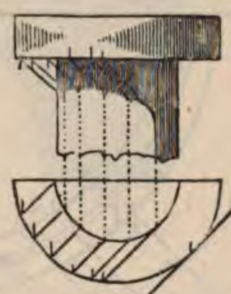


Fig. 444.

581.—To Find the Shadow Thrown on a Column by a Square Abacus.—From *a* and *b* (*Fig.* 443) draw the rays *ac* and *be*, and from *c* erect the perpendicular *ce*; tangical to the curve at *d* draw the ray *df*, and from *h*, corresponding to *f* in the plan, draw the ray *ho*; take any point between *a* and *f*, as *i*, and from this, as also from a corresponding point

n, draw the rays *i r* and *n s*; from *r* and from *d* erect the perpendiculars *r s* and *d o*; through the points *e*, *s*, and *o* trace the curve as shown in the figure; then the extent of the shadow will be defined.

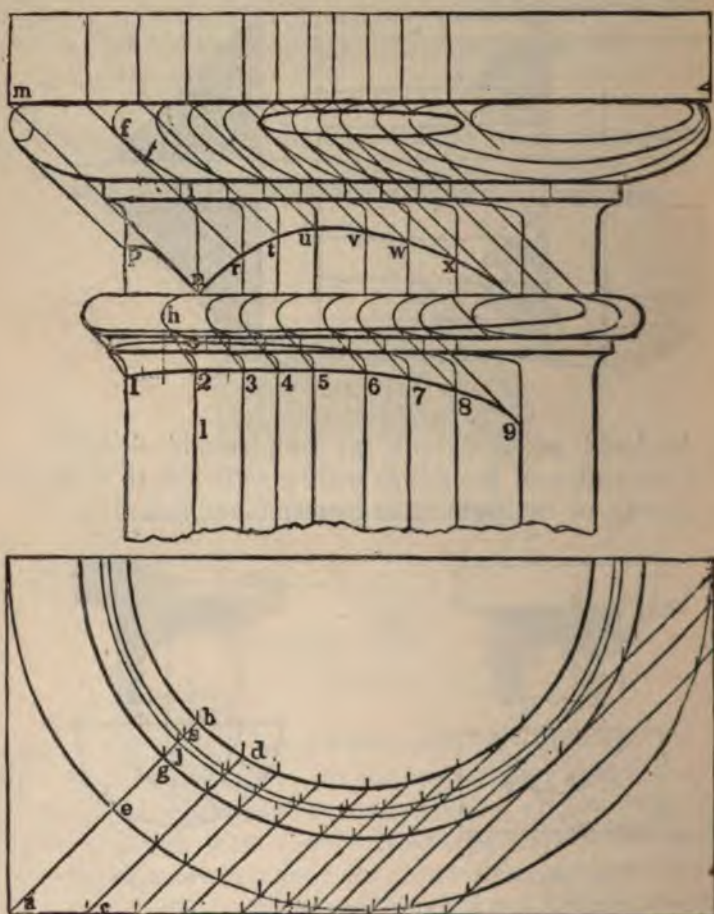


FIG. 445.

582.—To Find the Shadow Thrown on a Column by a Circular Abacus.—This is so nearly like the last example that no explanation will be necessary, farther than a reference to the preceding article.

583.—To Find the Shadows on the Capital of a Column. —This may be done according to the principles explained in the examples already given; a quicker way of doing it, however, is as follows: if we take into consideration one ray of light in connection with all those perpendicularly under and over it, it is evident that these several rays would form a vertical plane, standing at an angle of 45 degrees with the face of the elevation. Now we may suppose the column to be *sliced*, so to speak, with planes of this nature—cutting it in the lines *a b*, *c d*, etc. (*Fig.* 445), and, in the ele-

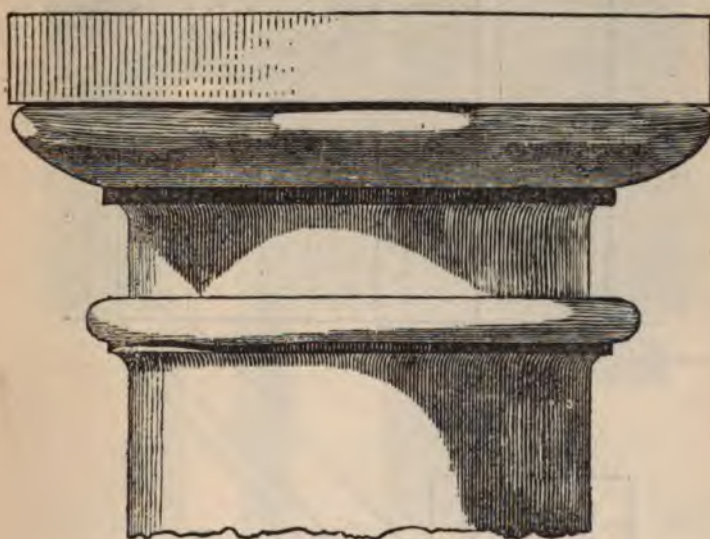


FIG. 446.

vation, find by squaring up from the plan, the *lines of section* which these planes would make thereupon. For instance: in finding upon the elevation the line of section *a b*, the plane cuts the ovolo at *e*, and therefore *f* will be the corresponding point upon the elevation; *h* corresponds with *g*, *i* with *j*, *o* with *s*, and *l* with *b*. Now, to find the shadows upon this line of section, cast from *m* the ray *m n*, from *h* the ray *h o*, etc.; then that part of the section indicated by the letters *m f i n*, and that part also between *h* and *o* will be under

shadow. By an inspection of the figure, it will be seen that the same process is applied to each line of section, and in that way the points p, r, t, u, v, w, x, as also 1, 2, 3, etc., are

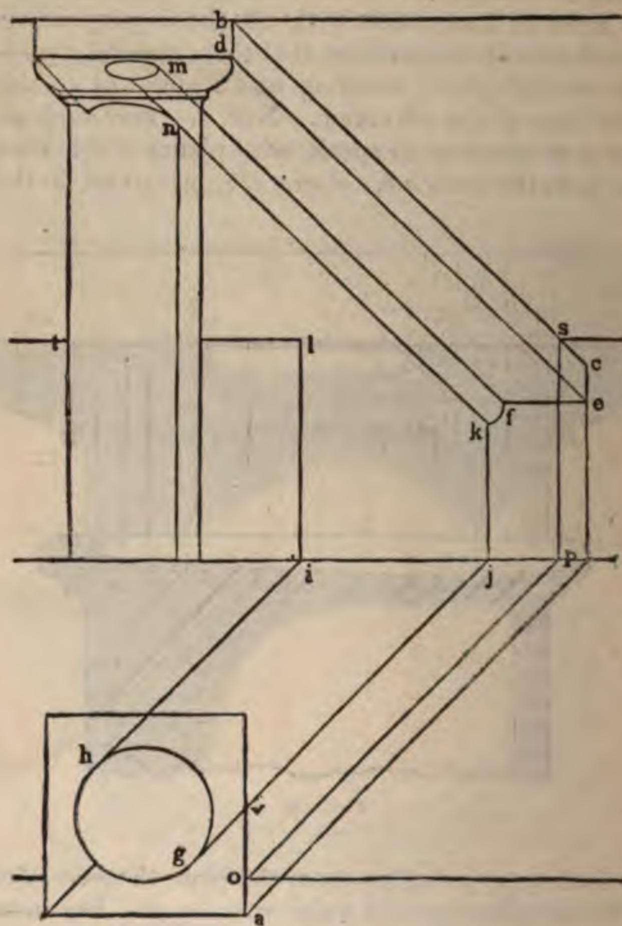


FIG. 447.

successively found, and the lines of shadow traced through them.

Fig. 446 is an example of the same capital with all the shadows finished in accordance with the lines obtained on *Fig.* 445.

584.—To Find the Shadow Thrown on a Vertical Wall by a Column and Entablature Standing in Advance of said Wall.—Cast rays from *a* and *b* (*Fig.* 447), and find the point *c* as in the previous examples; from *d* draw the ray *d e*, and from *e* the horizontal line *e f*; tangical to the curve at *g* and *h* draw the rays *g j* and *h i*, and from *i* and *j* erect the perpendiculars *i l* and *j k*; from *m* and *n* draw the rays *m f* and *n k*, and trace the curve between *k* and *f*; cast a ray from *o* to *p*, a vertical line from *p* to *s*, and through *s* draw the horizontal line *s t*; the shadow as required will then be completed.

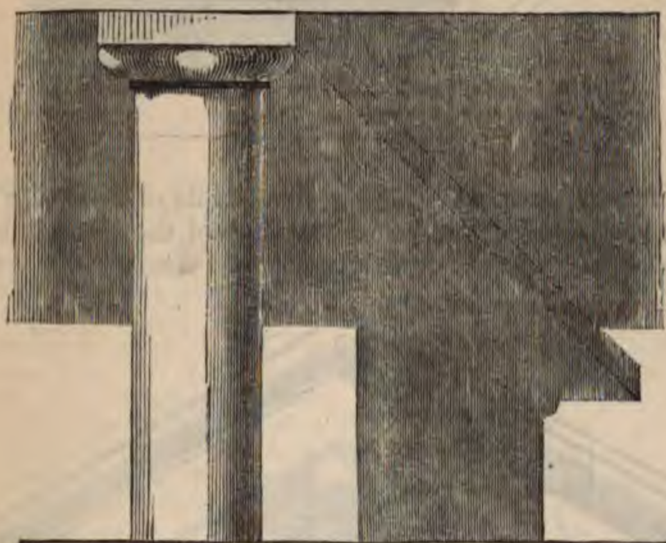


Fig. 448.

Fig. 448 is an example of the same kind as the last, with all the shadows filled in, according to the lines obtained in the preceding figure.

585.—Shadows on a Cornice.—*Figs.* 449 and 450 are examples of the Tuscan cornice. The manner of obtaining the shadows is evident.

586.—Reflected Light.—In shading, the finish and life of an object depend much on reflected light. This is seen to advantage in *Fig.* 446, and on the column in *Fig.* 448. Re-

flected rays are thrown in a direction exactly the reverse of direct rays; therefore, on that part of an object which is subject to reflected light, the shadows are reversed. The

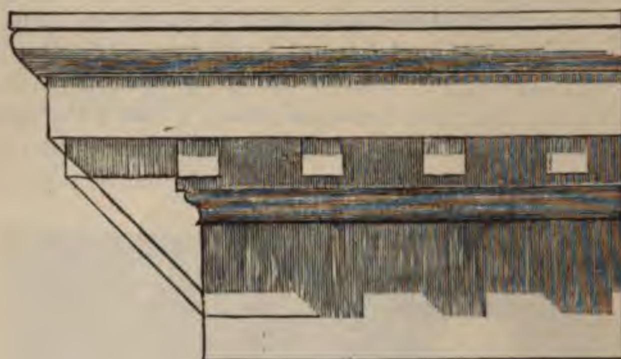


FIG. 449.

fillet of the ovolo in *Fig.* 446 is an example of this. On the right hand side of the column, the face of the fillet is much darker than the cove directly under it. The reason of this

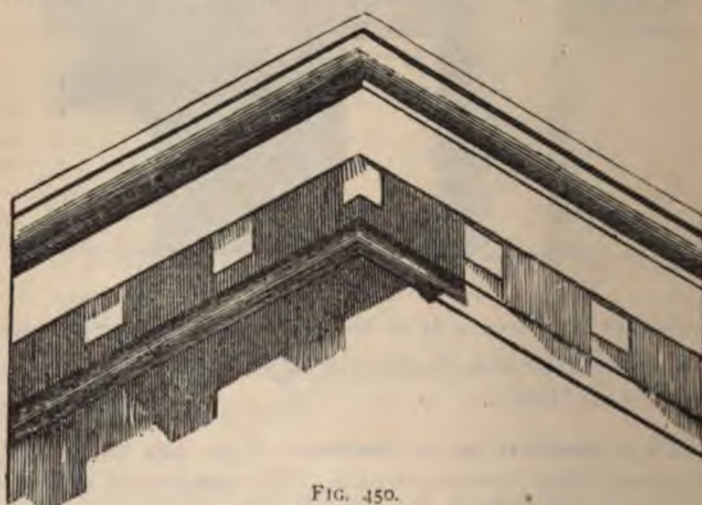


FIG. 450.

is, the face of the fillet is deprived both of direct and reflected light, whereas the cove is subject to the latter. Other instances of the effect of reflected light will be seen in the other examples.

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GLOSSARY.

Terms not found here can be found in the lists of definitions in other parts of this book, or in common dictionaries.

Abacus.—The uppermost member of a capital.

Abattoir.—A slaughter-house.

Abbey.—The residence of an abbot or abbess.

Abutment.—That part of a pier from which the arch springs.

Acanthus.—A plant called in English *bear's-breech*. Its leaves are employed for decorating the Corinthian and the Composite capitals.

Acropolis.—The highest part of a city ; generally the citadel.

Acroteria.—The small pedestals placed on the extremities and apex of a pediment, originally intended as a base for sculpture.

Aisle.—Passage to and from the pews of a church. In Gothic architecture, the lean-to wings on the sides of the *nave*.

Alcove.—Part of a chamber separated by an *estrade*, or partition of columns. Recess with seats, etc., in gardens.

Altar.—A pedestal whereon sacrifice was offered. In modern churches, the area within the railing in front of the pulpit.

Alto-relievo.—High relief ; sculpture projecting from a surface so as to appear nearly isolated.

Amphitheatre.—A double theatre, employed by the ancients for the exhibition of gladiatorial fights and other shows.

Ancones.—Trusses employed as an apparent support to a cornice upon the flanks of the architrave.

Annulet.—A small square moulding used to separate others ; the fillets in the Doric capital under the ovolo, and those which separate the flutings of columns, are known by this term.

Antæ.—A pilaster attached to a wall.

Apiary.—A place for keeping beehives.

Arabesque.—A building after the Arabian style.

Areostyle.—An intercolumniation of from four to five diameters.

Arcade.—A series of arches.

Arch.—An arrangement of stones or other material in a curvilinear form, so as to perform the office of a lintel and carry superincumbent weights.

Architrave.—That part of the entablature which rests upon the capital of a column, and is beneath the frieze. The casing and mouldings about a door or window.

Archivolt.—The ceiling of a vault ; the under surface of an arch.

Area.—Superficial measurement. An open space, below the level of the ground, in front of basement windows.

Arsenal.—A public establishment for the deposition of arms and warlike stores.

Astragal.—A small moulding consisting of a half-round with a fillet on each side.

Attic.—A low story erected over an order of architecture. A low additional story immediately under the roof of a building.

Aviary.—A place for keeping and breeding birds.

Balcony.—An open gallery projecting from the front of a building.

Baluster.—A small pillar or pilaster supporting a rail.

Balustrade.—A series of balusters connected by a rail.

Barge-course.—That part of the covering which projects over the gable of a building.

Base.—The lowest part of a wall, column, etc.

Basement-story.—That which is immediately under the principal story, and included within the foundation of the building.

Basso-relievo.—Low relief; sculptured figures projecting from a surface one half their thickness or less. See *Alto-relievo.*

Battering.—See *Talus.*

Battlement.—Indentations on the top of a wall or parapet.

Bay-window.—A window projecting in two or more planes, and not forming the segment of a circle.

Bazaar.—A species of mart or exchange for the sale of various articles of merchandise.

Bead.—A circular moulding.

Bed-mouldings.—Those mouldings which are between the corona and the frieze.

Belfry.—That part of the steeple in which the bells are hung; anciently called *campanile.*

Belvedere.—An ornamental turret or observatory commanding a pleasant prospect.

Bow-window.—A window projecting in curved lines.

Bressummer.—A beam or iron tie supporting a wall over a gateway or other opening.

Brick-nogging.—The brickwork between studs of partitions.

Buttress.—A projection from a wall to give additional strength.

Cable.—A cylindrical moulding placed in flutes at the lower part of the column.

Camber.—To give a convexity to the upper surface of a beam.

Campanile.—A tower for the reception of bells, usually, in Italy, separated from the church.

Canopy.—An ornamental covering over a seat of state.

Cantalivers.—The ends of rafters under a projecting roof. Pieces of wood or stone supporting the eaves.

Capital.—The uppermost part of a column included between the shaft and the architrave.

Caravansera.—In the East, a large public building for the reception of travellers by caravans in the desert.

Carpentry.—(From the Latin *carpentum*, carved wood.) That department of science and art which treats of the disposition, the construction, and the relative strength of timber. The first is called descriptive, the second constructive, and the last mechanical carpentry.

Caryatides.—Figures of women used instead of columns to support an entablature.

Casino.—A small country-house.

Castellated.—Built with battlements and turrets in imitation of ancient castles.

Castle.—A building fortified for military defence. A house with towers, usually encompassed with walls and moats, and having a donjon, or keep, in the centre.

Catacombs.—Subterraneous places for burying the dead.

Cathedral.—The principal church of a province or diocese, wherein the throne of the archbishop or bishop is placed.

Cavetto.—A concave moulding comprising the quadrant of a circle.

Cemetery.—An edifice or area where the dead are interred.

Cenotaph.—A monument erected to the memory of a person buried in another place.

Centring.—The temporary woodwork, or framing, whereon any vaulted work is constructed.

Cesspool.—A well under a drain or pavement to receive the waste water and sediment.

Chamfer.—The bevelled edge of anything originally right angled.

Chancel.—That part of a Gothic church in which the altar is placed.

Chantry.—A little chapel in ancient churches, with an endowment for one or more priests to say mass for the relief of souls out of purgatory.

Chapel.—A building for religious worship, erected separately from a church, and served by a chaplain.

Chaplet.—A moulding carved into beads, olives, etc.

Cincture.—The ring, listel, or fillet, at the top and bottom of a column, which divides the shaft of the column from its capital and base.

Circus.—A straight, long, narrow building used by the Romans for the exhibition of public spectacles and chariot races. At the present day, a building enclosing an arena for the exhibition of feats of horsemanship.

Clere-story.—The upper part of the nave of a church above the roofs of the aisles.

Cloister.—The square space attached to a regular monastery or large church, having a peristyle or ambulatory around it, covered with a range of buildings.

Coffer-dam.—A case of piling, water-tight, fixed in the bed of a river, for the purpose of excluding the water while any work, such as a wharf, wall, or the pier of a bridge, is carried up.

Collar-beam.—A horizontal beam framed between two principal rafters above the tie-beam.

Colonnade.—A range of columns.

Columbarium.—A pigeon-house.

Column.—A vertical cylindrical support under the entablature of an order.

Common-rafters.—The same as *jack-rafters*, which see.

Conduit.—A long, narrow, walled passage underground, for secret communication between different apartments. A canal or pipe for the conveyance of water.

Conservatory.—A building for preserving curious and rare exotic plants.

Consoles.—The same as *ancones*, which see.

Contour.—The external lines which bound and terminate a figure.

Convent.—A building for the reception of a society of religious persons.

Coping.—Stones laid on the top of a wall to defend it from the weather.

Corbels.—Stones or timbers fixed in a wall to sustain the timbers of a floor or roof.

Cornice.—Any moulded projection which crowns or finishes the part to which it is affixed.

Corona.—That part of a cornice which is between the crown-moulding and the bed-mouldings.

Cornucopia.—The horn of plenty.

Corridor.—An open gallery or communication to the different apartments of a house.

Cove.—A concave moulding.

Cripple-rafters.—The short rafters which are spiked to the hip-rafter of a roof.

Crockets.—In Gothic architecture, the ornaments placed along the angles of pediments, pinnacles, etc.

Crosettes.—The same as *ancones*, which see.

Crypt.—The under or hidden part of a building.

Culvert.—An arched channel of masonry or brickwork, built beneath the bed of a canal for the purpose of conducting water under it. Any arched channel for water underground.

Cupola.—A small building on the top of a dome.

Curtail-step.—A step with a spiral end, usually the first of the flight.

Cusps.—The pendants of a pointed arch.

Cyma.—An ogee. There are two kinds; the *cyma-recta*, having the upper part concave and the lower convex, and the *cyma-reversa*, with the upper part convex and the lower concave.

Dado.—The die, or part between the base and cornice of a pedestal.

Dairy.—An apartment or building for the preservation of milk, and the manufacture of it into butter, cheese, etc.

Dead-shoar.—A piece of timber or stone stood vertically in brickwork, to support a superincumbent weight until the brickwork which is to carry it has set or become hard.

Decastyle.—A building having ten columns in front.

Dentils.—(From the Latin, *dentes*, teeth.) Small rectangular blocks used in the bed-mouldings of some of the orders.

Diastyle.—An intercolumniation of three, or, as some say, four diameters.

Die.—That part of a pedestal included between the base and the cornice; it is also called a *dado*.

Dodecastyle.—A building having twelve columns in front.

Donjon.—A massive tower within ancient castles, to which the garrison might retreat in case of necessity.

Dooks.—A Scotch name given to wooden bricks.

Dormer.—A window placed on the roof of a house, the frame being placed vertically on the rafters.

Dormitory.—A sleeping-room.

Dovecote.—A building for keeping tame pigeons. A columbarium.

Echinus.—The Grecian ovolo.

Elevation.—A geometrical projection drawn on a plane at right angles to the horizon.

Entablature.—That part of an order which is supported by the columns; consisting of the architrave, frieze, and cornice.

Eustyle.—An intercolumniation of two and a quarter diameters.

Exchange.—A building in which merchants and brokers meet to transact business.

Extrados.—The exterior curve of an arch.

Façade.—The principal front of any building.

Face-mould.—The pattern for marking the plank out of which hand-railing is to be cut for stairs, etc.

Facia, or *Fascia.*—A flat member, like a band or broad fillet.

Falling-mould.—The mould applied to the convex, vertical surface of the rail-piece, in order to form the back and under surface of the rail, and finish the squaring.

Festoon.—An ornament representing a wreath of flowers and leaves.

Fillet.—A narrow flat band, listel, or annulet, used for the separation of one moulding from another, and to give breadth and firmness to the edges of mouldings.

Flutes.—Upright channels on the shafts of columns.

Flyers.—Steps in a flight of stairs that are parallel to each other.

Forum.—In ancient architecture a public market; also, a place where the common courts were held and law pleadings carried on.

Foundry.—A building in which various metals are cast into moulds or shapes.

Frieze.—That part of an entablature included between the architrave and the cornice.

Gable.—The vertical, triangular piece of wall at the end of a roof, from the level of the eaves to the summit.

Gain.—A recess made to receive a tenon or tusk.

Gallery.—A common passage to several rooms in an upper story. A long room for the reception of pictures. A platform raised on columns, pilasters, or piers.

Girder.—The principal beam in a floor, for supporting the binding and other joists, whereby the bearing or length is lessened.

Glyph.—A vertical, sunken channel. From their number, those in the Doric order are called *triglyphs.*

Granary.—A building for storing grain, especially that intended to be kept for a considerable time.

Groin.—The line formed by the intersection of two arches, which cross each other at any angle.

Guttæ.—The small cylindrical pendent ornaments, otherwise called *drops*, used in the Doric order under the triglyphs, and also pendent from the mutuli of the cornice.

Gymnasium.—Originally, a place measured out and covered with sand for the exercise of athletic games; afterward, spacious buildings devoted to the mental as well as corporeal instruction of youth.

Hall.—The first large apartment on entering a house. The public room of a corporate body. A manor-house.

Ham.—A house or dwelling-place. A street or village: hence Notting*ham*, Bucking*ham*, etc. *Hamlet*, the diminutive of *ham*, is a small street or village.

Helix.—The small volute, or twist, under the abacus in the Corinthian capital.

Hem.—The projecting spiral fillet of the Ionic capital.

Hexastyle.—A building having six columns in front.

Hip-rafter.—A piece of timber placed at the angle made by two adjacent inclined roofs.

Homestall.—A mansion-house, or seat in the country.

Hotel, or *Hostel.*—A large inn or place of public entertainment. A large house or palace.

Hot-house.—A glass building used in gardening.

Hovel.—An open shed.

Hut.—A small cottage or hovel, generally constructed of earthy materials, as strong loamy clay, etc.

Impost.—The capital of a pier or pilaster which supports an arch.

Intaglio.—Sculpture in which the subject is hollowed out, so that the impression from it presents the appearance of a bas-relief.

Intercolumniation.—The distance between two columns.

Intrados.—The interior and lower curve of an arch.

Jack-rafters.—Rafters that fill in between the principal rafters of a roof; called also *common-rafters*.

Jail.—A place of legal confinement.

Jambs.—The vertical sides of an aperture.

Joggle-piece.—A post to receive struts.

Joists.—The timbers to which the boards of a floor or the laths of a ceiling are nailed.

Keep.—The same as *donjon*, which see.

Key-stone.—The highest central stone of an arch.

Kiln.—A building for the accumulation and retention of heat, in order to dry or burn certain materials deposited within it.

King-post.—The centre-post in a trussed roof.

Knee.—A convex bend in the back of a hand-rail. See *Ramp*.

Lactarium.—The same as *dairy*, which see.

Lantern.—A cupola having windows in the sides for lighting an apartment beneath.

Larmier.—The same as *corona*, which see.

Lattice.—A reticulated window for the admission of air, rather than light, as in dairies and cellars.

Lever-boards.—Blind-slats; a set of boards so fastened that they may be turned at any angle to admit more or less light, or to lap upon each other so as to exclude all air or light through apertures.

Lintel.—A piece of timber or stone placed horizontally over a door, window, or other opening.

Listel.—The same as *fillet*, which see.

Lobby.—An enclosed space, or passage, communicating with the principal room or rooms of a house.

Lodge.—A small house near and subordinate to the mansion. A cottage placed at the gate of the road leading to a mansion.

Loop.—A small narrow window. *Loophole* is a term applied to the vertical series of doors in a warehouse, through which goods are delivered by means of a crane.

Luffer-boarding.—The same as *lever-boards*, which see.

Luthern.—The same as *dormer*, which see.

Mausoleum.—A sepulchral building—so called from a very celebrated one erected to the memory of Mausolus, king of Caria, by his wife Artemisia.

Metopa.—The square space in the frieze between the triglyphs of the Doric order.

Mezzanine.—A story of small height introduced between two of greater height.

Minaret.—A slender, lofty turret having projecting balconies, common in Mohammedan countries.

Minster.—A church to which an ecclesiastical fraternity has been or is attached.

Moat.—An excavated reservoir of water, surrounding a house, castle, or town.

Modillion.—A projection under the corona of the richer orders, resembling a bracket.

Module.—The semi-diameter of a column, used by the architect as a measure by which to proportion the parts of an order.

Monastery.—A building or buildings appropriated to the reception of monks.

Monopteron.—A circular colonnade supporting a dome without an enclosing wall.

Mosaic.—A mode of representing objects by the inlaying of small cubes of glass, stone, marble, shells, etc.

Mosque.—A Mohammedan temple or place of worship.

Mullions.—The upright posts or bars which divide the lights in a Gothic window.

Muniment-house.—A strong, fire-proof apartment for the keeping and preservation of evidences, charters, seals, etc., called muniments.

Museum.—A repository of natural, scientific, and literary curiosities or of works of art.

Mutule.—A projecting ornament of the Doric cornice supposed to represent the ends of rafters.

Nave.—The main body of a Gothic church.

Newel.—A post at the starting or landing of a flight of stairs.

Niche.—A cavity or hollow place in a wall for the reception of a statue, vase, etc.

Nogs.—Wooden bricks.

Nosing.—The rounded and projecting edge of a step in stairs.

Nunnery.—A building or buildings appropriated for the reception of nuns.

Obelisk.—A lofty pillar of a rectangular form.

Octastyle.—A building with eight columns in front.

Odeum.—Among the Greeks, a species of theatre wherein the poets and musicians rehearsed their compositions previous to the public production of them.

Ogee.—See *cyma*.

Orangery.—A gallery or building in a garden or parterre fronting the south.

Oriel-window.—A large bay or recessed window in a hall, chapel, or other apartment.

Ovolo.—A convex projecting moulding whose profile is the quadrant of a circle.

Pagoda.—A temple or place of worship in India.

Palisade.—A fence of pales or stakes driven into the ground.

Parapet.—A small wall of any material for protection on the sides of bridges, quays, or high buildings.

Pavilion.—A turret or small building generally insulated and comprised under a single roof.

Pedestal.—A square foundation used to elevate and sustain a column, statue, etc.

Pediment.—The triangular crowning part of a portico or aperture which terminates vertically the sloping parts of the roof; this, in Gothic architecture, is called a *gable*.

Penitentiary.—A prison for the confinement of criminals whose crimes are not of a very heinous nature.

Piazza.—A square, open space surrounded by buildings. This term is often improperly used to denote a *portico*.

Pier.—A rectangular pillar without any regular base or capital. The upright, narrow portions of walls between doors and windows are known by this term.

Pilaster.—A square pillar, sometimes insulated, but more commonly engaged in a wall, and projecting only a part of its thickness.

Piles.—Large timbers driven into the ground to make a secure foundation in marshy places, or in the bed of a river.

Pillar.—A column of irregular form, always disengaged, and always deviating from the proportions of the orders; whence the distinction between a pillar and a column.

Pinnacle.—A small spire used to ornament Gothic buildings.

Planceer.—The same as *soffit*, which see.

Plinth.—The lower square member of the base of a column, pedestal, or wall.

Porch.—An exterior appendage to a building, forming a covered approach to one of its principal doorways.

Portal.—The arch over a door or gate; the framework of the gate; the lesser gate, when there are two of different dimensions at one entrance.

Portcullis.—A strong timber gate to old castles, made to slide up and down vertically.

Portico.—A colonnade supporting a shelter over a walk, or ambulatory.

Priory.—A building similar in its constitution to a monastery or abbey, the head whereof was called a prior or prioress.

Prism.—A solid bounded on the sides by parallelograms, and on the ends by polygonal figures in parallel planes.

Prostyle.—A building with columns in front only.

Purlines.—Those pieces of timber which lie under and at right angles to the rafters to prevent them from sinking.

Pycnostyle.—An intercolumniation of one and a half diameters.

Pyramid.—A solid body standing on a square, triangle, or polygonal basis and terminating in a point at the top.

Quarry.—A place whence stones and slates are procured.

Quay.—(Pronounced *key.*) A bank formed towards the sea or on the side of a river for free passage, or for the purpose of unloading merchandise.

Quoin.—An external angle. See *Rustic quoins.*

Rabbet, or *Rebate.*—A groove or channel in the edge of a board.

Ramp.—A concave bend in the back of a hand-rail.

Rampant arch.—One having abutments of different heights.

Regula.—The band below the tænia in the Doric order.

Riser.—In stairs, the vertical board forming the front of a step.

Rostrum.—An elevated platform from which a speaker addresses an audience.

Rotunda.—A circular building.

Rubble-wall.—A wall built of unhewn stone.

Rudenture.—The same as *cable*, which see.

Rustic quoins.—The stones placed on the external angle of a building, projecting beyond the face of the wall, and having their edges bevelled.

Rustic-work.—A mode of building masonry wherein the faces of the stones are left rough, the sides only being wrought smooth where the union of the stones takes place.

Salon, or Saloon.—A lofty and spacious apartment comprehending the height of two stories with two tiers of windows.

Sarcophagus.—A tomb or coffin made of one stone.

Scantling.—The measure to which a piece of timber is to be or has been cut.

Scarfing.—The joining of two pieces of timber by bolting or nailing transversely together, so that the two appear but one.

Scotia.—The hollow moulding in the base of a column, between the fillets of the tori.

Scroll.—A carved curvilinear ornament, somewhat resembling in profile the turnings of a ram's horn.

Sepulchre.—A grave, tomb, or place of interment.

Sewer.—A drain or conduit for carrying off soil or water from any place.

Shaft.—The cylindrical part between the base and the capital of a column.

Shoar.—A piece of timber placed in an oblique direction to support a building or wall.

Sill.—The horizontal piece of timber at the bottom of framing ; the timber or stone at the bottom of doors and windows.

Soffit.—The underside of an architrave, corona, etc. The underside of the heads of doors, windows, etc.

Summer.—The lintel of a door or window ; a beam tenoned into a girder to support the ends of joists on both sides of it.

Systyle.—An intercolumniation of two diameters.

Tænia.—The fillet which separates the Doric frieze from the architrave.

Talus.—The slope or inclination of a wall, among workmen called *battering.*

Terrace.—An area raised before a building, above the level of the ground, to serve as a walk.

Tesselated pavement.—A curious pavement of mosaic work, composed of small square stones.

Tetrastyle.—A building having four columns in front.

Thatch.—A covering of straw or reeds used on the roofs of cottages, barns, etc.

Theatre.—A building appropriated to the representation of dramatic spectacles.

Tile.—A thin piece or plate of baked clay or other material used for the external covering of a roof.

Tomb.—A grave, or place for the interment of a human body, including also any commemorative monument raised over such a place.

Torus.—A moulding of semi-circular profile used in the bases of columns.

Tower.—A lofty building of several stories, round or polygonal.

Transept.—The transverse portion of a cruciform church.

Transom.—The beam across a double-lighted window ; if the window have no transom, it is called a *clere-story* window.

Thread.—That part of a step which is included between the face of its riser and that of the riser above.

Trellis.—A reticulated framing made of thin bars of wood for screens, windows, etc.

Triglyph.—The vertical tablets in the Doric frieze, chamfered on the two vertical edges, and having two channels in the middle.

Tripod.—A table or seat with three legs.

Trochilus.—The same as *scotia*, which see.

Truss.—An arrangement of timbers for increasing the resistance to cross-strains, consisting of a tie, two struts, and a suspending-piece.

Turret.—A small tower, often crowning the angle of a wall, etc.

Tusk.—A short projection under a tenon to increase its strength.

Tympanum.—The naked face of a pediment, included between the level and the raking mouldings.

Underpinning.—The wall under the ground-sills of a building.

University.—An assemblage of colleges under the supervision of a senate, etc.

Vault.—A concave arched ceiling resting upon two opposite parallel walls.

Venetian-door.—A door having side-lights.

Venetian-window.—A window having three separate apertures.

Veranda.—An awning. An open portico under the extended roof of a building.

Vestibule.—An apartment which serves as a medium of communication to another room or series of rooms.

Vestry.—An apartment in a church, or attached to it, for the preservation of the sacred vestments and utensils.

Villa.—A country-house for the residence of an opulent person.

Vinery.—A house for the cultivation of vines.

Volute.—A spiral scroll, which forms the principal feature of the Ionic and the Composite capitals.

Voussoirs.—Arch-stones.

Wainscoting.—Wooden lining of walls, generally in panels.

Water-table.—The stone covering to the projecting foundation or other walls of a building.

Well.—The space occupied by a flight of stairs. The space left beyond the ends of the steps is called the *well-hole.*

Wicket.—A small door made in a gate.

Winders.—In stairs, steps not parallel to each other.

Zophorus.—The same as *frieze*, which see.

Zystos.—Among the ancients, a portico of unusual length, commonly appropriated to gymnastic exercises.

TABLE OF SQUARES, CUBES, AND ROOTS.

(From Hutton's Mathematics.)

No.	Square.	Cube.	Sq. Root.	CubeRoot.	No.	Square.	Cube.	Sq. Root.	CubeRoot.
1	1	1	1·0000000	1·000000	68	4624	314432	8·2462113	4·081655
2	4	8	1·4142136	1·259921	69	4761	328509	8·3066239	4·101566
3	9	27	1·7320508	1·442250	70	4900	343000	8·3666003	4·121285
4	16	64	2·0000000	1·587401	71	5041	357911	8·4261498	4·140818
5	25	125	2·2360680	1·709976	72	5184	373248	8·4852814	4·160168
6	36	216	2·4494897	1·817121	73	5329	389017	8·5440037	4·179339
7	49	343	2·6457513	1·912931	74	5476	405224	8·6023253	4·198336
8	64	512	2·8284271	2·000000	75	5625	421875	8·6602540	4·217163
9	81	729	3·0000000	2·080084	76	5776	438976	8·7177979	4·235824
10	100	1000	3·1622777	2·154435	77	5929	456533	8·7749644	4·254321
11	121	1331	3·3166248	2·223980	78	6084	474552	8·8317609	4·272659
12	144	1728	3·4641016	2·289429	79	6241	493039	8·8881944	4·290840
13	169	2197	3·6055513	2·351335	80	6400	512000	8·9442719	4·308869
14	196	2744	3·7416574	2·410142	81	6561	531441	9·0000000	4·326749
15	225	3375	3·8729833	2·466212	82	6724	551368	9·0553851	4·344481
16	256	4096	4·0000000	2·519842	83	6889	571787	9·1104336	4·362071
17	289	4913	4·1231056	2·571282	84	7056	592704	9·1651514	4·379519
18	324	5832	4·2426407	2·620741	85	7225	614125	9·2195445	4·396830
19	361	6859	4·3588989	2·668402	86	7396	636056	9·2736185	4·414005
20	400	8000	4·4721360	2·714418	87	7569	658503	9·3273791	4·431048
21	441	9261	4·5825757	2·758924	88	7744	681472	9·3808315	4·447960
22	484	10648	4·6904158	2·802039	89	7921	704969	9·4339811	4·464745
23	529	12167	4·7958315	2·843867	90	8100	729000	9·4868330	4·481405
24	576	13824	4·8989795	2·884499	91	8281	753571	9·5393920	4·497941
25	625	15625	5·0000000	2·924018	92	8464	778688	9·5916630	4·514357
26	676	17576	5·0990195	2·962496	93	8649	804357	9·6436508	4·530655
27	729	19683	5·1961524	3·000000	94	8836	830584	9·6953597	4·546836
28	784	21952	5·2915026	3·036589	95	9025	857375	9·7467943	4·562903
29	841	24389	5·3851648	3·072317	96	9216	884736	9·7979590	4·578857
30	900	27000	5·4772256	3·107232	97	9409	912673	9·8488578	4·594701
31	961	29791	5·5677644	3·141381	98	9604	941192	9·8994949	4·610436
32	1024	32768	5·6568542	3·174802	99	9801	970299	9·9498744	4·626065
33	1089	35937	5·7445626	3·207534	100	10000	1000000	10·0000000	4·641589
34	1156	39304	5·8309519	3·239612	101	10201	1030301	10·0498756	4·657010
35	1225	42875	5·9160798	3·271066	102	10404	1061208	10·0995049	4·672330
36	1296	46656	6·0000000	3·301927	103	10609	1092727	10·1488916	4·687548
37	1369	50653	6·0827625	3·332222	104	10816	1124864	10·1980390	4·702669
38	1444	54872	6·1644140	3·361975	105	11025	1157625	10·2469508	4·717694
39	1521	59319	6·2449980	3·391211	106	11236	1191016	10·2956301	4·732624
40	1600	64000	6·3245553	3·419952	107	11449	1225043	10·3440804	4·747459
41	1681	68921	6·4031242	3·448217	108	11664	1259712	10·3923048	4·762203
42	1764	74088	6·4807407	3·476027	109	11881	1295029	10·4403065	4·776856
43	1849	79507	6·5574385	3·503398	110	12100	1331000	10·4880885	4·791420
44	1936	85184	6·6332496	3·530348	111	12321	1367631	10·5356538	4·805896
45	2025	91125	6·7082039	3·556893	112	12544	1404928	10·5830052	4·820284
46	2116	97336	6·7823300	3·583048	113	12769	1442897	10·6301458	4·834588
47	2209	103823	6·8556546	3·608826	114	12996	1481544	10·6770783	4·848808
48	2304	110592	6·9282032	3·634241	115	13225	1520875	10·7238053	4·862944
49	2401	117649	7·0000000	3·659306	116	13456	1560896	10·7703296	4·876999
50	2500	125000	7·0710678	3·684031	117	13689	1601613	10·8166538	4·890973
51	2601	132651	7·1414284	3·708430	118	13924	1643032	10·8627805	4·904868
52	2704	140608	7·2111026	3·732511	119	14161	1685159	10·9087121	4·918685
53	2809	148877	7·2801099	3·756286	120	14400	1728000	10·9544512	4·932424
54	2916	157464	7·3484692	3·779763	121	14641	1771561	11·0000000	4·946087
55	3025	166375	7·4161985	3·802952	122	14884	1815848	11·0453610	4·959676
56	3136	175616	7·4833148	3·825862	123	15129	1860867	11·0905365	4·973190
57	3249	185193	7·5498344	3·848501	124	15376	1906624	11·1355287	4·986631
58	3364	195112	7·6157731	3·870877	125	15625	1953125	11·1803399	5·000000
59	3481	205379	7·6811457	3·892996	126	15876	2000376	11·2249722	5·013298
60	3600	216000	7·7459667	3·914868	127	16129	2048383	11·2694277	5·026526
61	3721	226981	7·8102497	3·936497	128	16384	2097152	11·3137085	5·039684
62	3844	238328	7·8740079	3·957892	129	16641	2146689	11·3578167	5·052774
63	3969	250047	7·9372539	3·979057	130	16900	2197000	11·4017543	5·065797
64	4096	262144	8·0000000	4·000000	131	17161	2248091	11·4455231	5·078753
65	4225	274625	8·0622577	4·020726	132	17424	2299968	11·4891253	5·091643
66	4356	287496	8·1240384	4·041240	133	17689	2352637	11·5325626	5·104469
67	4489	300763	8·1853528	4·061548	134	17956	2406104	11·5758369	5·117230

No.	Square.	Cube.	Sq. Root.	CubeRoot.	No.	Square.	Cube.	Sq. Root.	CubeRoot.
135	18225	2460375	11·6189500	5·129928	202	40804	8242408	14·2126704	5·867464
136	18496	2515456	11·6619038	5·142563	203	41209	8365427	14·2478068	5·877131
137	18769	2571353	11·7046999	5·155137	204	41616	8489664	14·2828569	5·886765
138	19044	2628072	11·7473401	5·167649	205	42025	8615125	14·3178211	5·896368
139	19321	2685619	11·7898261	5·180101	206	42436	8741816	14·3527001	5·905941
140	19600	2744000	11·8321596	5·192494	207	42849	8869743	14·3874946	5·915482
141	19881	2803221	11·8743422	5·204828	208	43264	8998912	14·4222051	5·924992
142	20164	2863288	11·9163753	5·217103	209	43681	9129329	14·4568323	5·934473
143	20449	2924207	11·9582607	5·229321	210	44100	9261000	14·4913767	5·943922
144	20736	2985984	12·0000000	5·241483	211	44521	9393931	14·5258390	5·953342
145	21025	3048625	12·0415946	5·253588	212	44944	9528128	14·5602198	5·962732
146	21316	3112136	12·0830460	5·265637	213	45369	9663597	14·5945195	5·972093
147	21609	3176523	12·1243557	5·277632	214	45796	9800344	14·6287338	5·981424
148	21904	3241792	12·1655251	5·289572	215	46225	9938375	14·6628783	5·990726
149	22201	3307949	12·2065556	5·301459	216	46656	10077696	14·6969385	6·000000
150	22500	3375000	12·2474487	5·313293	217	47089	10218313	14·7309199	6·009245
151	22801	3442951	12·2882057	5·325074	218	47524	10360232	14·7648231	6·018462
152	23104	3511808	12·3288280	5·336803	219	47961	10503459	14·7986486	6·027650
153	23409	3581577	12·3693169	5·348481	220	48400	10648000	14·8323970	6·036811
154	23716	3652264	12·4096736	5·360108	221	48841	10793861	14·8660687	6·045943
155	24025	3723875	12·4498996	5·371685	222	49284	10941048	14·8996644	6·055049
156	24336	3796416	12·4899960	5·383213	223	49729	11089567	14·9331845	6·064127
157	24649	3869893	12·5299641	5·394691	224	50176	11239424	14·9666295	6·073178
158	24964	3944312	12·5698051	5·406120	225	50625	11390625	15·0000000	6·082202
159	25281	4019679	12·6095202	5·417501	226	51076	11543176	15·0332964	6·091199
160	25600	4096000	12·6491106	5·428835	227	51529	11697083	15·0665192	6·100170
161	25921	4173281	12·6885775	5·440122	228	51984	11852352	15·0996689	6·109115
162	26244	4251528	12·7279221	5·451362	229	52441	12008989	15·1327460	6·118033
163	26569	4330747	12·7671453	5·462556	230	52900	12167000	15·1657509	6·126926
164	26896	4410944	12·8062485	5·473704	231	53361	12326391	15·1986842	6·135792
165	27225	4492125	12·8452326	5·484807	232	53824	12487168	15·2315462	6·144634
166	27556	4574296	12·8840987	5·495865	233	54289	12649337	15·2643375	6·153449
167	27889	4657463	12·9228480	5·506878	234	54756	12812904	15·2970585	6·162240
168	28224	4741632	12·9614814	5·517848	235	55225	12977875	15·3297097	6·171006
169	28561	4826809	13·0000000	5·528775	236	55696	13144256	15·3622915	6·179747
170	28900	4913000	13·0384048	5·539658	237	56169	13312053	15·3948043	6·188463
171	29241	5000211	13·0766968	5·550499	238	56644	13481272	15·4272486	6·197154
172	29584	5088448	13·1148770	5·561298	239	57121	13651919	15·4596248	6·205822
173	29929	5177717	13·1529464	5·572055	240	57600	13824000	15·4919334	6·214465
174	30276	5268024	13·1909060	5·582770	241	58081	13997521	15·5241747	6·223084
175	30625	5359375	13·2287566	5·593145	242	58564	14172488	15·5563492	6·231680
176	30976	5451776	13·2664992	5·604079	243	59049	14348907	15·5884573	6·240251
177	31329	5545233	13·3041347	5·614672	244	59536	14526784	15·6204994	6·248800
178	31684	5639752	13·3416641	5·625226	245	60025	14706125	15·6524758	6·257325
179	32041	5735339	13·3790882	5·635741	246	60516	14886936	15·6843871	6·265827
180	32400	5832000	13·4164079	5·646216	247	61009	15069223	15·7162336	6·274305
181	32761	5929741	13·4536240	5·656653	248	61504	15252992	15·7480157	6·282761
182	33124	6028568	13·4907376	5·667051	249	62001	15438249	15·7797338	6·291195
183	33489	6128487	13·5277493	5·677411	250	62500	15625000	15·8113883	6·299605
184	33856	6229504	13·5646600	5·687734	251	63001	15813251	15·8429795	6·307994
185	34225	6331625	13·6014705	5·698019	252	63504	16003008	15·8745079	6·316360
186	34596	6434856	13·6381817	5·708267	253	64009	16194277	15·9059737	6·324704
187	34969	6539203	13·6747943	5·718479	254	64516	16387064	15·9373775	6·333026
188	35344	6644672	13·7113092	5·728654	255	65025	16581375	15·9687194	6·341326
189	35721	6751269	13·7477271	5·738794	256	65536	16777216	16·0000000	6·349604
190	36100	6859000	13·7840488	5·748897	257	66049	16974593	16·0312195	6·357861
191	36481	6967871	13·8202750	5·758965	258	66564	17173512	16·0623784	6·366097
192	36864	7077888	13·8564065	5·768998	259	67081	17373979	16·0934769	6·374311
193	37249	7189057	13·8924440	5·778996	260	67600	17576000	16·1245155	6·382504
194	37636	7301384	13·9283883	5·788960	261	68121	17779581	16·1554944	6·390676
195	38025	7414875	13·9642400	5·798890	262	68644	17984728	16·1864141	6·398828
196	38416	7529536	14·0000000	5·808786	263	69169	18191447	16·2172747	6·406958
197	38809	7645373	14·0356688	5·818648	264	69696	18399744	16·2480763	6·415069
198	39204	7762392	14·0712473	5·828477	265	70225	18609625	16·2788206	6·423158
199	39601	7880599	14·1067360	5·838272	266	70756	18821096	16·3095064	6·431228
200	40000	8000000	14·1421356	5·848035	267	71289	19034163	16·3401346	6·439277
201	40401	8120601	14·1774469	5·857766	268	71824	19248832	16·3707055	6·447306

No.	Square.	Cube.	Sq. Root.	Cube Root.	No.	Square.	Cube.	Sq. Root.	Cube Root.
269	72361	19465109	16·4012195	6·455315	336	112896	37933056	18·3303028	6·952053
270	72900	19683000	16·4316767	6·463304	337	113569	38272753	18·3575598	6·958943
271	73441	19902511	16·4620776	6·471274	338	114244	38614472	18·3847763	6·965820
272	73984	20123648	16·4924225	6·479224	339	114921	38958219	18·4119526	6·972683
273	74529	20346417	16·5227116	6·487154	340	115600	39304000	18·4390889	6·979532
274	75076	20570824	16·5529454	6·495065	341	116281	39651821	18·4661853	6·986368
275	75625	20796875	16·5831240	6·502957	342	116964	40001688	18·4932420	6·993191
276	76176	21024576	16·6132477	6·510830	343	117649	40353607	18·5202592	7·000000
277	76729	21253933	16·6433170	6·518684	344	118336	40707584	18·5472370	7·006796
278	77284	21484952	16·6733320	6·526519	345	119025	41063625	18·5741756	7·013579
279	77841	21717639	16·7032931	6·534335	346	119716	41421736	18·6010752	7·020349
280	78400	21952000	16·7332005	6·542133	347	120409	41781923	18·6279360	7·027106
281	78961	22188041	16·7630546	6·549912	348	121104	42144192	18·6547581	7·033850
282	79524	22425768	16·7928556	6·557672	349	121801	42508549	18·6815417	7·040581
283	80089	22665187	16·8226038	6·565414	350	122500	42875000	18·7082869	7·047299
284	80656	22906304	16·8522995	6·573138	351	123201	43243551	18·7349940	7·054004
285	81225	23149125	16·8819430	6·580844	352	123904	43614208	18·7616630	7·060697
286	81796	23393656	16·9115345	6·588532	353	124609	43986977	18·7882942	7·067377
287	82369	23639903	16·9410743	6·596202	354	125316	44361864	18·8148877	7·074044
288	82944	23887872	16·9705627	6·603854	355	126025	44738875	18·8414437	7·080699
289	83521	24137569	17·0000000	6·611489	356	126736	45118016	18·8679623	7·087341
290	84100	24389000	17·0293864	6·619106	357	127449	45499293	18·8944436	7·093971
291	84681	24642171	17·0587221	6·626705	358	128164	45882712	18·9208879	7·100588
292	85264	24897088	17·0880075	6·634287	359	128881	46268279	18·9472953	7·107194
293	85849	25153757	17·1172428	6·641852	360	129600	46656000	18·9736660	7·113787
294	86436	25412184	17·1464282	6·649400	361	130321	47045881	19·0000000	7·120367
295	87025	25672375	17·1755640	6·656930	362	131044	47437928	19·0262976	7·126936
296	87616	25934336	17·2046505	6·664444	363	131769	47832147	19·0525589	7·133492
297	88209	26198073	17·2336879	6·671940	364	132496	48228544	19·0787840	7·140037
298	88804	26463592	17·2626765	6·679420	365	133225	48627125	19·1049732	7·146569
299	89401	26730899	17·2916165	6·686883	366	133956	49027896	19·1311265	7·153090
300	90000	27000000	17·3205081	6·694330	367	134689	49430863	19·1572441	7·159599
301	90601	27270901	17·3493516	6·701759	368	135424	49836032	19·1833261	7·166096
302	91204	27543608	17·3781472	6·709173	369	136161	50243409	19·2093727	7·172581
303	91809	27818127	17·4068952	6·716570	370	136900	50653000	19·2353841	7·179054
304	92416	28094464	17·4355958	6·723951	371	137641	51064811	19·2613603	7·185516
305	93025	28372625	17·4642492	6·731316	372	138384	51478848	19·2873015	7·191966
306	93636	28652616	17·4928557	6·738664	373	139129	51895117	19·3132079	7·198405
307	94249	28934443	17·5214155	6·745997	374	139876	52313624	19·3390796	7·204832
308	94864	29218112	17·5499288	6·753313	375	140625	52734375	19·3649167	7·211248
309	95481	29503629	17·5783958	6·760614	376	141376	53157376	19·3907194	7·217652
310	96100	29791000	17·6068169	6·767899	377	142129	53582633	19·4164878	7·224045
311	96721	30080231	17·6351921	6·775169	378	142884	54010152	19·4422221	7·230427
312	97344	30371328	17·6635217	6·782423	379	143641	54439939	19·4679223	7·236797
313	97969	30664297	17·6918060	6·789661	380	144400	54872000	19·4935887	7·243156
314	98596	30959144	17·7200451	6·796884	381	145161	55306341	19·5192213	7·249504
315	99225	31255875	17·7482393	6·804092	382	145924	55742968	19·5448203	7·255841
316	99856	31554496	17·7763888	6·811285	383	146689	56181887	19·5703858	7·262167
317	100489	31855013	17·8044938	6·818462	384	147456	56623104	19·5959179	7·268482
318	101124	32157432	17·8325545	6·825624	385	148225	57066625	19·6214169	7·274786
319	101761	32461759	17·8605711	6·832771	386	148996	57512456	19·6468827	7·281079
320	102400	32768000	17·8885438	6·839904	387	149769	57960603	19·6723156	7·287362
321	103041	33076161	17·9164729	6·847021	388	150544	58411072	19·6977156	7·293633
322	103684	33386248	17·9443584	6·854124	389	151321	58863869	19·7230829	7·299894
323	104329	33698267	17·9722008	6·861212	390	152100	59319000	19·7484177	7·306144
324	104976	34012224	18·0000000	6·868285	391	152881	59776471	19·7737199	7·312383
325	105625	34328125	18·0277564	6·875344	392	153664	60236288	19·7989899	7·318611
326	106276	34645976	18·0554701	6·882389	393	154449	60698457	19·8242276	7·324829
327	106929	34965783	18·0831413	6·889419	394	155236	61162984	19·8494332	7·331037
328	107584	35287552	18·1107703	6·896435	395	156025	61629875	19·8746069	7·337234
329	108241	35611289	18·1383571	6·903436	396	156816	62099136	19·8997487	7·343420
330	108900	35937000	18·1659021	6·910423	397	157609	62570773	19·9248588	7·349597
331	109561	36264691	18·1934054	6·917396	398	158404	63044792	19·9499373	7·355762
332	110224	36594368	18·2208672	6·924356	399	159201	63521199	19·9749844	7·361918
333	110889	36926037	18·2482876	6·931301	400	160000	64000000	20·0000000	7·368063
334	111556	37259704	18·2756669	6·938232	401	160801	64481201	20·0249844	7·374198
335	112225	37595375	18·3030052	6·945150	402	161604	64964808	20·0499377	7·380323

No.	Square.	Cube.	Sq. Root.	CubeRoot.	No.	Square.	Cube.	Sq. Root.	CubeRoot.
403	162409	65450827	20·0748599	7·386437	470	220900	103823000	21·6794834	7·774980
404	163216	65939264	20·0997512	7·392542	471	221841	104487111	21·7025344	7·780490
405	164025	66430125	20·1246118	7·398636	472	222784	105154048	21·7255610	7·785993
406	164836	66923416	20·1494417	7·404721	473	223729	105823817	21·7485632	7·791487
407	165649	67419143	20·1742410	7·410795	474	224676	106496424	21·7715411	7·796974
408	166464	67917312	20·1990099	7·416859	475	225625	107171875	21·7944947	7·802454
409	167281	68417929	20·2237484	7·422914	476	226576	107850176	21·8174242	7·807925
410	168100	68921000	20·2484567	7·428959	477	227529	108531333	21·8403297	7·813389
411	168921	69426531	20·2731349	7·434994	478	228484	109215352	21·8632111	7·818846
412	169744	69934528	20·2977831	7·441019	479	229441	109902239	21·8860686	7·824294
413	170569	70444997	20·3224014	7·447034	480	230400	110592000	21·9089023	7·829735
414	171396	70957944	20·3469899	7·453040	481	231361	111284641	21·9317122	7·835169
415	172225	71473375	20·3715488	7·459036	482	232324	111980168	21·9544984	7·840595
416	173056	71991296	20·3960781	7·465022	483	233289	112678587	21·9772610	7·846013
417	173889	72511713	20·4205779	7·470999	484	234256	113379904	22·0000000	7·851424
418	174724	73034632	20·4450483	7·476966	485	235225	114084125	22·0227155	7·856828
419	175561	73560059	20·4694895	7·482924	486	236196	114791256	22·0454077	7·862224
420	176400	74088000	20·4939015	7·488872	487	237169	115501303	22·0680765	7·867613
421	177241	74618461	20·5182845	7·494811	488	238144	116214272	22·0907220	7·872994
422	178084	75151448	20·5426386	7·500741	489	239121	116930169	22·1133444	7·878368
423	178929	75686967	20·5669638	7·506661	490	240100	117649000	22·1359436	7·883735
424	179776	76225024	20·5912603	7·512571	491	241081	118370771	22·1585198	7·889095
425	180625	76765625	20·6155281	7·518473	492	242064	119095488	22·1810730	7·894447
426	181476	77308776	20·6397674	7·524365	493	243049	119823157	22·2036033	7·899792
427	182329	77854483	20·6639783	7·530248	494	244036	120553784	22·2261108	7·905129
428	183184	78402752	20·6881609	7·536122	495	245025	121287375	22·2485955	7·910460
429	184041	78953589	20·7123152	7·541987	496	246016	122023936	22·2710575	7·915783
430	184900	79507000	20·7364414	7·547842	497	247009	122763473	22·2934968	7·921099
431	185761	80062991	20·7605395	7·553689	498	248004	123505992	22·3159136	7·926408
432	186624	80621568	20·7846097	7·559526	499	249001	124251499	22·3383079	7·931710
433	187489	81182737	20·8086520	7·565355	500	250000	125000000	22·3606798	7·937005
434	188356	81746504	20·8326667	7·571174	501	251001	125751501	22·3830293	7·942293
435	189225	82312875	20·8566536	7·576985	502	252004	126506008	22·4053565	7·947574
436	190096	82881856	20·8806130	7·582786	503	253009	127263527	22·4276615	7·952848
437	190969	83453453	20·9045450	7·588579	504	254016	128024064	22·4499443	7·958114
438	191844	84027672	20·9284495	7·594363	505	255025	128787625	22·4722051	7·963374
439	192721	84604519	20·9523268	7·600138	506	256036	129554216	22·4944438	7·968627
440	193600	85184000	20·9761770	7·605905	507	257049	130323843	22·5166605	7·973873
441	194481	85766121	21·0000000	7·611663	508	258064	131096512	22·5388553	7·979112
442	195364	86350888	21·0237960	7·617412	509	259081	131872229	22·5610283	7·984344
443	196249	86938307	21·0475652	7·623152	510	260100	132651000	22·5831796	7·989570
444	197136	87528384	21·0713075	7·628884	511	261121	133432831	22·6053091	7·994788
445	198025	88121125	21·0950231	7·634607	512	262144	134217728	22·6274170	8·000000
446	198916	88716536	21·1187121	7·640321	513	263169	135005697	22·6495033	8·005205
447	199809	89314623	21·1423745	7·646027	514	264196	135796744	22·6715681	8·010403
448	200704	89915392	21·1660105	7·651725	515	265225	136590875	22·6936114	8·015595
449	201601	90518849	21·1896201	7·657414	516	266256	137388096	22·7156334	8·020779
450	202500	91125000	21·2132034	7·663094	517	267289	138188413	22·7376340	8·025957
451	203401	91733851	21·2367606	7·668766	518	268324	138991832	22·7596134	8·031129
452	204304	92345408	21·2602916	7·674430	519	269361	139798359	22·7815715	8·036293
453	205209	92959677	21·2837967	7·680086	520	270400	140608000	22·8035085	8·041451
454	206116	93576664	21·3072758	7·685733	521	271441	141420761	22·8254244	8·046603
455	207025	94196375	21·3307290	7·691372	522	272484	142236648	22·8473193	8·051748
456	207936	94818816	21·3541565	7·697002	523	273529	143055667	22·8691933	8·056886
457	208849	95443993	21·3775583	7·702625	524	274576	143877824	22·8910463	8·062018
458	209764	96071912	21·4009346	7·708239	525	275625	144703125	22·9128785	8·067143
459	210681	96702579	21·4242853	7·713845	526	276676	145531576	22·9346899	8·072262
460	211600	97336000	21·4476106	7·719443	527	277729	146363183	22·9564806	8·077374
461	212521	97972181	21·4709106	7·725032	528	278784	147197952	22·9782506	8·082480
462	213444	98611128	21·4941853	7·730614	529	279841	148035889	23·0000000	8·087579
463	214369	99252847	21·5174348	7·736188	530	280900	148877000	23·0217289	8·092672
464	215296	99897344	21·5406592	7·741753	531	281961	149721291	23·0434372	8·097759
465	216225	100544625	21·5638587	7·747311	532	283024	150568768	23·0651252	8·102839
466	217156	101194696	21·5870331	7·752861	533	284089	151419437	23·0867928	8·107913
467	218089	101847563	21·6101828	7·758402	534	285156	152273304	23·1084400	8·112980
468	219024	102503232	21·6333077	7·763936	535	286225	153130375	23·1300670	8·118041
469	219961	103161709	21·6564078	7·769462	536	287296	153990656	23·1516738	8·123096

No.	Square.	Cube.	Sq. Root.	CubeRoot.	No.	Square.	Cube.	Sq. Root.	CubeRoot.
537	288369	154854153	23·1732605	8·128145	604	364816	220348864	24·5764115	8·453028
538	289444	155720872	23·1948270	8·133187	605	366025	221445125	24·5967478	8·457691
539	290521	156590819	23·2163735	8·138223	606	367236	222545016	24·6170673	8·462348
540	291600	157464000	23·2379001	8·143253	607	368449	223648543	24·6373700	8·466999
541	292681	158340421	23·2594067	8·148276	608	369664	224755712	24·6576560	8·471647
542	293764	159220088	23·2808935	8·153294	609	370881	225866529	24·6779254	8·476289
543	294849	160103007	23·3023604	8·158305	610	372100	226981000	24·6981781	8·480926
544	295936	160989184	23·3238076	8·163310	611	373321	228099131	24·7184142	8·485558
545	297025	161878625	23·3452351	8·168309	612	374544	229220928	24·7386338	8·490185
546	298116	162771336	23·3666429	8·173302	613	375769	230346397	24·7588368	8·494806
547	299209	163667323	23·3880311	8·178289	614	376996	231475544	24·7790234	8·499423
548	300304	164566592	23·4093998	8·183269	615	378225	232608375	24·7991935	8·504035
549	301401	165469149	23·4307490	8·188244	616	379456	233744896	24·8193473	8·508642
550	302500	166375000	23·4520788	8·193213	617	380689	234885113	24·8394847	8·513243
551	303601	167284151	23·4733892	8·198175	618	381924	236029032	24·8596058	8·517840
552	304704	168196608	23·4946802	8·203132	619	383161	237176659	24·8797106	8·522432
553	305809	169112377	23·5159520	8·208082	620	384400	238328000	24·8997992	8·527019
554	306916	170031464	23·5372046	8·213027	621	385641	239483061	24·9198716	8·531601
555	308025	170953875	23·5584380	8·217966	622	386884	240641848	24·9399278	8·536178
556	309136	171879616	23·5796522	8·222898	623	388129	241804367	24·9599679	8·540750
557	310249	172808693	23·6008474	8·227825	624	389376	242970624	24·9799920	8·545317
558	311364	173741112	23·6220236	8·232746	625	390625	244140625	25·0000000	8·549880
559	312481	174676879	23·6431808	8·237661	626	391876	245314376	25·0199920	8·554437
560	313600	175616000	23·6643191	8·242571	627	393129	246491883	25·0399681	8·558990
561	314721	176558481	23·6854386	8·247474	628	394384	247673152	25·0599282	8·563538
562	315844	177504328	23·7065392	8·252371	629	395641	248858189	25·0798724	8·568081
563	316969	178453547	23·7276210	8·257263	630	396900	250047000	25·0998008	8·572619
564	318096	179406144	23·7486842	8·262149	631	398161	251239591	25·1197134	8·577152
565	319225	180362125	23·7697286	8·267029	632	399424	252435968	25·1396102	8·581681
566	320356	181321496	23·7907545	8·271904	633	400689	253636137	25·1594913	8·586205
567	321489	182284263	23·8117618	8·276773	634	401956	254840104	25·1793566	8·590724
568	322624	183250432	23·8327506	8·281635	635	403225	256047875	25·1992063	8·595238
569	323761	184220009	23·8537209	8·286493	636	404496	257259456	25·2190404	8·599748
570	324900	185193000	23·8746728	8·291344	637	405769	258474853	25·2388589	8·604252
571	326041	186169411	23·8956063	8·296190	638	407044	259694072	25·2586619	8·608753
572	327184	187149248	23·9165215	8·301030	639	408321	260917119	25·2784493	8·613248
573	328329	188132517	23·9374184	8·305865	640	409600	262144000	25·2982213	8·617739
574	329476	189119224	23·9582971	8·310694	641	410881	263374721	25·3179778	8·622225
575	330625	190109375	23·9791576	8·315517	642	412164	264609288	25·3377189	8·626706
576	331776	191102976	24·0000000	8·320335	643	413449	265847707	25·3574447	8·631183
577	332929	192100033	24·0208243	8·325147	644	414736	267089984	25·3771551	8·635655
578	334084	193100552	24·0416306	8·329954	645	416025	268336125	25·3968502	8·640123
579	335241	194104539	24·0624188	8·334755	646	417316	269586136	25·4165301	8·644585
580	336400	195112000	24·0831891	8·339551	647	418609	270840023	25·4361947	8·649044
581	337561	196122941	24·1039416	8·344341	648	419904	272097792	25·4558441	8·653497
582	338724	197137368	24·1246762	8·349126	649	421201	273359449	25·4754784	8·657946
583	339889	198155287	24·1453929	8·353905	650	422500	274625000	25·4950976	8·662391
584	341056	199176704	24·1660919	8·358678	651	423801	275894451	25·5147016	8·666831
585	342225	200201625	24·1867732	8·363447	652	425104	277167808	25·5342907	8·671266
586	343396	201230056	24·2074369	8·368209	653	426409	278445077	25·5538647	8·675697
587	344569	202262003	24·2280829	8·372967	654	427716	279726264	25·5734237	8·680124
588	345744	203297472	24·2487113	8·377719	655	429025	281011375	25·5929678	8·684546
589	346921	204336469	24·2693222	8·382465	656	430336	282300416	25·6124969	8·688963
590	348100	205379000	24·2899156	8·387206	657	431649	283593393	25·6320112	8·693376
591	349281	206425071	24·3104916	8·391942	658	432964	284890312	25·6515107	8·697784
592	350464	207474688	24·3310501	8·396673	659	434281	286191179	25·6709953	8·702188
593	351649	208527857	24·3515913	8·401398	660	435600	287496000	25·6904652	8·706588
594	352836	209584584	24·3721152	8·406118	661	436921	288804781	25·7099203	8·710983
595	354025	210644875	24·3926218	8·410833	662	438244	290117528	25·7293607	8·715373
596	355216	211708736	24·4131112	8·415542	663	439569	291434247	25·7487864	8·719760
597	356409	212776173	24·4335834	8·420246	664	440896	292754944	25·7681975	8·724141
598	357604	213847192	24·4540385	8·424945	665	442225	294079625	25·7875939	8·728518
599	358801	214921799	24·4744765	8·429638	666	443556	295408296	25·8069758	8·732892
600	360000	216000000	24·4948974	8·434327	667	444889	296740963	25·8263431	8·737260
601	361201	217081801	24·5153013	8·439010	668	446224	298077632	25·8456960	8·741625
602	362404	218167208	24·5356883	8·443688	669	447561	299418309	25·8650343	8·745985
603	363609	219256227	24·5560583	8·448360	670	448900	300763000	25·8843582	8·750340

No.	Square.	Cube.	Sq. Root.	CubeRoot.	No.	Square.	Cube.	Sq. Root.	CubeRoot.
671	450241	302111711	25·9035677	8·754691	738	544644	401947272	27·1661554	9·036886
672	451584	303464448	25·9229628	8·759033	739	546121	403583419	27·1845544	9·040965
673	452929	304821217	25·9422435	8·763331	740	547600	405224000	27·2029410	9·045042
674	454276	306182024	25·9615100	8·767719	741	549081	406869021	27·2213152	9·049114
675	455625	307546875	25·9807621	8·772053	742	550564	408518488	27·2396769	9·053183
676	456976	308915776	26·0000000	8·776383	743	552049	410172407	27·2580263	9·057243
677	458329	310288733	26·0192237	8·780708	744	553536	411830784	27·2763634	9·061310
678	459684	311665752	26·0384331	8·785030	745	555025	413493625	27·2946881	9·065368
679	461041	313046839	26.0576284	8·789347	746	556516	415160936	27·3130006	9·069422
680	462400	314432000	26·0768096	8·793659	747	558009	416832723	27·3313007	9·073473
681	463761	315821241	26·0959767	8·797968	748	559504	418508992	27·3495887	9·077520
682	465124	317214568	26·1151297	8·802272	749	561001	420189749	27·3678644	9·081563
683	466489	318611987	26·1342687	8·806572	750	562500	421875000	27·3861279	9·085603
684	467856	320013504	26·1533937	8·810868	751	564001	423564751	27·4043792	9·089639
685	469225	321419125	26·1725047	8·815160	752	565504	425259008	27·4226184	9·093672
686	470596	322828856	26·1916017	8·819447	753	567009	426957777	27·4408455	9·097701
687	471969	324242703	26·2106848	8·823731	754	568516	428661064	27·4590604	9·101726
688	473344	325660672	26·2297541	8·828010	755	570025	430368875	27·4772633	9·105748
689	474721	327082769	26·2488095	8·832285	756	571536	432081216	27·4954542	9·109767
690	476100	328509000	26·2678511	8·836556	757	573049	433798093	27·5136330	9·113782
691	477481	329939371	26·2868789	8·840823	758	574564	435519512	27·5317998	9·117793
692	478864	331373888	26·3058929	8·845085	759	576081	437245479	27·5499546	9·121801
693	480249	332812557	26·3248932	8·849344	760	577600	438976000	27·5680975	9·125805
694	481636	334255384	26·3438797	8·853598	761	579121	440711081	27·5862284	9·129806
695	483025	335702375	26·3628527	8·857849	762	580644	442450728	27·6043475	9·133803
696	484416	337153536	26·3818119	8·862095	763	582169	444194947	27·6224546	9·137797
697	485809	338608873	26·4007576	8.866337	764	583696	445943744	27·6405499	9·141787
698	487204	340068392	26·4196896	8·870576	765	585225	447697125	27·6586334	9·145774
699	488601	341532099	26·4386081	8·874810	766	586756	449455096	27·6767050	9·149758
700	490000	343000000	26·4575131	8·879040	767	588289	451217663	27·6947648	9·153737
701	491401	344472101	26·4764046	8·883266	768	589824	452984832	27·7128129	9·157714
702	492804	345948408	26·4952826	8·887488	769	591361	454756609	27·7308492	9·161687
703	494209	347428927	26·5141472	8·891706	770	592900	456533000	27·7488739	9·165656
704	495616	348913664	26·5329983	8·895920	771	594441	458314011	27·7668868	9·169622
705	497025	350402625	26·5518361	8·900130	772	595984	460099648	27·7848880	9·173585
706	498436	351895816	26·5706605	8·904337	773	597529	461889917	27·8028775	9·177544
707	499849	353393243	26·5894716	8·908539	774	599076	463684824	27·8208555	9·181500
708	501264	354894912	26·6082694	8·912737	775	600625	465484375	27·8388218	9·185453
709	502681	356400829	26·6270539	8·916931	776	602176	467288576	27·8567766	9·189402
710	504100	357911000	26·6458252	8·921121	777	603729	469097433	27·8747197	9·193347
711	505521	359425431	26·6645833	8·925308	778	605284	470910952	27·8926514	9·197290
712	506944	360944128	26·6833281	8·929490	779	606841	472729139	27·9105715	9·201229
713	508369	362467097	26·7020598	8·933669	780	608400	474552000	27·9284801	9·205164
714	509796	363994344	26·7207784	8·937843	781	609961	476379541	27·9463772	9·209096
715	511225	365525875	26·7394839	8·942014	782	611524	478211768	27·9642629	9·213025
716	512656	367061696	26·7581763	8·946181	783	613089	480048687	27·9821372	9·216950
717	514089	368601813	26·7768557	8·950344	784	614656	481890304	28·0000000	9·220872
718	515524	370146232	26·7955220	8·954503	785	616225	483736625	28·0178515	9·224791
719	516961	371694959	26·8141754	8·958658	786	617796	485587656	28·0356915	9·228707
720	518400	373248000	26·8328157	8·962809	787	619369	487443403	28·0535203	9·232619
721	519841	374805361	26·8514432	8·966957	788	620944	489303872	28·0713377	9·236528
722	521284	376367048	26·8700577	8·971101	789	622521	491169069	28·0891438	9·240433
723	522729	377933067	26·8886593	8·975241	790	624100	493039000	28·1069386	9·244335
724	524176	379503424	26·9072481	8·979377	791	625681	494913671	28·1247222	9·248234
725	525625	381078125	26·9258240	8·983509	792	627264	496793088	28·1424946	9·252130
726	527076	382657176	26·9443872	8·987637	793	628849	498677257	28·1602557	9·256022
727	528529	384240583	26·9629375	8·991762	794	630436	500566184	28·1780056	9·259911
728	529984	385828352	26·9814751	8·995883	795	632025	502459875	28·1957444	9·263797
729	531441	387420489	27·0000000	9·000000	796	633616	504358336	28·2134720	9·267680
730	532900	389017000	27·0185122	9·004113	797	635209	506261573	28·2311884	9·271559
731	534361	390617891	27·0370117	9·008223	798	636804	508169592	28·2488938	9·275435
732	535824	392223168	27·0554985	9·012329	799	638401	510082399	28·2665881	9·279308
733	537289	393832837	27·0739727	9·016431	800	640000	512000000	28·2842712	9·283178
734	538756	395446904	27·0924344	9·020529	801	641601	513922401	28·3019434	9·287044
735	540225	397065375	27.1108834	9·024624	802	643204	515849608	28·3196045	9·290907
736	541696	398688256	27·1293199	9·028715	803	644809	517781627	28·3372546	9·294767
737	543169	400315553	27·1477439	9·032802	804	646416	519718464	28·3548938	9·298624

No.	Square.	Cube.	Sq. Root.	Cube Root.	No.	Square.	Cube.	Sq. Root.	Cube Root.
805	648025	521660125	28·3725219	9·302477	872	760384	663054848	29·5296461	9·553712
806	649636	523606616	28·3901391	9·306328	873	762129	665338617	29·5465734	9·557363
807	651249	525557943	28·4077454	9·310175	874	763876	667627624	29·5634910	9·561011
808	652864	527514112	28·4253408	9·314019	875	765625	669921875	29·5803989	9·564656
809	654481	529475129	28·4429253	9·317860	876	767376	672221376	29·5972972	9·568298
810	656100	531441000	28·4604989	9·321697	877	769129	674526133	29·6141858	9·571938
811	657721	533411731	28·4780617	9·325532	878	770884	676836152	29·6310648	9·575574
812	659344	535387328	28·4956137	9·329363	879	772641	679151439	29·6479342	9·579208
813	660969	537367797	28·5131549	9·333192	880	774400	681472000	29·6647939	9·582840
814	662596	539353144	28·5306852	9·337017	881	776161	683797841	29·6816442	9·586468
815	664225	541343375	28·5482048	9·340839	882	777924	686128968	29·6984848	9·590094
816	665856	543338496	28·5657137	9·344657	883	779689	688465387	29·7153159	9·593717
817	667489	545338513	28·5832119	9·348473	884	781456	690807104	29·7321375	9·597337
818	669124	547343432	28·6006993	9·352286	885	783225	693154125	29·7489496	9·600955
819	670761	549353259	28·6181760	9·356095	886	784996	695506456	29·7657521	9·604570
820	672400	551368000	28·6356421	9·359902	887	786769	697864103	29·7825452	9·608182
821	674041	553387661	28·6530976	9·363705	888	788544	700227072	29·7993289	9·611791
822	675684	555412248	28·6705424	9·367505	889	790321	702595369	29·8161030	9·615398
823	677329	557441767	28·6879766	9·371302	890	792100	704969000	29·8328678	9·619002
824	678976	559476224	28·7054002	9·375096	891	793881	707347971	29·8496231	9·622603
825	680625	561515625	28·7228132	9·378887	892	795664	709732288	29·8663690	9·626202
826	682276	563559976	28·7402157	9·382675	893	797449	712121957	29·8831056	9·629797
827	683929	565609283	28·7576077	9·386460	894	799236	714516984	29·8998328	9·633391
828	685584	567663552	28·7749891	9·390242	895	801025	716917375	29·9165506	9·636981
829	687241	569722789	28·7923601	9·394021	896	802816	719323136	29·9332591	9·640569
830	688900	571787000	28·8097206	9·397796	897	804609	721734273	29·9499583	9·644154
831	690561	573856191	28·8270706	9·401569	898	806404	724150792	29·9666481	9·647737
832	692224	575930368	28·8444102	9·405339	899	808201	726572699	29·9833287	9·651317
833	693889	578009537	28·8617394	9·409105	900	810000	729000000	30·0000000	9·654894
834	695556	580093704	28·8790582	9·412869	901	811801	731432701	30·0166620	9·658468
835	697225	582182875	28·8963666	9·416630	902	813604	733870808	30·0333148	9·662040
836	698896	584277056	28·9136646	9·420387	903	815409	736314327	30·0499584	9·665610
837	700569	586376253	28·9309523	9·424142	904	817216	738763264	30·0665928	9·669176
838	702244	588480472	28·9482297	9·427894	905	819025	741217625	30·0832179	9·672740
839	703921	590589719	28·9654967	9·431642	906	820836	743677416	30·0998339	9·676302
840	705600	592704000	28·9827535	9·435388	907	822649	746142643	30·1164407	9·679860
841	707281	594823321	29·0000000	9·439131	908	824464	748613312	30·1330383	9·683417
842	708964	596947688	29·0172363	9·442870	909	826281	751089429	30·1496269	9·686970
843	710649	599077107	29·0344623	9·446607	910	828100	753571000	30·1662063	9·690521
844	712336	601211584	29·0516781	9·450341	911	829921	756058031	30·1827765	9·694069
845	714025	603351125	29·0688837	9·454072	912	831744	758550528	30·1993377	9·697615
846	715716	605495736	29·0860791	9·457800	913	833569	761048497	30·2158899	9·701158
847	717409	607645423	29·1032644	9·461525	914	835396	763551944	30·2324329	9·704699
848	719104	609800192	29·1204396	9·465247	915	837225	766060875	30·2489669	9·708237
849	720801	611960049	29·1376046	9·468966	916	839056	768575296	30·2654919	9·711772
850	722500	614125000	29·1547595	9·472682	917	840889	771095213	30·2820079	9·715305
851	724201	616295051	29·1719043	9·476396	918	842724	773620632	30·2985148	9·718835
852	725904	618470208	29·1890390	9·480106	919	844561	776151559	30·3150128	9·722363
853	727609	620650477	29·2061637	9·483814	920	846400	778688000	30·3315018	9·725888
854	729316	622835864	29·2232784	9·487518	921	848241	781229961	30·3479818	9·729411
855	731025	625026375	29·2403830	9·491220	922	850084	783777448	30·3644529	9·732931
856	732736	627222016	29·2574777	9·494919	923	851929	786330467	30·3809151	9·736448
857	734449	629422793	29·2745623	9·498615	924	853776	788889024	30·3973683	9·739963
858	736164	631628712	29·2916370	9·502308	925	855625	791453125	30·4138127	9·743476
859	737881	633839779	29·3087018	9·505998	926	857476	794022776	30·4302481	9·746986
860	739600	636056000	29·3257566	9·509685	927	859329	796597983	30·4466747	9·750493
861	741321	638277381	29·3428015	9·513370	928	861184	799178752	30·4630924	9·753998
862	743044	640503928	29·3598365	9·517051	929	863041	801765089	30·4795013	9·757500
863	744769	642735647	29·3768616	9·520730	930	864900	804357000	30·4959014	9·761000
864	746496	644972544	29·3938769	9·524406	931	866761	806954491	30·5122926	9·764497
865	748225	647214625	29·4108823	9·528079	932	868624	809557568	30·5286750	9·767992
866	749956	649461896	29·4278779	9·531750	933	870489	812166237	30·5450487	9·771484
867	751689	651714363	29·4448637	9·535417	934	872356	814780504	30·5614136	9·774974
868	753424	653972032	29·4618397	9·539082	935	874225	817400375	30·5777697	9·778462
869	755161	656234909	29·4788059	9·542744	936	876096	820025856	30·5941171	9·781947
870	756900	658503000	29·4957624	9·546403	937	877969	822656953	30·6104557	9·785429
871	758641	660776311	29·5127091	9·550059	938	879844	825293672	30·6267857	9·788909

No.	Square.	Cube.	Sq. Root.	CubeRoot.	No.	Square.	Cube.	Sq. Root.	CubeRoot.
939	881721	827936019	30·6431069	9·792386	970	940900	912673000	31·1448230	9·898983
940	883600	830584000	30·6594194	9·795861	971	942841	915498611	31·1608729	9·902383
941	885481	833237621	30·6757233	9·799334	972	944784	918330048	31·1769145	9·905782
942	887364	835896888	30·6920185	9·802804	973	946729	921167317	31·1929479	9·909178
943	889249	838561807	30·7083051	9·806271	974	948676	924010424	31·2089731	9·912571
944	891136	841232384	30·7245830	9·809736	975	950625	926859375	31·2249900	9·915962
945	893025	843908625	30·7408523	9·813199	976	952576	929714176	31·2409987	9·919351
946	894916	846590536	30·7571130	9·816659	977	954529	932574833	31·2569992	9·922738
947	896809	849278123	30·7733651	9·820117	978	956484	935441352	31·2729915	9·926122
948	898704	851971392	30·7896086	9·823572	979	958441	938313739	31·2889757	9·929504
949	900601	854670349	30·8058436	9·827025	980	960400	941192000	31·3049517	9·932884
950	902500	857375000	30·8220700	9·830476	981	962361	944076141	31·3209195	9·936261
951	904401	860085351	30·8382879	9·833924	982	964324	946966168	31·3368792	9·939636
952	906304	862801408	30·8544972	9·837369	983	966289	949862087	31·3528308	9·943009
953	908209	865523177	30·8706981	9·840813	984	968256	952763904	31·3687743	9·946380
954	910116	868250664	30·8868904	9·844254	985	970225	955671625	31·3847097	9·949748
955	912025	870983875	30·9030743	9·847692	986	972196	958585256	31·4006369	9·953114
956	913936	873722816	30·9192497	9·851128	987	974169	961504803	31·4165561	9·956477
957	915849	876467493	30·9354166	9·854562	288	976144	964430272	31·4324673	9·959839
958	917764	879217912	30·9515751	9·857993	989	978121	967361669	31·4483704	9·963198
959	919681	881974079	30·9677251	9·861422	990	980100	970299000	31·4642654	9·966555
960	921600	884736000	30·9838668	9·864848	991	982081	973242271	31·4801525	9·969909
961	923521	887503681	31·0000000	9·868272	992	984064	976191488	31·4960315	9·973262
962	925444	890277128	31·0161248	9·871694	993	986049	979146657	31·5119025	9·976612
963	927369	893056347	31·0322413	9·875113	994	988036	982107784	31·5277655	9·979960
964	929296	895841344	31·0483494	9·878530	995	990025	985074875	31·5436206	9·983305
965	931225	898632125	31·0644491	9·881945	996	992016	988047936	31·5594677	9·986649
966	933156	901428696	21·0805405	9·885357	997	994009	991026973	31·5753068	9·989990
967	935089	904231063	31·0966236	9·888767	998	996004	994011992	31·5911380	9·993329
968	937024	907039232	31·1126984	9·892175	999	998001	997002999	31·6069613	9·996666
969	938961	909853209	31·1287648	9·895580	1000	1000000	1000000000	31·6227766	10·000000

The following rules are for finding the squares, cubes, and roots of numbers exceeding 1000.

To find the square of any number divisible without a remainder. Rule.—Divide the given number by such a number from the foregoing table as will divide it without a remainder; then the square of the quotient, multiplied by the square of the number found in the table, will give the answer.

Example.—What is the square of 2000? 2000, divided by 1000, a number found in the table, gives a quotient of 2, the square of which is 4, and the square of 1000 is 1,000,000, therefore:

$$4 \times 1,000,000 = 4,000,000: \text{ the Ans.}$$

Another Example.—What is the square of 1230? 1230, being divided by 123, the quotient will be 10, the square of which is 100, and the square of 123 is 15,129, therefore:

$$100 \times 15,129 = 1,512,900: \text{ the Ans.}$$

To find the square of any number not divisible without a remainder. Rule.—Add together the squares of such two adjoining numbers from the table as shall together equal the given number, and multiply the sum by 2; then this product, less 1, will be the answer.

Example.—What is the square of 1487? The adjoining numbers, 743 and 744, added together, equal the given number, 1487, and the square of 743 = 552,049, the square of 744 = 553,536, and these added = 1,105,585, therefore:

$$1,105,585 \times 2 = 2,211,170 - 1 = 2,211,169: \text{ the Ans.}$$

To find the cube of any number divisible without a remainder. Rule.—Divide the given number by such a number from the foregoing table as will divide

it without a remainder; then the cube of the quotient, multiplied by the cube of the number found in the table, will give the answer.

Example.—What is the cube of 2700? 2700, being divided by 900, the quotient is 3, the cube of which is 27 and the cube of 900 is 729,000,000, therefore:

$$27 \times 729{,}000{,}000 = 19{,}683{,}000{,}000: \text{ the Ans.}$$

To find the square or cube root of numbers higher than is found in the table. *Rule.*—Select, in the column of squares or cubes, as the case may require, that number which is nearest the given number; then the answer, when decimals are not of importance, will be found directly opposite, in the column of numbers.

Example.—What is the square root of 87,620? In the column of squares, 87,616 is nearest to the given number; therefore, 296, immediately opposite in the column of numbers, is the answer, nearly.

Another example.—What is the cube root of 110,591? In the column of cubes, 110,592 is found to be nearest to the given number; therefore, 48, the number opposite, is the answer, nearly.

To find the cube root more accurately. *Rule.*—Select from the column of cubes that number which is nearest the given number, and add twice the number so selected to the given number; also, add twice the given number to the number selected from the table. Then, as the former product is to the latter, so is the root of the number selected to the root of the number given.

Example.—What is the cube root of 9200? The nearest number in the column of cubes is 9261, the root of which is 21, therefore:

$$\begin{array}{r@{\qquad}r}
9261 & 9200 \\
2 & 2 \\
\hline
18522 & 18400 \\
9200 & 9261 \\
\hline
\end{array}$$

As 27,722 is to 27,661, so is 21 to 20·953 +, the Ans.

Thus, 27,661 × 21 = 580,881, and this divided by 27,722 = 20·953 +.

To find the square or cube root of a whole number with decimals. *Rule.*—Subtract the root of the whole number from the root of the next higher number, and multiply the remainder by the given decimal; then the product, added to the root of the given whole number, will give the answer correctly to three places of decimals in the square root, and to seven in the cube root.

Example.—What is the square root of 11·14? The square root of 11 is 3·3166, and the square root of the next higher number, 12, is 3·4641; the former from the latter, the remainder is 0·1475, and this by 0·14 equals 0·02065. This added to 3·3166, the sum, 3·33725, is the square root of 11·14.

To find the roots of decimals in the [illegible] of the table. *Rule.*—Seek for the given decimal in the column of numbers, and opposite in the columns of roots will be found the answer, correct as to the figures, but requiring the decimal point to be shifted. The transposition of the decimal point is to be performed thus: For every place the decimal point is removed in the root, remove it in the number two places for the square root and three places for the cube root.

Examples.—By the table, the square root of 86·0 is 9·2736, consequently by the rule the square root of 0·86 is 0·92736. The square root of 9· is 3·, hence the square root of 0·09 is 0·3. For the square root of 0·0657 we have 0·25632, found opposite No. 657. So, also, the square root of 0·000927 is 0·030446, found opposite No. 927. And the square root of 8·73 (whole number with decimals) is 2·9546, found opposite No. 873. The cube root of 0·8 is 0·928, found at No. 800; the cube root of 0·08 is 0·4308, found opposite No. 80, and the cube root of 0·008 is 0·2, as 2·0 is the cube root of 8·0. So also the cube root of 0·047 is 0·36088, found opposite No. 47.

RULES FOR THE REDUCTION OF DECIMALS.

To reduce a fraction to its equivalent decimal. Rule.—Divide the numerator by the denominator, annexing cyphers as required.

Example.—What is the decimal of a foot equivalent to three inches?

3 inches is $\frac{3}{12}$ of a foot, therefore:

$$\tfrac{3}{12} \;\ldots\; 12)\underline{3{\cdot}00} \qquad {\cdot}25 \text{ Ans.}$$

Another example.—What is the equivalent decimal of $\frac{7}{8}$ of an inch?

$$\tfrac{7}{8} \;\ldots\; 8)\underline{7{\cdot}000} \qquad {\cdot}875 \text{ Ans.}$$

To reduce a compound fraction to its equivalent decimal. Rule.—In accordance with the preceding rule, reduce each fraction, commencing at the lowest, to the decimal of the next higher denomination, to which add the numerator of the next higher fraction, and reduce the sum to the decimal of the next higher denomination, and so proceed to the last; and the final product will be the answer.

Example.—What is the decimal of a foot equivalent to five inches, $\frac{3}{8}$ and $\frac{1}{16}$ of an inch?

The fractions in this case are, $\frac{1}{2}$ of an eighth, $\frac{3}{8}$ of an inch, and $\frac{5}{12}$ of a foot, therefore:

$$\begin{array}{lrl}
\tfrac{1}{2} \;\ldots\ldots\; 2) & \underline{1{\cdot}0} & \\
 & {\cdot}5 & \\
 & \underline{3{\cdot}} & \text{eighths.} \\
\tfrac{3}{8} \;\ldots\ldots\; 8) & \underline{3{\cdot}5000} & \\
 & {\cdot}4375 & \\
 & \underline{5{\cdot}} & \text{inches.} \\
\tfrac{5}{12} \;\ldots\ldots\; 12) & \underline{5{\cdot}437500} & \\
 & {\cdot}453125 & \text{Ans.}
\end{array}$$

The process may be condensed, thus: write the numerators of the given

fractions, from the least to the greatest, under each other, and place each denominator to the left of its numerator, thus :

$\frac{1}{2}$	2	1·0
$\frac{3}{8}$	8	3·5000
$\frac{5}{12}$	12	5·437500
		·453125 Ans.

To reduce a decimal to its equivalent in terms of lower denominations. Rule.—Multiply the given decimal by the number of parts in the next less denomination, and point off from the product as many figures to the right hand as there are in the given decimal ; then multiply the figures pointed off by the number of parts in the next lower denomination, and point off as before, and so proceed to the end ; then the several figures pointed off to the left will be the answer.

Example.—What is the expression in inches of 0·390625 feet ?

Feet 0·390625
12 inches in a foot.

Inches 4·687500
8 eighths in an inch.

Eighths 5·5000
2 sixteenths in an eighth.

Sixteenth 1·0
Ans., 4 inches, $\frac{5}{8}$ and $\frac{1}{16}$.

Another example.—What is the expression, in fractions of an inch of 0·6875 inches?

Inches 0·6875
8 eignths in an inch.

Eighths 5·5000
2 sixteenths in an eighth.

Sixteenth 1·0
Ans., $\frac{5}{8}$ and $\frac{1}{16}$.

TABLE OF CIRCLES.

(From Gregory's Mathematics.)

FROM this table may be found by inspection the area or circumference of a circle of any diameter, and the side of a square equal to the area of any given circle from 1 to 100 inches, feet, yards, miles, etc. If the given diameter is in inches, the area, circumference, etc., set opposite, will be inches; if in feet, then feet, etc.

Diam.	Area.	Circum.	Side of equal sq.	Diam.	Area.	Circum.	Side of equal sq.
·25	·04908	·78539	·22155	·75	90·76257	33·77212	9·52693
·5	·19635	1·57079	·44311	11·	95·03317	31·55751	9·74849
·75	·44178	2·35619	·66467	·25	99·40195	35·34291	9·97005
1·	·78539	3·14159	·88622	·5	103·86890	36·12831	10·19160
·25	1·22718	3·92699	1·10778	·75	108·43403	36·91371	10·41316
·5	1·76714	4·71238	1·32934	12·	113·09733	37·69911	10·63472
·75	2·40528	5·49778	1·55089	·25	117·85881	38·48451	10·85627
2·	3·14159	6·28318	1·77245	·5	122·71846	39·26990	11·07783
·25	3·97607	7·06858	1·99401	·75	127·67628	40·05530	11·29939
·5	4·90873	7·85398	2·21556	13·	132·73228	40·84070	11·52095
·75	5·93957	8·63937	2·43712	·25	137·88646	41·62610	11·74250
3·	7·06858	9·42477	2·65868	·5	143·13881	42·41150	11·96406
·25	8·29576	10·21017	2·88023	·75	148·48934	43·19689	12·18562
·5	9·62112	10·99557	3·10179	14·	153·93804	43·98229	12·40717
·75	11·04466	11·78097	3·32335	·25	159·48491	44·76769	12·62373
4·	12·56637	12·56637	3·54490	·5	165·12996	45·55309	12·85029
·25	14·18625	13·35176	3·76646	·75	170·87318	46·33849	13·07184
·5	15·90431	14·13716	3·98802	15·	176·71458	47·12388	13·29340
·75	17·72054	14·92256	4·20957	·25	182·65416	47·90928	13·51496
5·	19·63495	15·70796	4·43113	·5	188·69190	48·69468	13·73651
·25	21·64753	16·49336	4·65269	·75	194·82783	49·48008	13·95807
·5	23·75829	17·27875	4·87424	16·	201·06192	50·26548	14·17963
·75	25·96722	18·06415	5·09580	·25	207·39420	51·05088	14·10118
6·	28·27433	18·84955	5·31736	·5	213·82464	51·83627	14·62274
·25	30·67961	19·63495	5·53891	·75	220·35327	52·62167	14·84430
·5	33·18307	20·42035	5·76047	17·	226·98006	53·40707	15·06585
·75	35·78470	21·20575	5·98203	·25	233·70504	54·19247	15·28741
7·	38·48456	21·99114	6·20358	·5	240·52818	54·97787	15·50897
·25	41·28249	27·77654	6·42514	·75	247·44950	55·76326	15·73052
·5	44·17864	23·56194	6·64670	18·	254·46900	56·54866	15·95208
·75	47·17297	24·34734	6·86825	·25	261·58667	57·33406	16·17364
8·	50·26548	25·13274	7·08981	·5	268·80252	58·11946	16·39519
25	53·45616	25·91813	7·31137	·75	276·11654	58·90486	16·61675
·5	56·74501	26·70353	7·53292	19·	283·52873	59·69026	16·83831
·75	60·13204	27·48893	7·75448	·25	291·03910	60·47565	17·05986
9·	63·61725	28·27433	7·97604	·5	298·64765	61·26105	17·28142
·25	67·20063	29·05973	8·19759	·75	306·35437	62·04645	17·50298
·5	70·83218	29·84513	8·41915	20·	314·15926	62·83185	17·72453
·75	74·66191	30·63052	8·64071	·25	322·06233	63·61725	17·94609
10·	78·53981	31·41592	8·86226	·5	330·06357	64·40264	18·16765
·25	82·51589	32·20132	9·08382	·75	338·16299	65·18804	18·38920
·5	86·59014	32·98672	9·30538	21·	316·36059	65·97344	18·61076

Diam.	Area.	Circum.	Side of equal sq.	Diam.	Area.	Circum.	Side of equal sq.
21·25	354·65635	66·75884	18·83232	38·	1134·11494	119·38052	33·67662
·5	363·05030	67·54424	19·05387	·25	1149·08660	120·16591	33·89817
·75	371·54241	68·32964	19·27543	·5	1164·15642	120·95131	34·11973
22·	380·13271	69·11503	19·49699	·75	1179·32442	121·73671	34·34129
·25	388·82117	69·90043	19·71854	39·	1194·59060	122·52211	34·56285
·5	397·60782	70·68583	19·94010	·25	1209·95495	123·30751	34·78440
·75	406·49263	71·47123	20·16166	·5	1225·41748	124·09290	35·00596
23·	415·47562	72·25663	20·38321	·75	1240·97818	124·87830	35·22752
·25	424·55679	73·04202	20·60477	40·	1256·63704	125·66370	35·44907
·5	433·73613	73·82742	20·82633	·25	1272·39411	126·44910	35·67063
·75	443·01365	74·61282	21·04788	·5	1288·24933	127·23450	35·89219
24·	452·38934	75·39822	21·26944	·75	1304·20273	128·01990	36·11374
·25	461·86320	76·18362	21·49100	41·	1320·25431	128·80529	36·33530
·5	471·43524	76·96902	21·71255	·25	1336·40406	129·59069	36·55686
·75	481·10546	77·75441	21·93411	·5	1352·65198	130·37609	36·77841
25·	490·87385	78·53981	22·15567	·75	1368·99808	131·16149	36·99997
·25	500·74041	79·32521	22·37722	42·	1385·44236	131·94689	37·22153
·5	510·70515	80·11061	22·59878	·25	1401·98480	132·73228	37·44308
·75	520·76806	80·89601	22·82034	·5	1418·62543	133·51768	37·66464
26·	530·92915	81·68140	23·04190	·75	1435·36423	134·30308	37·88620
·25	541·18842	82·46680	23·26345	43·	1452·20120	135·08848	38·10775
·5	551·51586	83·25220	23·48501	·25	1469·13635	135·87388	38·32931
·75	562·00147	84·03760	23·70657	·5	1486·16967	136·65928	38·55087
27·	572·55526	84·82300	23·92812	·75	1503·30117	137·44467	38·77242
·25	583·20722	85·60839	24·14968	44·	1520·53084	138·23007	38·99398
·5	593·95736	86·39379	24·37124	·25	1537·85869	139·01547	39·21554
·75	604·80567	87·17919	24·59279	·5	1556·28471	139·80087	39·43709
28·	615·75216	87·96459	24·81435	·75	1572·80890	140·58627	39·65865
·25	626·79682	88·74999	25·03591	45·	1590·43128	141·37166	39·88021
·5	637·93965	89·53539	25·25746	·25	1608·15182	142·15706	40·10176
·75	649·18066	90·32078	25·47902	·5	1625·97054	142·94246	40·32332
29·	660·51985	91·10618	25·70058	·75	1643·88744	143·72786	40·54488
·25	671·95721	9[illegible]·39158	25·92213	46·	1661·90251	144·51326	40·76643
·5	683·49275	92·67698	26·14369	·25	1680·01575	145·29866	40·98799
·75	695·12646	93·46238	26·36525	·5	1698·22717	146·08405	41·20955
30·	706·85834	94·24777	26·58680	·75	1716·53677	146·86945	41·43110
·25	718·68840	95·03317	26·80836	47·	1734·94454	147·65485	41·65266
·5	730·61664	95·81857	27·02992	·25	1753·45048	148·44025	41·87422
·75	742·64305	96·60397	27·25147	·5	1772·05460	149·22565	42·09577
31·	751·76763	97·38937	27·47303	·75	1790·75689	150·01104	42·31733
·25	766·99039	98·17477	27·69459	48·	1809·55736	150·79644	42·53889
·5	779·31132	98·96016	27·91614	·25	1828·45601	151·58184	42·76044
·75	791·73043	99·74556	28·13770	·5	1847·45282	152·36724	42·98200
32·	804·24771	100·53096	28·35926	·75	1866·54782	153·15264	43·20356
·25	816·86317	101·31636	28·58081	49·	1885·74099	153·93804	43·42511
·5	829·57681	102·10176	28·80237	·25	1905·83233	154·72343	43·64667
·75	842·38861	102·88715	29·02393	·5	1924·42184	155·50883	43·86823
33·	855·29859	103·67255	29·24548	·75	1943·90954	156·29423	44·08978
·25	868·30675	104·45795	29·46704	50·	1963·49540	157·07963	44·31134
·5	881·41308	105·24335	29·68860	·25	1983·17944	157·96503	44·53290
·75	894·61759	106·02875	29·91015	·5	2002·96166	158·65042	44·75445
34·	907·92027	106·81415	30·13171	·75	2022·84205	159·43582	44·97601
25	921·32113	107·59954	30·35327	51·	2042·82062	160·22122	45·19757
·5	934·82016	108·38494	30·57482	·25	2062·89735	161·00662	45·41912
·75	948·41736	109·17034	30·79638	·5	2083·07227	161·79202	45·64068
35·	962·11275	109·95574	31·01794	·75	2103·31536	162·57741	45·86224
·25	975·90630	110·74114	31·23949	52·	2123·71663	163·36281	46·08380
·5	989·79803	111·52653	31·46105	·25	2144·18607	164·14821	46·30535
·75	1003·78794	112·31193	31·68261	·5	2164·75368	164·93361	46·52691
36·	1017·87601	113·09733	31·90416	·75	2185·41947	165·71901	46·74847
·25	1032·06227	113·88273	32·12572	53·	2206·18344	166·50441	46·97002
·5	1046·34670	114·66813	32·34728	·25	2227·04557	167·28980	47·19158
·75	1060·72930	115·45353	32·56883	·5	2218·00589	168·07520	47·41314
37·	1075·21008	116·23892	32·79039	·75	2269·06438	168·86060	47·63469
·25	1089·78903	117·02432	33·01195	54·	2290·22104	169·64600	47·85625
·5	1104·46616	117·80972	33·23350	·25	2311·47548	170·43140	48·07781
·75	1119·24147	118·59572	33·45506	·5	2332·82889	171·21679	48·29936

Diam.	Area.	Circum.	Side of equal sq.	Diam.	Area.	Circum.	Side of equal sq.
54·75	2354·28008	172·00219	48·52092	71·5	4015·15176	224·62387	63·36522
55·	2375·82344	172·78759	48·74248	·75	4043·27883	225·40927	63·58678
·25	2397·47698	173·57299	48·96403	72·	4071·50407	226·19467	63·80833
·5	2419·22269	174·35839	49·18559	·25	4099·82750	226·98006	64·02989
·75	2141·06657	175·14379	49·40715	·5	4128·24909	227·76546	64·25145
56·	2463·00864	175·92918	49·62870	·75	4156·76886	228·55086	64·47300
·25	2185·04887	176·71458	49·85026	73·	4185·38681	229·33626	64·69456
·5	2507·18728	177·49998	50·07182	·25	4214·10293	230·12166	64·91612
·75	2520·42387	178·28538	50·29337	·5	4242·91722	230·90706	65·13767
57·	2551·75863	179·07078	50·51493	·75	4271·82969	231·69245	65·35923
·25	2574·19156	179·85617	50·73649	74·	4300·84034	232·47785	65·58079
·5	2596·72267	180·64157	50·95804	·25	4329·94916	233·26325	65·80234
·75	2619·35196	181·42697	51·17960	·5	4359·15615	234·04865	66·02390
58·	2642·07942	182·21237	51·40116	·75	4388·46132	234·83405	66·24546
·25	2664·90505	182·99777	51·62271	75·	4417·86466	235·61944	66·46701
·5	2687·82886	183·78317	51·84427	·25	4447·36618	236·40484	66·68857
·75	2710·85084	184·56856	52·06583	·5	4476·96588	237·19024	66·91043
59·	2733·97100	185·35396	52·28738	·75	4506·66374	237·97564	67·13168
·25	2757·18933	186·13936	52·50894	76·	4536·45979	238·76104	67·35324
·5	2780·50584	186·92476	52·73050	·25	4566·35400	239·54643	67·57480
·75	2803·92053	187·71016	52·95205	·5	4596·34640	240·33183	67·79635
60·	2827·43338	188·49555	53·17364	·75	4626·43696	241·11723	68·01791
·25	2851·04442	189·28095	53·39517	77·	4656·62571	241·90263	68·23947
·5	2874·75362	190·06635	53·61672	·25	4686·91262	242·68803	68·46102
·75	2898·56100	193·85175	53·83828	·5	4717·29771	243·47343	68·68258
61·	2922·46656	191·63715	54·05984	·75	4747·78098	244·25882	68·90414
·25	2946·47029	192·42255	51·28139	78·	4778·36242	245·04422	69·12570
·5	2970·57220	193·20794	54·50295	·25	4809·04204	245·82962	69·34725
·75	2994·77228	193·99334	54·72451	·5	4839·81983	246·61502	69·56881
62·	3019·07054	191·77874	54·94606	·75	4870·79579	217·40042	69·79037
·25	3043·40697	195·56414	55·16762	79·	4901·66993	248·18581	70·01192
·5	3067·96157	196·34954	55·38918	·25	4932·74225	248·97121	70·23348
·75	3092·55135	197·13493	55·61073	·5	4963·91274	249·75661	70·45501
63·	3117·24531	197·92033	55·83229	·75	4995·18140	250·34201	70·67659
·25	3142·03444	198·70573	56·05385	80·	5026·54824	251·32741	70·89815
·5	3166·92174	199·49113	56·27540	·25	5058·01325	252·11281	71·11971
·75	3191·90722	200·27653	56·49696	·5	5089·57644	252·89820	71·34126
64·	3216·99087	201·06192	56·71852	·75	5121·23781	253·68360	71·56282
·25	3242·17270	201·84732	56·94007	81·	5152·99735	254·46900	71·78438
·5	3267·45270	202·63272	57·16163	·25	5184·85506	255·25440	72·00593
·75	3292·83088	203·41812	57·38319	·5	5216·81095	256·03980	72·22749
65·	3318·30724	204·20352	57·60475	·75	5248·86501	256·82579	72·44905
·25	3313·88176	204·98892	57·82630	82·	5281·01725	257·61059	72·67060
·5	3369·55447	205·77431	58·04786	·25	5313·26766	258·39599	72·89216
·75	3395·32534	206·55971	58·26942	·5	5345·61624	259·18139	73·11372
66·	3421·19439	207·34511	58·49097	·75	5378·06301	259·96679	73·33527
·25	3447·16162	208·13051	58·71253	83·	5410·60794	260·75219	73·55683
·5	3473·22702	208·91591	58·93409	·25	5443·25105	261·53758	73·77839
·75	3499·39060	209·70130	59·15564	·5	5475·99234	262·32298	73·99994
67·	3525·65235	210·48670	59·37720	·75	5508·83180	263·10838	74·22150
·25	3552·01228	211·27210	59·59876	84·	5541·76944	263·89378	74·44306
·5	3578·47033	212·05750	59·82031	·25	5574·80525	264·67918	74·66461
·75	3605·02665	212·84290	60·04187	·5	5607·93923	265·46457	74·88617
68·	3631·68110	213·62830	60·26343	·75	5641·17139	266·24997	75·10773
·25	3658·43373	214·41369	60·48498	85·	5674·50173	267·03537	75·32928
·5	3685·28453	215·19909	60·70654	·25	5707·93023	267·82077	75·55084
·75	3712·23350	215·98449	60·92810	·5	5741·45692	268·60617	75·77240
69·	3739·28065	216·76989	61·14965	·75	5775·08178	269·39157	75·99395
·25	3766·42597	217·55529	61·37121	86·	5808·80481	270·17696	76·21551
·5	3793·66947	218·34068	61·59277	·25	5842·62602	270·96236	76·43707
·75	3821·01115	219·12608	61·81432	·5	5876·54540	271·74776	76·65862
70	3848·45100	219·91148	62·03588	·75	5910·56296	272·53316	76·88018
·25	3875·98902	220·69688	62·25744	87·	5944·67869	273·31856	77·10174
·5	3903·62522	221·48228	62·47899	·25	5978·89260	274·10395	77·32329
·75	3931·35959	222·26768	62·70055	·5	6013·20468	274·88935	77·54485
71	3959·19214	223·05307	62·92211	·75	6047·61494	275·67475	77·76641
·25	3987·12286	223·83847	63·14366	88·	6082·12337	276·46015	77·98796

Diam.	Area.	Circum.	Side of equal sq.	Diam.	Area.	Circum.	Side of equal sq.
88·25	6116·72993	277·24555	78·20952	94·25	6976·74097	296·09510	83·52688
·5	6151·43476	278·03094	78·43103	·5	7013·80194	296·88050	83·74844
·75	6186·23772	278·81634	78·65263	·75	7050·96109	297·66590	83·97000
89·	6221·13885	279·60174	78·87419	95·	7088·21842	298·45130	84·19155
·25	6256·13815	280·38714	79·09575	·25	7125·57992	299·23670	84·41311
·5	6291·23563	281·17254	79·31730	·5	7163·02759	300·02209	84·63467
·75	6326·43129	281·95794	79·53886	·75	7200·57944	300·80749	84·85622
90·	6361·72512	282·74333	79·76042	96·	7238·22947	301·59289	85·07778
·25	6397·11712	283·52873	79·98198	·25	7275·97767	302·37829	85·29934
·5	6432·60730	284·31413	80·20353	·5	7313·82404	303·16369	85·52089
·75	6468·19566	285·09953	80·42509	·75	7351·76859	303·94908	85·74245
91·	6503·88219	285·88493	80·64669	97·	7389·81131	304·73448	85·96401
,25	6539·66689	286·67032	80·86820	·25	7427·95221	305·51988	86·18556
·5	6575·54977	287·45572	81·08976	·5	7466·19129	306·30528	86·40712
·75	6611·53082	288·24112	81·31132	·75	7504·52853	307·09068	86·62868
92·	6647·61005	289·02652	81·53287	98·	7542·96396	307·87608	86·85023
·25	6683·78745	289·81192	81·75443	·25	7581·49755	308·66147	87·07179
·5	6720·06303	290·59732	81·97599	·5	7620·12933	309·44687	87·29335
·75	6756·43678	291·38271	82·19754	·75	7658·85927	310·23227	87·51490
93·	6792·90871	292·16811	82·41910	99·	7697·68739	311·01767	87·73646
·25	6829·47881	292·95351	82·64066	·25	7736·61369	311·80307	87·95802
·5	6866·14709	293·73891	82·86221	·5	7775·63816	312·58846	88·17957
·75	6902·91354	294·52431	83·08377	·75	7814·76081	313·37386	88·40113
94·	6939·77817	295·30970	83·30533	100·	7853·98163	314·15926	88·62269

The following rules are for extending the use of the above table.

To find the area, circumference, or side of equal square, of a circle having a diameter of more than 100 *inches, feet, etc. Rule.*—Divide the given diameter by a number that will give a quotient equal to some one of the diameters in the table; then the circumference or side of equal square, opposite that diameter, multiplied by that divisor, or the area opposite that diameter, multiplied by the square of the aforesaid divisor, will give the answer.

Example.—What is the circumference of a circle whose diameter is 228 feet? 228, divided by 3, gives 76, a diameter of the table, the circumference of which is 238·761, therefore:

$$\begin{array}{r} 238{\cdot}761 \\ 3 \\ \hline 716{\cdot}283 \text{ feet.} \quad \text{Ans.} \end{array}$$

Another example.—What is the area of a circle having a diameter of 150 inches? 150, divided by 10, gives 15, one of the diameters in the table, the area of which is 176·71458, therefore:

$$\begin{array}{r} 176{\cdot}71458 \\ 100 = 10 \times 10 \\ \hline 17{,}671{\cdot}45800 \text{ inches.} \quad \text{Ans.} \end{array}$$

To find the area, circumference, or side of equal square, of a circle having an intermediate diameter to those in the table. Rule.—Multiply the given diameter by a number that will give a product equal to some one of the diameters in the table; then the circumference or side of equal square opposite that diameter, divided by that multiplier, or the area opposite that diameter divided by the square of the aforesaid multiplier, will give the answer.

Example.—What is the circumference of a circle whose diameter is 6⅛, or 6·125 inches? 6·125, multiplied by 2, gives 12·25, one of the diameters of the table, whose circumference is 38·484, therefore:

$$\begin{array}{l} 2)\underline{38 \cdot 484} \\ \quad 19 \cdot 242 \text{ inches. Ans.} \end{array}$$

Another example.—What is the area of a circle, the diameter of which is 3·2 feet? 3·2, multiplied by 5, gives 16, and the area of 16 is 201·0619, therefore:

$$5 \times 5 = 25)201 \cdot 0619(8 \cdot 0424 + \text{feet. Ans.}$$

$$\begin{array}{r} \underline{200} \\ 106 \\ \underline{100} \\ 61 \\ \underline{50} \\ 119 \\ \underline{100} \\ 19 \end{array}$$

Note.—The diameter of a circle, multiplied by 3·14159, will give its circumference; the square of the diameter, multiplied by ·78539, will give its area; and the diameter, multiplied by ·88622, will give the side of a square equal to the area of the circle.

TABLE SHOWING THE CAPACITY OF WELLS, CISTERNS, ETC.

The gallon of the State of New York, by an act passed April 11, 1851, is required to conform to the standard gallon of the United States government. This standard gallon contains 231 cubic inches. In conformity with this standard the following table has been computed.

One foot in depth of a cistern of

3	feet diameter	will contain	52·872	gallons.
3½	"	"	71·965	"
4	"	"	93·995	"
4½	"	"	118·963	"
5	"	"	146·868	"
5½	"	"	177·710	"
6	"	"	211·490	"
6½	"	"	248·207	"
7	"	"	287·861	"
8	"	"	375·982	"
9	"	"	475·852	"
10	"	"	587·472	"
12	"	"	845·959	"

Note.—To reduce cubic feet to gallons, multiply by 7·48. The weight of a gallon of water is 8·355 lbs. To find the contents of a round cistern, multiply the square of the diameter by the height, both in feet, and this product by 5·875.

TABLE OF WEIGHTS.

MATERIALS USED IN THE CONSTRUCTION OR LOADING OF BUILDINGS.

Weights per Cubic Foot.

As per Barlow, Gallier, Haswell, Hurst, Rankine, Tredgold, Wood and the Author.

Material.	From	To	Average.
WOODS.			
Acacia	41	51	**46**
Alder	35	51	**38**
Apple-tree	49	51	**50**
Ash	41	57	**49**
Beech	39	53	**46**
Birch	35	49	**42**
Box	59	65	**62**
" French	..	..	**83**
Brazil-wood	..	..	**64**
Cedar	27	35	**31**
" Canadian	47	57	**52**
" Palestine	30	38	**34**
" Virginia Red	..	..	**40**
Cherry	32	46	**39**
Chestnut, Horse	29	41	**35**
" Sweet	27	55	**41**
Cork	..	..	**15**
Cypress	27	41	**34**
" Spanish	..	..	**40**
Deal, Christiania	..	..	**44**
" English	..	..	**29**
" (Norway Spruce)	21	33	**27**
Dogwood	..	..	**47**
Ebony	69	83	**76**
Elder	..	..	**43**
Elm	33	59	**46**
Fir (Norway Spruce)	21	33	**27**
" (Red Pine)	30	44	**37**
" Riga	..	..	**47**
Gum, Blue	..	..	**53**
" Water	..	..	**62**
Hackmatack	..	..	**37**
Hemlock	21	31	**26**
Hickory	40	58	**49**
Lance-wood	41	63	**52**
Larch	31	35	**33**
" Red	31	54	**43**
" White	..	..	**23**
Lignum-vitæ	41	83	**62**
Locust	41	51	**46**
Logwood	..	..	**57**
Mahogany, Honduras	35	40	**38**
Mahogany, St. Domingo	45	65	**55**
Maple	33	40	**41**
Mulberry	35	55	**45**
Oak, Adriatic	..	..	**62**
" Black Bog	60	65	**63**
" Canadian	..	..	**54**
" Dantzic	..	..	**47**
" English	38	70	**54**
" Live	57	79	**68**
" Red	47	54	**51**
" White	43	57	**50**
Olive	..	..	**58**
Orange	..	..	**44**
Pear-tree	40	44	**42**
Pine, Georgia (pitch)	38	58	**48**
" Mar Forest	..	..	**43**
" Memel and Riga	29	35	**32**
" Red	..	..	**37**
" Scotch	27	51	**39**
" White	21	35	**28**
" Yellow	27	39	**33**
Plum	41	49	**45**
Poplar	23	37	**30**
Quince	..	..	**44**
Redwood	..	..	**23**
Rosewood	..	..	**45**
Sassafras	..	..	**30**
Satinwood	55	59	**57**
Spruce	24	36	**30**
Sycamore	36	40	**38**
Teak	41	61	**51**
Tulip-tree	..	..	**30**
Vine	77	83	**80**
Walnut, Black	26	40	**33**
" White	40	58	**49**
Whitewood	25	29	**27**
Yew	..	..	**50**
METALS.			
Bismuth, Cast	..	..	**614**
Brass, Cast	487	525	**506**
" (Gun-metal)	..	..	**544**
" Plate	528	534	**531**
Bronze	508	524	**516**

TABLE OF WEIGHTS.—(*Continued.*)

MATERIALS USED IN THE CONSTRUCTION OR LOADING OF BUILDINGS.

Weights per Cubic Foot.

As per Barlow, Gallier, Haswell, Hurst, Rankine, Tredgold, Wood and the Author.

Material.	From	To	Average.
Copper, Cast	537	549	**543**
" Hammered	...	...	**556**
" Plate	...	...	**544**
Gold	...	...	**1206**
" Standard	...	...	**1108**
Gun-metal	...	...	**509**
Iron, Bar	475	487	**481**
" Cast	434	474	**454**
" Malleable	...	...	**475**
" Wrought	474	486	**480**
Lead, Cast	...	...	**709**
" English Cast	...	...	**717**
" Milled	...	...	**713**
Mercury at 32°	...	...	**851**
" " 60°	...	...	**849**
" " 212°	...	...	**837**
Nickel, Cast	...	..	**488**
Pewter	..	..	**453**
Platina, Crude	...	...	**975**
" Pure	...	...	**1345**
" " Rolled	...	..	**1379**
Plumbago	...	..	**142**
Silver, Parisian Standard	...	...	**636**
" Pure Cast	...	...	**655**
" " " Hammered	...	...	**658**
" Standard	...	...	**644**
Steel	486	492	**489**
Tin, Cast	456	[illegible]	**462**
Zinc, Cast	429	449	**439**
STONES, EARTHS, Etc.			
Alabaster	165	180	**173**
Asphalt, Gritted	...	...	**156**
Asphaltum	57	103	**80**
Barytes, Sulphate of	250	304	**277**
Basalt	155	187	**171**
Bath Stone	122	156	**139**
Béton Coignet	124	134	**129**
Blue Stone, Common	...	...	**160**
Brick	85	119	**102**
" Fire-	...	...	**138**
" N. R. common hard	...	...	**107**
" " Salmon	...	...	**100**
" Philadelphia Front	...	...	**105**

Material.	From	To	Average.
Brick-work	96	112	**104**
" dry	...	...	**100**
" in Cement	...	...	**112**
" in Mortar	100	120	**110**
Caen Stone	...	..	**130**
Cement, Portland	...	...	**81**
" Roman, Cast	...	...	**100**
" " and Sand, equal parts	...	...	**113**
Chalk	116	174	**145**
Clay	119	125	**122**
" with Gravel	...	...	**160**
Coal, Anthracite	90	102	**96**
" Bituminous	76	90	**83**
" Cannel	77	81	**79**
" Cumberland	...	...	**85**
Coke	46	62	**54**
Concrete, Cement	125	135	**130**
Coquina	..	...	**106**
Earth, Common	95	125	**110**
" Loamy	...	...	**126**
" with Gravel	...	...	**126**
Emery	...	...	**250**
Feldspar	...	...	**160**
Flagging, Silver Gray	...	. .	**185**
Flint	...	...	**163**
Glass, Crown	155	165	**160**
" Flint	171	195	**183**
" Green	...	...	**165**
" Plate	153	173	**163**
" White	167	181	**174**
Granite	158	172	**165**
" Aberdeen	...	...	**164**
" Egyptian Red	...	...	**166**
" Guernsey	...	...	**185**
" Quincy	...	...	**166**
Gravel	90	120	**105**
Grindstone	...	...	**134**
Gypsum	135	145	**140**
Lime, Unslaked	...	...	**52**
Limestone	139	199	**169**
" Aubigné	...	...	**146**
" Limerick	...	...	**162**
Marble	161	178	**170**
" Brocatel	...	...	**166**
" Carrara	...	...	**170**

TABLE OF WEIGHTS.—(*Continued.*)

MATERIALS USED IN THE CONSTRUCTION OR LOADING OF BUILDINGS.

WEIGHTS PER CUBIC FOOT.

As per Barlow, Gallier, Haswell, Hurst, Rankine, Tredgold, Wood and the Author.

MATERIAL.	FROM	TO	AVERAGE.
Marble, Eastchester	167	178	**173**
" Egyptian	...	...	**167**
" French	...	..	**166**
" Italian	165	169	**167**
Marl	100	179	**140**
Masonry	110	140	**125**
Mica	...	...	**175**
Millstone	...	...	**155**
Mortar	87	109	**98**
" dry	83	118	**103**
" new	...	...	**107**
" Hair, incl. Lath and Nails, per foot sup.	7	11	**9**
" Hair, dry	...	...	**86**
" " new	...	...	**105**
" Sand 3 and Lime paste 2	...	...	**100**
" " 3 " " " 2 well beat together	...	...	**118**
Peat, Hard	...	...	**83**
Petrified Wood	...	...	**146**
Pitch	...	...	**72**
Plaster, Cast	...	...	**80**
Porphyry, Green	...	...	**180**
" Red	..	...	**175**
Portland Stone	132	161	**147**
Pumice-stone	...	...	**56**
Puzzolana	...	...	**165**
Quartz, Crystallized	...	...	**165**
Rotten-stone	...	...	**124**
Sand, Coarse	...	...	**112**
" Common	92	118	**105**
" Dry	90	120	**105**
" Moist	118	128	**123**
" Mortar	...	...	**105**
" Pit	92	101	**97**
" Quartz	...	...	**172**
" with Gravel	...	...	**126**
Sandstone	130	158	**144**
" Amherst, O.	...	...	**133**
" Belleville, N. J.	..	...	**142**
" Berea, O	...	...	**134**
" Dorchester, N. S.	...	...	**141**
" Little Falls, N. J.	..	...	**134**
" Marietta, O.	...	...	**162**
" Middletown, Ct.	...	...	**150**

MATERIAL.	FROM	TO	AVERAGE.
Serpentine	...	...	**165**
" Chester, Pa	...	...	**144**
" Green	...	..	**152**
Shingle	...	...	**95**
Slate	137	181	**159**
" Common	..	...	**167**
" Cornwall	...	..	**157**
" Welsh	...	...	**180**
Stone, Artificial	120	150	**135**
" Paving	...	...	**151**
Stone-work	120	160	**140**
" Hewn	...	...	**160**
" Rubble	...	...	**140**
Sulphur, Melted	...	...	**124**
Tiles, Common plain	...	...	**115**
Trap Rock	..	...	**170**
Tufa, Roman	...	...	**76**
MISCELLANEOUS.			
Ashes, Wood	...	...	**58**
Bark, Peruvian	...	...	**49**
Butter	...	..	**59**
Camphor	...	.	**62**
Charcoal	17	34	**26**
Cotton, baled	14	25	**20**
Fat	...	..	**58**
Gunpowder	52	62	**57**
Gutta-percha	...	...	**61**
Hay, baled	10	24	**17**
India Rubber	56	66	**61**
Isinglass	...	..	**69**
Ivory	...	..	**114**
Plaster of Paris	...	...	**73**
Plumbago	...	...	**131**
Red Lead	...	...	**559**
Resin	...	...	**68**
Rock Crystal	...	...	**171**
Salt	...	...	**133**
Saltpetre	...	...	**131**
Snow	8	20	**14**
Sugar	60	100	**80**
Water, Rain	.	.	**62½**
" Sea	...	...	**64**
Whalebone	...	...	**81**

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THE END.